Introduction to Mathematical Fluid Dynamics-III

The Navier-Stokes Equations

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Euler equations

For ideal and incompressible fluid, the Euler equations are

$$\begin{cases} \rho \frac{D\mathbf{u}}{Dt} & = -\text{grad} p + \rho b \quad \text{(Balance of Momentum)} \\ \frac{D\rho}{Dt} & = 0 \quad \text{(Conservation of Mass)} \\ \text{div}\mathbf{u} & = 0 \quad \text{(Incompressibility)} \end{cases}$$

with boundary conditions

$$\mathbf{u} \cdot \mathbf{n} = 0 \quad \text{on} \quad \partial D$$

The reason for imposing this type of boundary conditions will be explained in the end of this lecture.

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The reason for imposing this type of boundary conditions will be explained in the end of this lecture.

Balance of momentum

Recall the integral form for the balance of momentum,

$$\frac{d}{dt}\int_{W_t}\rho\mathbf{u}dV = S_{\partial W_t} + \int_{W_t}\rho b dV$$

Here $S_{\partial W_t}$ represents the force exerted on the surface of W. For a viscous fluid, we should also consider the tangential force on the surface. This is the difference between the Euler equation and the Navier-Stokes equations that we will derive in this lecture.

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Stress tensor

Stress tensor is the force that exerted on the surface of a body, which can be analyzed for fluid similar to solid. In general, stress tensor consists of normal stress and tangential stress, which cause change of volume and change of shape of the fluid respectively.

Denote by σ the stress tensor, it is a second order tensor written as a 3×3 matrix in 3 dimensional flow.

$$\sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}$$

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Normal stress tensor

For a fluid at rest,

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -p$$

Above formula actually gives the definition for pressure p, these normal components result in a change of volume for the fluid.

For a fluid under deformation, we need to consider tangential force then the normal tensor becomes

$$\sigma_{xx} = -p + \tau_{xx}$$

$$\sigma_{yy} = -p + \tau_{yy}$$

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The remaining components of the tensor are given by

$$\sigma_{ij} = \tau_{ij} \quad \text{for} \quad i \neq j$$

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Deviatoric tensor

τ is called the deviatoric tensor and it results in the change of shape of the fluid. It can also be written as a second order tensor.

$$\tau = \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix}$$

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Strain tensor

In Newtonian fluid, the rate of strain is linear to the deviatoric stress

$$\tau_{ij} = \mu\epsilon_{kl}, \quad \text{for} \quad i, j, k, l = 1, 2, 3,$$

where μ is the first viscosity coefficient and ϵ is the rate of strain tensor

$$\epsilon = \begin{pmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{pmatrix}$$

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Strain tensor

After analysis of the deformation, we have

$$\epsilon_{xx} = 2\frac{\partial u}{\partial x}, \epsilon_{yy} = 2\frac{\partial v}{\partial y}, \epsilon_{zz} = 2\frac{\partial w}{\partial z}$$

and

$$\epsilon_{xy} = \epsilon_{yx} = (\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})$$

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Normal direction

If we assume

$$\tau_{xx} = \mu\epsilon_{xx}, \tau_{yy} = \mu\epsilon_{yy}, \tau_{zz} = \mu\epsilon_{zz}$$

It only counts in the effect of extension and compression without change of volume.

In general, for a system undergoing both change of shape and volume,

$$\tau_{xx} = \mu\epsilon_{xx} + \lambda(\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}) \rightarrow \text{linear relation between } \tau \text{ and } \epsilon$$

Here λ is the second viscosity coefficient and

$$\frac{1}{2}(\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}) = \mathrm{div}\mathbf{u}$$

is the rate of expansion in volume.

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Thus,

$$\sigma_{xx} = \tau_{xx} - p = \mu\epsilon_{xx} + \lambda(\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}) - p$$
$$= 2\mu\frac{\partial u}{\partial x} + 2\lambda \text{div}\mathbf{u} - p$$

Similarly, for the other two normal directions,

$$\sigma_{yy} = \tau_{yy} - p = 2\mu\frac{\partial v}{\partial y} + 2\lambda \text{div}\mathbf{u} - p$$

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Written in vector form, we have

$$\sigma = 2\lambda \mathrm{div}\mathbf{u} \cdot I + \mu D - pI$$

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Deformation matrix

Here I is the identity matrix and D is the deformation matrix defined by

$$D = \frac{(\nabla \mathbf{u}) + (\nabla \mathbf{u})^T}{2}$$

and since

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{pmatrix}$$

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Now let us go back to the balance of momentum

$$\frac{d}{dt}\int_{W_t}\rho\mathbf{u}dV = S_{\partial W_t} + \int_{W_t}\rho b dV$$

