Mild Solutions of Stochastic Navier-Stokes Equations-II

Stochastic Integrals in Infinite Dimensions

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Outline

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1. Nuclear and Hilbert-Schmidt Operators
   - Nuclear Operators
   - Hilbert-Schmidt Operators

2. Stochastic Integrals in Hilbert Spaces
   - Infinite Dimensional Wiener Processes
   - Martingales in Banach Spaces
   - Construction of Stochastic Integrals
   - Properties of Stochastic Integrals
   - Stochastic Integrals for Cylindrical Wiener Processes

Space setting

Let us first introduce some notations of spaces and operators that we will work on.

Let $(U, <,>_U)$ and $(H, <,>)$ be two separable Hilbert spaces.

$L(U, H)$ denotes the space of bounded linear operators from U to H. L^* is its adjoint. $L(U) = L(U, U)$.

We say $L \in L(U)$ is symmetric if

$$< Lu, v >_U = < u, Lv >_U \quad \text{for all} \quad u, v \in U$$

$L \in L(U)$ is nonnegative if

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Definition

An element $T \in L(U, H)$ is said to be a nuclear operator if there exists a sequence $(a_j)_{j\in\mathbb{N}}$ in H and a sequence $(b_j)_{j\in\mathbb{N}}$ in U such that

$$Tx = \sum_{j=1}^{\infty} a_j < b_j, x >_U \quad \text{for all} \quad x \in U$$

and

$$\sum_{j\in\mathbb{N}} ||a_j|| \cdot ||b_j|| < \infty$$

Denote by $L_1(U, H)$ the space of all nuclear operators from U to H. If $U = H$, $T \in L_1(U, H)$ is nonnegative and symmetric, then T is called trace class.

Lemma

The space $L_1(U, H)$ endowed with the norm

$$||T||_{L_1(U,H)} = \inf\{\sum_{j\in\mathbb{N}} ||a_j||\cdot||b_j||_U | Tx = \sum_{j=1\infty} a_j < b_j, x >_U, x \in U\}$$

is a Banach space.

Definition

Let $T \in L(U)$, $e_k, k \in \mathbb{N}$ be an orthonormal basis of U. Then we define

$$\mathrm{tr}\, T := \sum_{k\in\mathbb{N}} < Te_k, e_k >_U$$

if the series converges.

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Trace and $L_1(U)$

The relation between a trace operator and nuclear operator is given by the following lemma.

Lemma

If $T \in L_1(U)$, then trT is well-defined independently of the choice of orthonormal basis $e_k, k \in \mathbb{N}$. Moreover we have

$$|trT| \leq ||T||_{L_1(U)}$$

Proof: Let $(a_j)_{j\in\mathbb{N}}$ and $(b_j)_{j\in\mathbb{N}}$ be sequences in U such that

$$Tx = \sum_{j\in\mathbb{N}} a_j < b_j, x >_U \quad \text{for all} \quad x \in U$$

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Then we get for any orthonormal basis $e_k, k \in \mathbb{N}$ of U that

$$< Te_k, e_k >_U = \sum_{j \in \mathbb{N}} < e_k, a_j >_U \cdot < e_k, b_j >_U$$

Therefore

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$$\leq \sum_{j \in \mathbb{N}} \left(\sum_{k \in \mathbb{N}} | < e_k, a_j >_U |^2 \right)^{1/2} \cdot \left(\sum_{k \in \mathbb{N}} | < e_k, b_j >_U |^2 \right)^{1/2}$$

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$|\mathrm{tr}\, T| \leq ||T||_{L_1(U)}$ follows and we can thus exchange the summation to get

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A bounded linear operator $T : U \to H$ is called Hilbert-Schmidt if

$$\sum_{k \in \mathbb{N}} ||Te_k||^2 < \infty$$

where $e_k, k \in \mathbb{N}$ is an orthonormal basis of U.

The space of all Hilbert-Schmidt operators from U to H is denoted by $L_2(U, H)$.

We are going to prove the following theorem on Hilbert-Schmidt operators.

Theorem

- *The definition of Hilbert-Schmidt operators and the number*

$$\|T\|^2_{L_2(U,H)} = \sum_{k \in \mathbb{N}} \|Te_k\|^2$$

  does not depend on the choice of the orthonormal basis e_k, $k \in \mathbb{N}$, *we have that* $\|T\|_{L_2(U,H)} = \|T^*\|_{L_2(H,U)}$. *For simplicity we write* $\|T\|_{L_2}$ *or* $\|T\|_{L_2(U,H)}$.
- $\|T\|_{L(U,H)} \leq \|T\|_{L_2(U,H)}$.
- *Let* G *be another Hilbert space,* $S_1 \in L(H, G)$, $S_2 \in L(G, U)$ *and* $T \in L_2(U, H)$. *Then* $S_1T \in L_2(U, G)$, $TS_2 \in L_2(G, H)$ *and*

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$$||S_1 T||_{L_2(U,G)} \leq ||S_1||_{L(H,G)} ||T||_{L_2(U,H)}$$
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Proof: If $e_k, k \in \mathbb{N}$ is an orthonormal basis of U and $f_k, k \in \mathbb{N}$ is an orthonormal basis of H. We obtain by Parseval identity that

$$\sum_{k\in\mathbb{N}} ||Te_k||^2 = \sum_{k\in\mathbb{N}} \sum_{j\in\mathbb{N}} | < Te_k, f_j > |^2$$
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So the first conclusion is proved.

