Mild Solutions of Stochastic Navier-Stokes Equations-I

Semigroup Theory

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Outline

1. Semigroups of Linear Operators
   - Preliminaries
   - C_0-semigroup
   - Hille Yosida Theorem
   - Analytic Semigroups
   - Cauchy problem

Semigroups of bounded linear operators

Definition

Let X be a Banach space. A one parameter family $T(t)$,$0 \leq t < \infty$ of bounded linear operators from X to X is a semigroup of bounded linear operator on X if

- $T(0) = I$, $\quad I$ is the identity operator on X
- $T(t+s) = T(s)T(t)$ for every $t, s \geq 0$ $\quad$ (semigroup property)

$T(t)$ is called a uniformly continuous semigroup if

$$\lim_{t \to 0} ||T(t) - I|| = 0$$

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Infinitesimal generator

Linear operator A defined by

$$D(A) = \{x \in X : \lim_{t \to 0} \frac{T(t)x - x}{t} \quad \text{exists}\}$$

and

$$Ax = \lim_{t \to 0} \frac{T(t)x - x}{t} = \frac{d^+ T(t)x}{dt}\Big|_{t=0} \quad \text{for} \quad x \in D(A)$$

is the infinitesimal generator of the semigroup $T(t)$. $D(A)$ is called the domain of A.

Definition

Let X be a Banach space

Definition

A semigroup $T(t)$, $0 \leq t < \infty$ of bounded linear operators on X is a strongly continuous semigroup of bounded linear operators if

$$\lim_{t \to 0} T(t)x = x \quad \text{for every} \quad x \in X \tag{1}$$

We usually call it C_0-semigroup.

Properties of C_0-semigroup

Theorem

Let $T(t)$ be a C_0-semigroup. Then there exist constants $w \geq 0$ and $M \geq 1$ such that

$$||T(t)|| \leq Me^{wt} \quad 0 \leq t < \infty$$

proof: First, there exists a constant $\eta > 0$ such that $||T(t)||$ is bounded for $t \in [0, \eta]$. Suppose this is false, then there exists a sequence $\{t_n\}$,$t_n \geq 0$ and $\lim_{n\to\infty} t_n = 0$ such that

$$||T(t_n)|| \geq n.$$

From uniform boundedness theorem, there exist some $x \in X$, such that $||T(t_n)x||$ is unbounded, which contradicts with definition (1).

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Proof continues...

Thus

$$||T(t)|| \leq M \quad \text{for} \quad t \in [0, \eta]$$

Since $||T(0)|| = 1$, $M \geq 1$. Let $w = \eta^{-1} \log M \geq 0$. Given $t \geq 0$, we have $t = n\eta + \delta$ with $0 \leq \delta < \eta$.

Therefore, by semigroup property

$$||T(t)|| = ||T(\delta)T(\eta)^n|| \leq M^{n+1} \leq MM^{t/\eta} = Me^{wt}$$

The proof is complete.

Corollary

If $T(t)$ is a C_0-semigroup then for every $x \in X$, $t \to T(t)x$ is a continuous function from R_0^+ into X.

Proof continues...

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Proof of corollary

Let $t, h \geq 0$

$$||T(t+h)x - T(t)x|| \leq ||T(t)|| \cdot ||T(h)x - x|| \leq Me^{wt}||T(h)x - x||$$

For $t \geq h \geq 0$,

$$||T(t-h)x - T(t)x|| \leq ||T(t-h)|| \cdot ||x - T(h)x|| \leq Me^{wt}||T(h)x - x||$$

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Main Theorem

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Let $T(t)$ be a C_0-semigroup, A be its infinitesimal generator. Then

1. *For $x \in X$, $\lim_{h\to 0} \frac{1}{h}\int_t^{t+h} T(s)xds = T(t)x$*
2. *For $x \in X$, $\int_0^t T(s)xds \in D(A)$ and $A\left(\int_0^t T(s)xds\right) = T(t)x - x$*
3. *For $x \in D(A)$, $T(t)x \in D(A)$ and $\frac{d}{dt}T(t)x = AT(t)x = T(t)Ax$*
4. *For $x \in D(A)$,*

$$T(t)x - T(s)x = \int_s^t T(\tau)Axd\tau = \int_s^t AT(\tau)xd\tau$$

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Proof of main theorem

(1): It follows from the continuity of $T(t)$

(2): Let $x \in X$ and $h > 0$. Then

$$\frac{T(h)-I}{h}\int_0^t T(s)xds = \frac{1}{h}\int_0^t (T(s+h)x - T(s)x)ds$$
$$= \frac{1}{h}\int_t^{t+h} T(s)xds - \frac{1}{h}\int_0^h T(s)xds$$

As $h \to 0$, by property (1), RHS$\to T(t)x - x$ and LHS is

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(3): Let $x \in D(A)$ and $h > 0$. Then

$$\frac{T(h)-I}{h}T(t)x = T(t)(\frac{T(h)-I}{h}) \to T(t)Ax \quad \text{as} \quad h \to 0$$

Thus $T(t)x \in D(A)$ and $AT(t)x = T(t)Ax$, we have

$$\frac{d^+}{dt}T(t)x = AT(t)x = T(t)Ax$$

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To prove (3) we need to show that for $t > 0$, the left derivative of $T(t)x$ exists and equals $T(t)Ax$.