Because of viscous property of the fluid,

$$\begin{aligned} S_{\partial W_t} &= \int_{\partial W_t}\sigma\cdot\mathbf{n}dA \\ &= \int_{\partial W_t}(2\lambda\mathrm{div}\mathbf{u}\cdot I + \mu D - pI)\cdot\mathbf{n}dA \\ &= 2\lambda\int_{\partial W_t}(\mathrm{div}\mathbf{u})I\cdot\mathbf{n}dA + \mu\int_{\partial W_t}D\cdot\mathbf{n}dA - \int_{\partial W_t}pI\cdot\mathbf{n}dA \end{aligned}$$

Force on the surface

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Force on the boundary

By divergence theorem,

$$\int_{\partial W_t} (\mathrm{div}\mathbf{u}) I \cdot \mathbf{n} dA = \int_{W_t} \nabla \cdot ((\mathrm{div}\mathbf{u}) I) dV$$
$$= \int_{W_t} \nabla (\mathrm{div}\mathbf{u}) dV$$

$$\int_{\partial W_t} D \cdot \mathbf{n} dA = \int_{W_t} \nabla \cdot (\frac{\nabla \mathbf{u} + (\nabla \mathbf{u})^T}{2}) dV$$

and

$$\int_{\partial W_t} pI \cdot \mathbf{n} dA = \int_{W_t} \nabla \cdot (pI) dV = \int_{W_t} \nabla p dV$$

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The second term

$$\int_{W_t} \nabla\cdot(\frac{\nabla \mathbf{u} + (\nabla \mathbf{u})^T}{2})dV$$
$$= \frac{1}{2}\int_{W_t}(\triangle \mathbf{u})dV + \frac{1}{2}\int_{W_t}\nabla(\text{div}\mathbf{u})dV$$

Using the transport theorem and substitute above terms into the balance of momentum equation, we have

$$\rho\frac{D\mathbf{u}}{Dt} = (2\lambda + \frac{\mu}{2})\nabla(\text{div}\mathbf{u}) - \nabla p + \frac{\mu}{2}\triangle \mathbf{u} + \rho b$$

Formulation of balance of momentum

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The Navier-Stokes equations

After normalizing the density ρ to be 1, the previous equation reduces to

$$\frac{D\mathbf{u}}{Dt} = -\nabla p + (2\lambda + \frac{\mu}{2})(\text{div}\mathbf{u}) + \frac{\mu}{2}\triangle\mathbf{u} + b$$

For incompressible fluid, denote $\nu = \frac{\mu}{2}$, we have

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We end up with the Navier-Stokes equations

$$\partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \nu\triangle\mathbf{u} + b$$

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Boundary conditions for Euler and N-S equations

We end this lecture with a discussion on the boundary conditions we should impose for the Euler and Navier-Stokes equations.

For Euler's equations for ideal flow we use $\mathbf{u} \cdot \mathbf{n} = 0$, that is, fluid does not cross the boundary but may move tangentially to the boundary.

For the Navier-Stokes equations, the extra term $\nu \triangle \mathbf{u}$ raises the number of derivatives of $\mathbf{u}$. For both experimental and mathematical reasons, this is accompanied by an increase in the number of boundary conditions.

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Mathematical reason

For instance, on a solid wall at rest we add the condition that the tangential velocity also be zero ("no-slip condition"), so the full boundary condition are simply

$$\mathbf{u} = 0 \quad \text{on solid walls at rest}$$

The mathematical need for extra boundary conditions lies on their role in proving that the equations are well posed, that is, a unique solution exists and depends continuously on the initial data.

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Physical reason

The physical need for the extra boundary conditions comes from simple experiments involving flow past a solid wall. For example, if dye is injected into flow down a pipe and is carefully watched near the boundary, one sees that the velocity approaches zero at the boundary to a high degree of precision.

The no-slip condition is also reasonable if one contemplates the physical mechanism responsible for the viscous terms, namely, molecular diffusion. Our example indicates that molecular interaction between the solid wall with zero tangential velocity should impart the same condition to the immediately adjacent fluid.

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Final remark

Another crucial feature of the boundary condition $\mathbf{u} = 0$ is that it provides a mechanism by which a boundary can produce vorticity in the fluid. But we will not talk about rotation and vorticity of fluid in this lecture.