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So the first conclusion is proved.

Proof continues...

Let $x \in U$ and $f_k, k \in \mathbb{N}$ be an orthonormal basis of H, then we get

$$\begin{aligned} ||Tx||^2 &= \sum_{k \in \mathbb{N}} < Tx, f_k >^2 \\ &\leq ||x||_U^2 \sum_{k \in \mathbb{N}} ||T^* f_k||_U^2 \\ &= ||T||^2_{L_2(U,H)} ||x||_U^2 \end{aligned}$$

where we used Cauchy-Schwarz inequality and property (1).

Therefore, we showed that

$$||T||_{L(U,H)} \leq ||T||_{L_2(U,H)}$$

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Let $e_k, k \in \mathbb{N}$ be an orthonormal basis of U, then

$$\sum_{k\in\mathbb{N}} ||S_1 Te_k||_G^2 \leq ||S_1||^2_{L(H,G)} ||T||^2_{L_2(U,H)}$$

Furthermore, since $(TS_2)^* = S_2^* T^*$. From above and (1), we have $TS_2 \in L_2(G,H)$ and

$$\begin{aligned} ||TS_2||_{L_2(G,H)} &= ||(TS_2)^*||_{L_2(H,G)} \\ &= ||S_2^* T^*||_{L_2(H,G)} \\ &\leq ||S_2||_{L_2(G,U)} ||T||_{L_2(U,H)} \end{aligned}$$

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A lemma

Lemma

Let $S, T \in L_2(U, H)$ and $e_k, k \in \mathbb{N}$ be an orthonormal basis of U. If we define

$$< T, S >_{L_2} := \sum_{k \in \mathbb{N}} < Se_k, Te_k >$$

we obtain that $(L_2(U, H), <, >_{L_2})$ is a separable Hilbert space.

If $f_k, k \in \mathbb{N}$ is an orthonormal basis of H, we get that $f_j \otimes e_k := f_j < e_k, \cdot >_U, j, k \in \mathbb{N}$ is an orthonormal basis of $L_2(U, H)$.

Proof: Let us first show $L_2(U,H)$ is complete.

Let $T_n, n \in \mathbb{N}$ be a Cauchy sequence in $L_2(U,H)$. Then it is clear that it is also a Cauchy sequence in $L(U,H)$. Because of the completeness of $L(U,H)$, there exists an element $T \in L(U,H)$, such that $||T_n - T||_{L(U,H)} \to 0$ as $n \to \infty$.

But by Fatou's lemma, we also have for any orthonormal basis $e_k, k \in \mathbb{N}$ of U that

$$\begin{aligned} ||T_n - T||_{L_2}^2 &= \sum_{k\in\mathbb{N}} < (T_n - T)e_k, (T_n, T)e_k > \\ &= \sum_{k\in\mathbb{N}} \liminf_{m\to\infty} ||(T_n - T_m)e_k||^2 \le \liminf_{m\to\infty} \sum_{k\in\mathbb{N}} ||(T_n - T_m)e_k||^2 \\ &= \liminf_{m\to\infty} ||T_n - T_m||_{L_2}^2 < \epsilon \end{aligned}$$

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If we define $f_j \otimes e_k := f_j < e_k, \cdot >_U$, $j, k \in \mathbb{N}$, then it is clear that $f_j \otimes e_k \in L_2(U, H)$ for all $j, k \in \mathbb{N}$ and for arbitrary $T \in L_2(U, H)$ we get

$$< f_j \otimes e_k, T >_{L_2} = \sum_{n \in \mathbb{N}} < e_k, e_n >_U \cdot < f_j, Te_n > = < f_j, Te_k >$$

Thus $f_j \otimes e_k$ is an orthonormal system.

Since $T = 0$ if $< f_j \otimes e_k, T >_{L_2} = 0$ for all $j, k \in \mathbb{N}$,

$$\text{span}(f_j \otimes e_k | j, k \in \mathbb{N}) \text{ is dense in } L_2(U, H).$$

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Since $T = 0$ if $< f_j \otimes e_k, T >_{L_2} = 0$ for all $j, k \in \mathbb{N}$,

$$\text{span}(f_j \otimes e_k | j, k \in \mathbb{N}) \text{ is dense in } L_2(U, H).$$

Proof continues...

Now we will prove that $L_2(U, H)$ is separable.

If we define $f_j \otimes e_k := f_j < e_k, \cdot >_U$, $j, k \in \mathbb{N}$, then it is clear that $f_j \otimes e_k \in L_2(U, H)$ for all $j, k \in \mathbb{N}$ and for arbitrary $T \in L_2(U, H)$ we get

$$< f_j \otimes e_k, T >_{L_2} = \sum_{n \in \mathbb{N}} < e_k, e_n >_U \cdot < f_j, Te_n > = < f_j, Te_k >$$

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Another lemma

Lemma

Let $(G, <, >_G)$ *be a separable Hilbert space. If* $T \in L_2(U, H)$ *and* $S \in L_2(H, G)$*, then* $ST \in L_2(U, G)$ *and*

$$||ST||_{L_2(U,G)} \leq ||S||_{L_2} ||T||_{L_2}$$

Proof: Let f_k be an orthonormal basis in H, then

$$STx = \sum_{k \in \mathbb{N}} < Tx, f_k > Sf_k, \quad x \in U$$

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Proof continues...