This follows from

$$\lim_{h\to 0}\left[\frac{T(t)x - T(t-h)x}{h} - T(t)Ax\right]$$
$$= \lim_{h\to 0} T(t-h)\left[\frac{T(h)x - x}{h} - Ax\right] + \lim_{h\to 0}\left[T(t-h)Ax - T(t)Ax\right]$$

The first limit vanishes because $T(t-h)$ is bounded and $x \in D(A)$. The second limit is zero because of the continuity of $T(t)Ax$.

(4): It is obvious by taking integration of (3).

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Corollary on A

Corollary

If A is the infinitesimal generator of a C_0-semigroup $T(t)$, then $D(A)$ is dense in X and A is a closed linear operator.

proof: For every $x \in X$, set $X_t = \frac{1}{t}\int_0^t T(s)xds$. By (2), $x_t \in D(A)$. By (1) $x_t \to x$ as $t \to 0$. Thus

$$\overline{D(A)} = X$$

Linearity of A follows from its definition.

Closedness: Let $x_n \in D(A)$ such that $x_n \to x$ and $Ax_n \to y$ as $n \to \infty$. By 4, we have

$$T(t)x_n - x_n = \int_0^t T(s)Ax_n ds \tag{2}$$

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Proof of corollary

The integrand of (2) converges to $T(s)y$ on bounded intervals.

Let $n \to \infty$ in (2), then

$$T(t)x - x = \int_0^t T(s)yds$$

Divide (10) by t and let $t \to 0$. From (1) we have

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Uniqueness of semigroup

Theorem

Let $T(t)$, $S(t)$ be C_0-semigroups of bounded linear operators with infinitesimal generators A and B. If $A = B$, then $T(t) = S(t)$ for any $t \geq 0$.

proof: Let $x \in D(A) = D(B)$. From 3 it follows that $s \to T(t-s)S(s)x$ is differentiable and

$$\begin{aligned}\frac{d}{ds}T(t-s)S(s)x &= -AT(t-s)S(s)x + T(t-s)BS(s)x \\ &= -T(t-s)AS(s)x + T(t-s)BS(s)x = 0\end{aligned}$$

Thus $s \to T(t-s)S(s)x$ is constant and values at $s = 0$ or t are the same.

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$$T(t)x = S(t)x \quad \text{for every} \quad x \in X$$

We have the following stronger result on A comparing to the previous corollary.

Theorem

Let A be the infinitesimal generator of a C_0-semigroup $T(t)$. If $D(A^n)$ is the domain of A^n, then $\cap_{n=1}^{\infty} D(A^n)$ is dense in X.

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Let $T(t)$ be a C_0-semigroup. By theorem, it follows that there exist $w \geq 0$ and $M \geq 1$ such that

$$||T(t)|| \leq Me^{wt}, \quad t \geq 0$$

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Resolvent

If A is linear operator in X. the resolvent set $\rho(A)$ of A is the set of all complex number λ for which $\lambda I - A$ is invertible. i.e.

$(\lambda I - A)^{-1}$ is a bounded linear operator in X.

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Hille-Yosida Theorem

Theorem

A linear (unbounded) operator A is the infinitesimal generator of a C_0-semigroup of contractions $T(t)$, $t \geq 0$ if and only if

- *A is closed and* $\overline{D(A)} = X$
- $\mathbb{R}^+ \subset \rho(A)$ *and for every* $\lambda > 0$, $||R(\lambda : A)|| \leq \frac{1}{\lambda}$

Lemma

Let A satisfy conditions of the above theorem, then

$$\lim_{n \to \infty} \lambda R(\lambda : A)x = x \quad \text{for} \quad x \in X$$

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A linear operator A is the infinitesimal generator of a C_0-semigroup $T(t)$, satisfying $||T(t)|| \leq M$ $(M \geq 1)$, if and only if

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Let A be the infinitesimal generator of a C_0-semigroup $T(t)$ on X. If A_λ is the Yosida approximation of A, i.e. $A_\lambda = \lambda AR(\lambda : A)$, then $T(t)x = \lim_{\lambda \to \infty} e^{tA_\lambda} x$ where $e^{tA_\lambda} = \sum_{n=0}^{\infty} \frac{(tA_\lambda)^n}{n!}$.