Thus

$$\begin{aligned} ||ST||_{L_1(U,G)} &\leq \sum_{k\in\mathbb{N}} ||T^* f_k||_U \cdot ||Sf_k||_G \\ &\leq (\sum_{k\in\mathbb{N}} ||T^* f_k||_U^2)^{1/2} \cdot (\sum_{k\in\mathbb{N}} ||Sf_k||_G^2)^{1/2} \\ &= ||S||_{L_2} \cdot ||T||_{L_2} \end{aligned}$$

The proof is complete.

Remark: Let $e_k, k \in \mathbb{N}$ be an orthonormal basis of U. If $T \in L(U)$ is symmetric, nonnegative with $\sum_{k\in\mathbb{N}} < Te_k, e_k > < \infty$, then $T \in L_1(U)$.

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Third lemma

Lemma

Let $L \in L(H)$ and $B \in L_2(U, H)$, then $LBB^* \in L_1(H)$, $B^*LB \in L_1(U)$ and we have

$$trLBB^* = trB^*LB$$

Proof: By previous theorem and lemma, $LBB^* \in L_1(H)$ and $B^*LB \in L_1(U)$.

Let $e_k, f_k, k \in \mathbb{N}$ be an orthonormal basis of U and H respectively. By Parseval identity

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Proof continues...

$$\sum_{k\in\mathbb{N}}\sum_{n\in\mathbb{N}} || < f_k, Be_n > \cdot < f_k, LBe_n > ||$$
$$\leq \sum_{n\in\mathbb{N}} (\sum_{k\in\mathbb{N}} | < f_k, Be_n > |^2)^{1/2} \cdot (\sum_{k\in\mathbb{N}} | < f_k, LBe_n > |^2)^{1/2}$$
$$= \sum_{n\in\mathbb{N}} ||Be_n|| \cdot ||LBe_n||$$
$$\leq ||L||_{L(H)} \cdot ||B||_{L_2}^2$$

and we can exchange the summation to get

Proof continues...

$$\begin{aligned}
&\sum_{k\in\mathbb{N}}\sum_{n\in\mathbb{N}} || < f_k, Be_n > \cdot < f_k, LBe_n > || \\
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&= \sum_{n\in\mathbb{N}} ||Be_n|| \cdot ||LBe_n|| \\
&\leq ||L||_{L(H)} \cdot ||B||_{L_2}^2
\end{aligned}$$

and we can exchange the summation to get

$$
\begin{aligned}
\mathrm{tr} LBB^* &= \sum_{k\in\mathbb{N}} < LBB^* f_k, f_k >= \sum_{k\in\mathbb{N}} < B^* f_k, B^* l^* f_k >_U \\
&= \sum_{k\in\mathbb{N}} \sum_{n\in\mathbb{N}} < B^* f_k, e_n >_U \cdot < B^* L^* f_k, e_n >_U \\
&= \sum_{n\in\mathbb{N}} \sum_{k\in\mathbb{N}} < f_k, Be_n > \cdot < f_k, LBe_n > \\
&= \sum_{n\in\mathbb{N}} < Be_n, LBe_n > \\
&= \sum_{n\in\mathbb{N}} < e_n, B^* LBe_n >_U \\
&= \mathrm{tr} B^* LB
\end{aligned}
$$

The proof is complete.

Gaussian measure

Consider two separable Hilbert spaces $(U, <,>_U)$ and $(H, <,>)$. Denote $\mathcal{B}(X)$ as the Borel σ-algebra of X.

Definition

A probability measure μ on $(U, \mathcal{B}(U))$ is called *Gaussian* if for all $v \in U$, the bounded linear mapping $v' : U \to \mathbb{R}$ defined by $u \mapsto < u, v >_U, \quad u \in U$ has a Gaussian law, i.e. for all $v \in U$, there exists $m := m(v) \in \mathbb{R}$ and $\sigma := \sigma(v) \in [0, \infty[$ such that if $\sigma(v) > 0$

$$(\mu \circ (v')^{-1})(A) = \mu(v' \in A) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_A e^{-\frac{(x-m)^2}{2\sigma^2}} dx$$

for all $A \in \mathcal{B}(\mathbb{R})$. If $\sigma(v) = 0$, $\mu \circ (v')^{-1} = \delta_{m(v)}$.

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Mean and Covariance

Theorem

A measure μ on $(U, \mathcal{B}U)$ is Gaussian if and only if

$$\hat{m}u(u) := \int_U e^{i<u,v>_U} \mu(dv) = e^{i<m,u>_U - \frac{1}{2}<Qu,u>_U}, \quad u \in U,$$

where $m \in U$ and $Q \in L(U)$ is nonnegative, symmetric with finite trace (trace class).

In this case μ is denoted by $N(m, Q)$ and m is called mean and Q is called covariance. The measure μ is uniquely determined by m and Q.

Mean and Covariance

Theorem

Furthermore, for all $h, g \in U$

$$\int < x, h >_U \mu(dx) =< m, h >_U,$$

$$\int (< x, h >_U - < m, h >_U)(< x, g >_U - < m, g >_U)\mu(dx) =< Qh, g >_U,$$

$$\int ||x - m||_U^2 \mu(dx) = trQ.$$

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Furthermore, for all $h, g \in U$

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$$\int ||x - m||_U^2 \mu(dx) = trQ.$$

Representation of Gaussian random variables

Lemma

If $Q \in L(U)$ is of trace class, then there exists an orthonormal basis $e_k, k \in \mathbb{N}$ of U such that

$$Qe_k = \lambda_k e_k, \quad \lambda_k \geq 0, k \in \mathbb{N}$$

0 is the only accumulation point of $(\lambda_k)_{k\in\mathbb{N}}$.

Theorem

Let $m \in U$, $Q \in L(U)$ of trace class. In addition, assume $\{e_k\}$ is an orthonormal basis of U with eigenvectors of Q and corresponding eigenvalues λ_k.

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Theorem

Then a U-valued random variable X on $(\Omega, \mathcal{F}, P)$ is Gaussian with $P \circ X^{-1} = N(m, Q)$ if and only if $X = \sum_{k \in \mathbb{N}} \sqrt{\lambda_k} \beta_k e_k + m$. Here β_k are independent real-valued random variables with $P \circ \beta_k^{-1} = N(0, 1)$ for all $k \in \mathbb{N}$ with $\lambda_k > 0$. The series converges in $\mathrm{L}^2(\Omega, \mathcal{F}, P; U)$.

Proof: Let X be a Gaussian random variable with mean m and covariance Q. Set $<,>=<,>_U$, then $X = \sum_{k \in \mathbb{N}} < X, e_k > e_k$ in U, $< X, e_k >$ is normally distributed with mean $< m, e_k >$ and variance $\lambda_k, k \in \mathbb{N}$ by lemma.

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Proof continues...

Define

$$\beta_k := \begin{cases} \frac{<X,e_k>-<m,e_k>}{\sqrt{\lambda_k}} & \text{if} \quad k \in \mathbb{N}, \lambda_k > 0 \\ 0 & \text{otherwise} \end{cases}$$

then $P \circ \beta_k^{-1} = N(0,1)$ and $X = \sum_{k \in \mathbb{N}} \sqrt{\lambda_k} \beta_k e_k + m$.

To prove the independence of β_k, take an arbitrary $n \in \mathbb{N}$, $a_k \in \mathbb{R}$, $1 \leq k \leq n$ for

$$c := -\sum_{k=1, \lambda_k \neq 0}^{n} \frac{a_k}{\sqrt{\lambda_k}} < m, e_k >$$

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$$\sum_{k=1}^{n} a_k \beta_k = \sum_{k=1,\lambda_k \neq 0}^{n} \frac{a_k}{\sqrt{\lambda_k}} < X, e_k > + c$$

$$< X, \sum_{k=1,\lambda_k \neq 0}^{n} \frac{a_k}{\sqrt{\lambda_k}} e_k > + c$$

which is normally distributed since X is a Gaussian random variable. Thus β_k form a Gaussian family.

$$E(\beta_i \beta_j) = \frac{1}{\sqrt{\lambda_i \lambda_j}} E(< X - m, e_i >< X - m, e_j >)$$

$$\frac{1}{\sqrt{\lambda_i \lambda_j}} < Qe_i, e_j >= \frac{\lambda_i}{\sqrt{\lambda_i \lambda_j}} < e_i, e_j >$$

Proof continues...

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Besides, $\sum_{k=1}^{n} \sqrt{\lambda_k} \beta_k e_k$, $n \in \mathbb{N}$ converges in $\mathrm{L}^2(\Omega, \mathcal{F}, P; U)$ since the space is complete and

$$E(||\sum_{k=m}^{n} \sqrt{\lambda_k} \beta_k e_k||^2) = \sum_{k=m}^{n} \lambda_k E(||\beta_k||^2)$$

$$= \sum_{k=m}^{n} \lambda_k \to 0 \quad \text{as} \quad n, m \to 0$$

because $\sum_{k \in \mathbb{N}} \lambda_k = \mathrm{tr} Q < \infty$. This complete one direction of the proof.

Proof continues...

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Proof continues...

Let e_k be an orthonormal basis of U such that $Qe_k = \lambda_k e_k, k \in \mathbb{N}$. Let β_k be a family of independent real-valued Gaussian random variables with mean 0 and variance 1. Then

$$\sum_{k=1}^{n} \sqrt{\lambda_k}\beta_k e_k + m, n \in \mathbb{N} \to x := \sum_{k \in \mathbb{N}} \sqrt{\lambda_k}\beta_k e_k + m$$

in $\mathrm{L}^2(\Omega, \mathcal{F}, P; U)$ from the first part of the proof.

Proof continues...

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Proof continues...

Fix $u \in U$, we get

$$< \sum_{k=1}^{n} \sqrt{\lambda_k}\beta_k e_k + m, u >= \sum_{k=1}^{n} \sqrt{\lambda_k}\beta_k < e_k, u > + < m, u >$$

is normally distributed for all $n \in \mathbb{N}$ and the sequence converges in $\mathrm{L}^2(\Omega, \mathcal{F}, P)$. This implies that the limit $< X, u >$ is also normally distributed.

$$E(< X, u >) = E(\sum_{k \in \mathbb{N}} \sqrt{\lambda_k}\beta_k < e_k, u > + < m, u >)$$

$$= \lim_{n \to \infty} E(\sum_{k=1}^{n} \sqrt{\lambda_k}\beta_k < e_k, u >) + < m, u > = < m, u >$$

Proof continues...