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A sufficient condition for C_0-semigroup

An easier to use theorem showing A is the infinitesimal generator of a C_0-semigroup is given below.

Theorem

Let A be a densely defined operator in X satisfying the following conditions.

1. *For some $0 < \delta < \pi/2$, $\rho(A) \supset \Sigma_\delta = \{\lambda : |\arg\lambda| < \pi/2 + \delta\} \cup \{0\}$.*
2. *There exists a constant M such that $\|R(\lambda : A)\| \leq \frac{M}{|\lambda|}$ for $\lambda \in \Sigma_\delta, \lambda \neq 0$.*

Then, A is the infinitesimal generator of a C_0 semigroup $T(t)$ satisfying $\|T(t)\| \leq C$ for some constant C.

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Definition of analytic semigroups

Definition

Let $\triangle = \{z : \varphi_1 < \arg z < \varphi_2, \varphi_1 < 0 < \varphi_2\}$ and for $z \in \triangle$ let $T(z)$ be a bounded linear operator. The family $T(z)$, $z \in \triangle$ is an analytic semigroup in $\triangle$ if

1. $z \to T(z)$ is analytic in *triangle*.
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A semigroup $T(t)$ will be called analytic if it is analytic in some sector $\triangle$ containing the nonnegative real axis.

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Main theorem

Theorem

Let $T(t)$ be a uniformly bounded C_0 semigroup. Let A be the infinitesimal generator of $T(t)$ and assume $0 \in \rho(A)$. The following statements are equivalent:

- *$T(t)$ can be extended to an analytic semigroup in a sector $\triangle_\delta = \{z : |argz| < \delta\}$ and $||T(z)||$ is uniformly bounded in every closed subsector $\bar{\triangle}_{\delta'}$, $\delta' < \delta$, of $\triangle_\delta$.*
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Characterization of analytic semigroups

Theorem

Let A be the infinitesimal generator of a C_0 semigroup $T(t)$ satisfying $||T(t)|| \leq Me^{wt}$. Then $T(t)$ is analytic if and only if there are constants $C > 0$ and $\Lambda \geq 0$ such that

$$||AR(\lambda : A)^{n+1}|| \leq \frac{C}{n\lambda^n} \quad \text{for} \quad \lambda > n\Lambda, \quad n = 1, 2, \cdots .$$

Homogeneous Cauchy problem

Consider

$$\begin{cases} \frac{du}{dt} = Au(t), \quad t > 0 \\ u(0) = x \end{cases} \tag{3}$$

Definition

An X-valued function $u(t)$ is called a solution of above problem if: $u(t)$ is continuous and continuously differentiable for $t \geq 0$, $u(t) \in D(A)$ for $t > 0$ and (3) is satisfied.

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Definition

An X-valued function $u(t)$ is called a solution of above problem if: $u(t)$ is continuous and continuously differentiable for $t \geq 0$, $u(t) \in D(A)$ for $t > 0$ and (3) is satisfied.

Relation with C_0-semigroup

If A is the infinitesimal generator of a C_0-semigroup $T(t)$, then (3) has a solution $u(t) = T(t)x$, for every $x \in D(A)$.

Theorem

Let A be a densely defined linear operator with a nonempty resolvent set $\rho(A)$. Then (3) has a unique solution which is continuously differentiable on $[0, \infty)$ for every initial value $x \in D(A)$ if and only if

$$A \text{ is the infinitesimal generator of a } C_0 - semigroup T(t)$$

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Mild solutions

Definition

Let $T(t)$ be a C_0-semigroup on X. $T(t)$ is called differentiable for $t > t_0$ if for every $x \in X$, $t \to T(t)x$ is differentiable for $t > t_0$.

Definition

If A is the infinitesimal generator of a C_0-semigroup which is not differentiable, then in general, if $x \notin D(A)$, (3) does not have a solution. The function $t \to T(t)x$ is called a mild solution.

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nonhomogeneous Cauchy problem

Theorem

Let A be the infinitesimal generator of a C_0-semigroup $T(t)$. Let $x \in X$, $f \in \mathrm{L}^1(0, T; X)$. $u \in C([0, T], X)$ given by

$$u(t) = T(t)x + \int_0^t T(t-s)f(s)ds, \quad 0 \leq t \leq T$$

is the mild solution on $[0, T]$ for the nonhomogeneous Cauchy problem

$$\begin{cases} \frac{du(t)}{dt} = Au(t) + f(t), \quad t > 0 \\ u(0) = x \end{cases}$$