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Proof continues...

and

$$\begin{aligned} & E((< X, u > - < m, u >)(< X, v > - < m, v >)) \\ & = \lim_{n\to\infty} E(\sum_{k=1}^{n} \sqrt{\lambda_k}\beta_k < e_k, u > \sum_{k=1}^{n} \sqrt{\lambda_k}\beta_k < e_k, v >) \\ & = \sum_{k\in\mathbb{N}} \lambda_k < e_k, u >< e_k, v >= \sum_{k\in\mathbb{N}} < Qe_k, u >< e_k, v > \\ & = \sum_{k\in\mathbb{N}} < e_k, Qu >< e_k, v >=< Qu, v > \end{aligned}$$

for all $u, v \in U$. The proof is complete.

Proof continues...

and

$$
\begin{aligned}
& E\left((< X, u > - < m, u >)(< X, v > - < m, v >)\right) \\
&= \lim_{n\to\infty} E\Big(\sum_{k=1}^{n} \sqrt{\lambda_k}\beta_k < e_k, u > \sum_{k=1}^{n} \sqrt{\lambda_k}\beta_k < e_k, v >\Big) \\
&= \sum_{k\in\mathbb{N}} \lambda_k < e_k, u >< e_k, v >= \sum_{k\in\mathbb{N}} < Qe_k, u >< e_k, v > \\
&= \sum_{k\in\mathbb{N}} < e_k, Qu >< e_k, v >=< Qu, v >
\end{aligned}
$$

for all $u, v \in U$. The proof is complete.

Existence result

From the proof above, we have the following existence result for Gaussian measure.

Corollary

Let $Q \in L(U)$ be trace class and $m \in U$. Then there exists a Gaussian measure $\mu = N(m, Q)$ on $(U, \mathcal{B}U)$.

Q-Wiener Processes

Definition

A U-valued stochastic process $W(t)$, $t \in [0, T]$ on probability space $(\Omega, \mathcal{F}, P)$ is called a (standard) Q-Wiener process if

- $W(0) = 0$
- W has P-a.s. continuous trajectories
- $W(t_1), W(t_2) - W(t_1), \cdots, W(t_n) - W(t_{n-1})$ are independent for all $0 \leq t_1 < \cdots < t_n \leq T$, $n \in \mathbb{N}$
- the increments have the following Gaussian laws

$$P \circ (W(t) - W(s))^{-1} = N(0, (t-s)Q) \quad \text{for all} \quad 0 \leq s \leq t \leq T$$

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Representation of Q-Wiener processes

Lemma

Let e_k be an orthonormal basis of U consisting of eigenvectors of Q with corresponding eigenvalues $\lambda_k, k \in \mathbb{N}$. Then a U-valued stochastic process $W(t)$, $t \in [0, T]$ is a Q-Wiener process if and only if

$$W(t) = \sum_{k \in \mathbb{N}} \sqrt{\lambda_k} \beta_k(t) e_k, \quad t \in [0, T]$$

where $\beta_k(t)$, $k \in \{n \in \mathbb{N} | \lambda_n > 0\}$ are independent real-valued Brownian motion on $(\Omega, \mathcal{F}, P)$. The series converges in $\mathrm{L}^2(\Omega, \mathcal{F}, P, C([0, T], U))$, thus has a P-a.s. continuous modification. In particular, for any Q as above there exists a Q-Wiener process on U.

Normal filtration

Definition

A filtration $\mathcal{F}_t$, $t \in [0,T]$ on $(\Omega, \mathcal{F}, P)$ is called normal if

- $\mathcal{F}_0$ contains all elements $A \in \mathcal{F}$ with $P(A) = 0$
- $\mathcal{F}_t = \mathcal{F}_{t+} = \cap_{s>t}\mathcal{F}_s$ for all $t \in [0,T]$

Definition

A Q-Wiener process $W(t)$, $t \in [0,T]$ is called a Q-Wiener process w.r.t. a filtration $\mathcal{F}_t$, $t \in [0,T]$ if

- $W(t)$, $t \in [0,T]$ is adapted to $\mathcal{F}_t, t \in [0,T]$
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Existence of normal filtration Q-Wiener processes

Define

$$\mathcal{N} := \{A \in \mathcal{F} | P(A) = 0\}, \quad \tilde{\mathcal{F}}_t := \sigma(W(s) | s \leq t)$$

$$\tilde{\mathcal{F}}_t^0 := \sigma(\tilde{\mathcal{F}}_t \cup \mathcal{N})$$

Then we get

$$\mathcal{F}_t := \cap_{s>t} \tilde{\mathcal{F}}_s^0, \quad t \in [0, T]$$

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Existence of normal filtration Q-Wiener processes

From above we have

Lemma

Let $W(t)$, $t \in [0, T]$ be an arbitrary U-valued Q-Wiener process on $(\Omega, \mathcal{F}, P)$. Then it is a Q-Wiener process w.r.t. the normal filtration $\mathcal{F}_t$, $t \in [0, T]$ defined as above.

Bochner integrable random variables

Lemma

Assume that E is a separable real Banach space. Let X be a Bochner integrable E-valued random variable defined on $(\Omega, \mathcal{F}, P)$. Let $\mathcal{G}$ be a σ-field contained in $\mathcal{F}$.

Then there exists a unique Bochner integrable E-valued random variable z a.s., measurable with respect to $\mathcal{G}$ such that

$$\int_A X dP = \int_A z dP \quad \text{for all} \quad A \in \mathcal{G}$$

The random variable z is denoted by $E(X|\mathcal{G})$ and is called the conditional expectation of X given $\mathcal{G}$. Furthermore

$$||E(X|\mathcal{G})|| \leq E(||X|||\mathcal{G}).$$

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Property of conditional expectation

Lemma

Let (E_1, ε_1) and (E_2, ε_2) be two-separable spaces and $\psi : E_1 \times E_2 \to \mathbb{R}$ a bounded measurable function. Let X_1, X_2 be two random variables on $(\Omega, \mathcal{F}, P)$ with values in (E_1, ε_1), (E_2, ε_2) respectively and let $\mathcal{G} \subset \mathcal{F}$ be a fixed σ-field.

Assume that X_1 is $\mathcal{G}$-measurable and X_2 is independent of $\mathcal{G}$, then

$$E(\phi(X_1, X_2)|\mathcal{G}) = \phi \hat{X}_1$$

where $\phi(\hat{x}_1) = E(\phi(x_1, x_2))$, $x_1 \in E_1$.

$\mathcal{F}_t$-martingale

Definition

Let $M(t)$, $t \geq 0$ be a stochastic process on $(\Omega, \mathcal{F}, P)$ with values in a separable Banach space E, let $\mathcal{F}_t$,$t \geq 0$ be a filtration on $(\Omega, \mathcal{F}, P)$. The process M is called an $\mathcal{F}$-martingale, if

- $E(||M(t)||) < \infty$ for all $t \geq 0$
- $M(t)$ is $\mathcal{F}_t$-measurable for all $t \geq 0$
- $E(M(t)|\mathcal{F}_s) = M(s)$ P-a.s. for all $0 \leq s \leq t < \infty$.

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Properties of $\mathcal{F}_t$-martingale

Lemma

If $M(t), t \geq 0$ is an E-valued $\mathcal{F}_t$-martingale and $p \in [1, \infty)$, then $||M(t)||^p, t \geq 0$ is a real-valued $\mathcal{F}_t$-martingale.

Proof: Since E is separable, there exists $l_k \in E^*$, $k \in \mathbb{N}$ such that $||z|| = \sup l_k(z)$ for all $z \in E$. Then for $s < t$,

$$\begin{aligned} E(||M_t|| \,|\, \mathcal{F}_s) &\geq \sup_k E(l_k(M_t) | \mathcal{F}_s) \\ &= \sup_k l_k(E(M_t | \mathcal{F}_s)) \\ &= \sup_k l_k(M_s) = ||M_s|| \end{aligned}$$

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Properties of $\mathcal{F}_t$-martingale

This proves the lemma for $p = 1$. Jensen's inequality implies that for all $p \in [1, \infty)$,

$$E(||M_t||^p|\mathcal{F}_s) \geq (E(||M_t|||\mathcal{F}_s))^p$$

Thus the lemma holds for any $p \in [1, \infty)$.

Theorem

Let $p > 1$. Let E be a separable Banach space. If $M(t), t \in [0, T]$ is a right-continuous E-valued $\mathcal{F}_t$-martingale, then

$$E(\sup_{t\in[0,T]} ||M(t)||^p)^{1/p} \leq \frac{p}{p-1} \sup_{t\in[0,T]} (E(||M(t)||^p))^{1/p}$$

$$= \frac{p}{p-1}(E(||M(T)||^p))^{1/p}$$

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Square integrable martingales

Denote by $\mathcal{M}_T^2(E)$ the space of all E-valued continuous square integrable martingales $M(t), t \in [0, T]$.

Lemma

The space $\mathcal{M}_T^2(E)$ equipped with the norm

$$||M(t)||_{\mathcal{M}_T^2} := \sup_{t\in[0,T]} (E(||M(t)||^2))^{1/2} = (E(||M(T)||^2))^{1/2}$$
$$= (E(\sup_{t\in[0,T]} ||M(t)||^2))^{1/2} \leq 2E(||M(T)||^2)^{1/2}$$

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Properties of square integrable martingales

Lemma

Let $T > 0$, $W(t)$,$t \in [0, T]$ be a U-valued Q-Wiener process with respect to a normal filtration $\mathcal{F}_t$, $t \in [0, T]$ on $(\Omega, \mathcal{F}, P)$. Then $W(t)$, $t \in [0, T]$ is a continuous square integrable $\mathcal{F}_t$-martingale, i.e. $W \in \mathcal{M}_T^2(U)$.

Proof: The continuity follows from the definition of Q-Wiener processes.

For each $t \in [0, T]$, we have

$$E(||W(t)||_U^2) = t\mathrm{tr}Q < \infty.$$

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Proof of the lemma

Hence let $0 \leq s \leq t \leq T$, $A \in \mathcal{F}_s$. By proposition,

$$< \int_A W(t) - W(s) dP, u >_U = \int_A < W(t) - W(s), u >_U dP$$
$$= P(A) \int < W(t) - W(s), u >_U dP = 0$$

for all $u \in U$ as $\mathcal{F}_s$ is independent of $W(t) - W(s)$ and

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\end{aligned}
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Proof continues...

Therefore

$$\begin{aligned}\int_A W(t)dP &= \int_A W(s) + (W(A) - W(s))dP \\ &= \int_A W(s)dP + \int_A W(t) - W(s)dP \\ &= \int_A W(s)dP = W(s)\end{aligned}$$

for all $A \in \mathcal{F}_s$.

The proof is complete.

Proof continues...

Therefore

$$\begin{aligned}\int_A W(t)dP &= \int_A W(s) + (W(A) - W(s))dP \\ &= \int_A W(s)dP + \int_A W(t) - W(s)dP \\ &= \int_A W(s)dP = W(s)\end{aligned}$$

for all $A \in \mathcal{F}_s$.

The proof is complete.

Elementary process

First we consider the following class of processes:

Definition

An $L = L(U, H)$-valued process $\phi(t), t \in [0, T]$ on $(\Omega, \mathcal{F}, P)$ with normal filtration $\mathcal{F}_t, t \in [0, T]$ is said to be elementary if there exists $0 = t_0 < \cdots < t_k = T, k \in \mathbb{N}$ such that

$$\phi(t) = \sum_{m=1}^{k-1} \phi_m \mathbf{1}_{]t_m, t_{m+1}]}(t), \quad t \in [0, T]$$

where $\phi_m : \Omega \to L(U, H)$ is $\mathcal{F}_{t_m}$-measurable with respect to strong σ-algebra on $L(U, H)$, $0 \le m \le k-1$. ϕ_m takes only a finite number of values in $L(U, H)$, $0 \le m \le k-1$.

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ϕ_m

Denote the space of elementary process defined as ε. Define

$$\begin{aligned} \text{Int}(\phi)(t) :&= \int_0^t \phi(s)dW(s) \\ &:= \sum_{m=0}^{k-1} \phi_m[W(t_{m+1} \wedge t) - W(t_m \wedge t)], t \in [0, T] \end{aligned}$$

For all $\phi \in \varepsilon$, we have

Lemma

The stochastic integral $\int_0^t \phi(s)dW(s)$, $t \in [0, T]$ is a continuous square integrable martingale w.r.t. $\mathcal{F}_t$, $t \in [0, T]$, i.e.

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To show the mapping above is an isometry and extend the class ε to its completion, we recall the Hilbert-Schmidt operators:

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Let $e_k, k \in \mathbb{N}$ be an orthonormal basis of U. An operator $A \in L(U, H)$ is called Hilbert-Schmidt if

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Important lemmas

Lemma

If $Q \in L(U)$ is nonnegative and symmetric then there exists a unique $Q^{1/2} \in L(U)$ nonnegative and symmetric such that $Q^{1/2} \circ Q^{1/2} = Q$

If in addition $trQ < \infty$, we have that $Q^{1/2} \in L_2(U)$ where $||Q^{1/2}||_{L_2}^2 = trQ$ and $L \circ Q^{1/2} \in L_2(U, H)$ for all $L \in L(U, H)$.

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Remark

If two elementary processes ϕ and $\bar{\phi}$ belong to one equivalence class with respect to $||\cdot||_T$, it does not follow that they are equal P_t-a.e. Because their values only have to correspond on $Q^{1/2}(U)$ P_t-a.e.

$$\text{Int:}(\varepsilon, ||\cdot||_T) \to (\mathcal{M}_T^2, ||\cdot||_{\mathcal{M}_T^2})$$

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A Hilbert space

In this section we will seek an explicit representation of the completion $\bar{\varepsilon}$.

First, define $U_0 := Q^{1/2}(U)$ with inner product

$$< u_0, v_0 >_0 := < Q^{-1/2} u_0, Q^{-1/2} v_0 >_U, \quad u_0, v_0 \in U_0$$

where $Q^{-1/2}$ is the pseudo-inverse of $Q^{1/2}$ in the case that Q is not one-to-one.

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The separable Hilbert space $L_2(U_0, H)$ is called L_2^0. By proposition, we know $Q^{1/2}g_k, k \in \mathbb{N}$ is an orthonormal basis of $(U_0, <,>_0)$ if $g_k, k \in \mathbb{N}$ is an orthonormal basis of $(\ker Q^{1/2})^{\perp}$. This basis can be supplemented to a basis of U by elements of $\ker Q^{1/2}$.

Thus

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Representation of $\bar{\varepsilon}$

Define $L(U,H)_0 := \{T|_{U_0} | T \in L(U,H)\}$. Since $Q^{1/2} \in L_2(U)$, it is clear that $L(U,H)_0 \subset L_2^0$ and $||\cdot||_T$-norm of $\phi \in \varepsilon$ can be written as

$$||\phi||_T = (E(\int_0^T ||\phi(s)||^2_{L_2^0} ds))^{1/2}$$

Define

$$\begin{aligned}\mathcal{N}^2_W(0,T;H) : &= \{\phi : [0,T] \times \Omega \to L_2^0 | \phi \text{is predictable and} ||\phi||_t < \infty\} \\ &= L^2([0,T] \times \Omega, \mathcal{P}_T, dt \otimes P; L_2^0)\end{aligned}$$

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To prove $\bar{\varepsilon} = \mathcal{N}_W^2$, we need

- Since $L(U,H)_0 \subset L_2^0$ and $\phi \in \varepsilon$ is L_2^0-predictable, we have $\varepsilon \subset \mathcal{N}_W^2$.
- By completeness of L_2^0, we have $\mathcal{N}_W^2$ is complete.
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Extension of stochastic integrals

Finally we will extend the definition of stochastic integrals from $\mathcal{N}_W^2$ to

$$\mathcal{N}_W(0,T;H) :=$$

$$\{\phi : \Omega_T \to L_2^0 | \phi \text{is predictable with} P(\int_0^T ||\phi(s)||_{L_2^0}^2 ds < \infty) = 1\}$$

We call $\mathcal{N}_W(0,T;H)$ the space of stochastically integrable processes.

First lemma

Let $W(t)$ be a Q-Wiener process, $T > 0$.

Lemma

Let ϕ be a L_2^0-valued stochastically integrable process. $(\tilde{H}, ||\cdot||_{\tilde{H}})$ is a separable Hilbert space and $L \in L(H, \tilde{H})$. Then

$L(\phi(t))$, $t \in [0, T]$ is an element of $\mathcal{N}_W(0, T; \tilde{H})$ and

$$L\left(\int_0^T \phi(t)dW(t)\right) = \int_0^T L(\phi(t))dW(t), \quad P - a.s.$$

Second lemma

Lemma

Let $\phi \in \mathcal{N}_W(0,T)$ and f is an $\mathcal{F}_t$-adapted continuous H-valued process. Set

$$\int_0^T < f(t), \phi(t)dW(t) >:= \int_0^T \tilde{\phi}_f(t)dW(t)$$

with $\tilde{\phi}_f(t) :=< f(t), \phi(t)u >$, $u \in U_0$. Then this integral is well-defined as a continuous $\mathbb{R}$-valued stochastic process.

Second lemma

Lemma

More precisely, $\tilde{\phi}_f$ *is a* $P_T/\mathcal{B}(L_2(U_o,\mathbb{R}))$*-measurable map from* $[0,T]\times\Omega$ *to* $L_2(U_0,\mathbb{R})$, $||\tilde{\phi}_f(t,w)||_{L_2(U_0,\mathbb{R})} = ||\phi^*(t,w)f(t,w)||_{U_0}$ *for all* $(t,w)\in[0,T]\times\Omega$ *and*

$$\int_0^T ||\tilde{\phi}_f(t)||^2_{L_2(U_0,\mathbb{R})}dt \le \sup_{t\in[0,T]} ||f(t)|| \int_0^T ||\phi(t)||^2_{L_2^0}dt < \infty \quad P-a.s.$$

Third lemma

Lemma

Let $\phi \in \mathcal{N}_W(0,T)$ and $M(t) := \int_0^t \phi(s)dW(s)$,$t \in [0,T]$. Define

$$< M >_t := \int_0^t ||\phi(s)||^2_{L_2^0} ds, \quad t \in [0,T].$$

Then $< M >$ is the unique continuous increasing $\mathcal{F}_t$-adapted process starting at zero such that $||M(t)||^2 - < M >_t$,$t \in [0,T]$ is a local martingale.

Third lemma

Lemma

If $\phi \in \mathcal{N}^2_W(0,T)$, *then for any sequence*

$$I_l := \{0 = t_0^l < t_1^l < \cdots < t_{k_l}^l = T\}, \quad l \in \mathbb{N},$$

of partitions with $\max_i(t_i^l - t_{i-1}^l) \to 0$ *as* $l \to \infty$

$$\lim_{l\to\infty} E\left(\left|\sum_{t_{j+1}^l \le t} ||M(t_{j+1}^l) - M(t_j^l)||^2 - < M >_t\right|\right) = 0$$

Cylindrical Wiener processes

In case that Q is not of finite trace, we need a Hilbert space $(U_1, <, >_1)$ and a Hilbert-Schmidt embedding

$$J : (U_0, <, >_0) \to (U_1, <, >_1)$$

Lemma

Let $e_k, k \in \mathbb{N}$ be an orthonormal basis of $U_0 = Q^{1/2}(U)$ and $\beta_k, k \in \mathbb{N}$ a family of independent real-valued Brownian motions. Define $Q_1 := JJ^*$. Then $Q_1 \in L(U_1)$, Q_1 is nonnegative definite and symmetric with finite trace and the series

$$W(t) = \sum_{k=1}^{\infty} \beta_k(t) J e_k, \quad t \in [0, T], \qquad (1)$$

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Lemma

Moreover, we have that $Q_1^{1/2}(U_1) = J(U_0)$ *and for all* $u_0 \in U_0$

$$||u_0||_0 = ||Q_1^{-1/2} J u_0||_1 = ||J u_0||_{Q_1^{1/2} U_1}$$

i.e. $J;\, U_0 \to Q_1^{1/2} U_1$ *is an isometry.*

Stochastic integrals

Fix $Q \in L(U)$ nonnegative, symmetric but not necessarily of finite trace. We integrate with respect to the standard U_1-valued Q_1-Wiener process given by the above lemma.

First we get a process $\phi(t), t \in [0, T]$ is integrable with respect to $W(t), t \in [0, T]$, if it takes values in $L_2(Q_1^{1/2}(U_1), H)$, is predictable and if

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It follows that $Je_k, k \in \mathbb{N}$ is an orthonormal basis of $Q_1^{1/2}(U_1)$. Hence

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Define

$$\int_0^t \phi(s) dW(s) := \int_0^t \phi(s) \circ J^{-1} dW(s), \quad t \in [0, T]. \qquad (2)$$

Then the class of all integrable processes is given by

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Remark

1. The stochastic integral defined in the last slide is independent of the choice of $(U_1, <,>_1)$ and J. This follows by construction, since by (1) for elementary processes (2) does not depend on J.
2. If $Q \in L(U)$ is trace class, the standard Q-Wiener process can also be considered as a cylindrical Q-Wiener process by setting $J = I : U_0 \to U$ where I is the identity map. In this case both definitions of the stochastic integral coincide.

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