Introduction to Mathematical Fluid Dynamics-II

Balance of Momentum

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Fluid path

Consider the path followed by a fluid particle flows inside a domain W.

$$\mathsf{x}(t) = (x(t), y(t), z(t))$$

Then the velocity field becomes

$$\mathbf{u}(x(t), y(t), z(t), t) = (\dot{x}(t), \dot{y}(t), \dot{z}(t))$$

or

$$\mathbf{u}(\mathsf{x}(t), t) = \frac{d\mathsf{x}}{dt}(t)$$

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Acceleration of fluid particle

Another physical quantity in fluid mechanics is the acceleration of the fluid particle

$$a(t) = \frac{d^2}{dt^2}x(t) = \frac{d}{dt}\mathbf{u}(x(t), y(t), z(t))$$
$$= \frac{\partial \mathbf{u}}{\partial x}\dot{x} + \frac{\partial \mathbf{u}}{\partial y}\dot{y} + \frac{\partial \mathbf{u}}{\partial z}\dot{z} + \frac{\partial \mathbf{u}}{\partial t}$$

Denote $\mathbf{u}_x = \frac{\partial \mathbf{u}}{\partial x}$,... $\mathbf{u}_t = \frac{\partial \mathbf{u}}{\partial t}$ and

$$\mathbf{u}(x, y, z, t) = (u(x, y, z, t), v(x, y, z, t), w(x, y, z, t))$$

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\end{aligned}
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\mathbf{u}(x, y, z, t) = (u(x, y, z, t), v(x, y, z, t), w(x, y, z, t))
$$

From the above notation, we can rewrite

$$\begin{aligned} a(t) &= u\mathbf{u}_x + v\mathbf{u}_y + w\mathbf{u}_z + \mathbf{u}_t \\ &= \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} \end{aligned}$$

We will frequently use the operator

$$\frac{D}{Dt} = \partial_t + \mathbf{u} \cdot \nabla \tag{1}$$

Operator (1) is called the material derivative.

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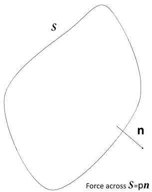
Ideal fluid



Ideal Fluid

For any motion of the fluid in a region W, there is a function $p(x, t)$ called the pressure, such that ∂W is a surface in the fluid with a chosen unit normal $\mathbf{n}$, the force of stress exerted across the surface ∂W per unit area at $x \in \partial W$ at time t is $p(x, t)\mathbf{n}$.

Remark: The absence of tangential forces implies that there is no rotation for fluid in W.

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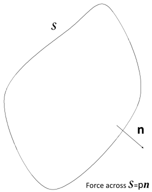

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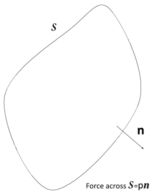

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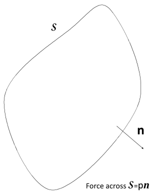

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Force on the boundary

For ideal fluid, the total force on the fluid inside W by means of stress on its boundary is

$$S_{\partial W} = \{\text{force} \quad \text{on} \quad W\} = -\int_{\partial W} p\mathbf{n}dA$$

For any fixed vector $\mathbf{e}$, divergence theorem gives us

$$\begin{aligned}
\mathbf{e} \cdot S_{\partial W} &= -\int_{\partial W} p\mathbf{e} \cdot \mathbf{n}dA \\
&= -\int_{W} \text{div}(p\mathbf{e})dV \\
&= -\int_{W} (\text{grad}p) \cdot \mathbf{e}dV
\end{aligned}$$

Hence

$$S_{\partial W} = -\int_{W} \text{grad}p dV$$

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Balance of momentum

Denote $b(x,t)$ as the given body force per unit mass, then the total body force is

$$B = \int_W \rho b dV$$

In all, force per unit volume is equal to

$$-\text{grad}p + \rho b$$

Balance of Momentum(Differential Form)

By the principle of momentum balance (Newton's second law),

$$\rho \frac{D\mathbf{u}}{Dt} = -\text{grad}p + \rho b$$

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$$\rho \frac{D\boldsymbol{u}}{Dt} = -gradp + \rho b$$

Integral form

An integral form of the balance of momentum can be derived for general fluid:

Balance of Momentum(Integral Form)

By the principle of momentum balance,

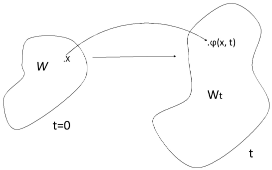
$$\frac{d}{dt}\int_{W_t} \rho \mathbf{u} dV = S_{\partial W_t} + \int_{W_t} \rho b dV$$

Here W_t is a region at time t and $S_{\partial W_t}$ represents the total force exerted on the surface ∂W_t.

Flow map

Write $\varphi(x, t)$ as the trajectory followed by the particle at point x and time t. Assume the flow is smooth enough. Then we can define a mapping

$$\varphi_t : \quad x \mapsto \varphi(x, t)$$



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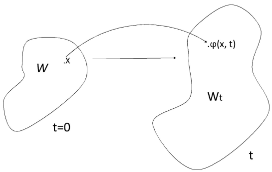

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First lemma

The first lemma before we continue is the following

Lemma 1

Define $J(x,t)$ as the Jacobian determinant of the map φ_t, we have

$$\frac{\partial}{\partial t}J(x,t) = J(x,t)\left[div\mathbf{u}(\varphi(x,t),t)\right]$$

We give a sketch of proof for this lemma.

Write the components of φ as $\xi(x,t)$,$\eta(x,t)$ and $\zeta(x,t)$. Then its Jacobian determinant can be written as

$$J(x,t) = \begin{pmatrix} \frac{\partial\xi}{\partial x} & \frac{\partial\eta}{\partial x} & \frac{\partial\zeta}{\partial x} \\ \frac{\partial\xi}{\partial y} & \frac{\partial\eta}{\partial y} & \frac{\partial\zeta}{\partial y} \\ \frac{\partial\xi}{\partial z} & \frac{\partial\eta}{\partial z} & \frac{\partial\zeta}{\partial z} \end{pmatrix}$$

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Proof of lemma 1

For fixed x,

$$\frac{\partial}{\partial t} J = \frac{\partial}{\partial t} \begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} & \frac{\partial \zeta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} & \frac{\partial \zeta}{\partial y} \\ \frac{\partial \xi}{\partial z} & \frac{\partial \eta}{\partial z} & \frac{\partial \zeta}{\partial z} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial}{\partial t}\frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} & \frac{\partial \zeta}{\partial x} \\ \frac{\partial}{\partial t}\frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} & \frac{\partial \zeta}{\partial y} \\ \frac{\partial}{\partial t}\frac{\partial \xi}{\partial z} & \frac{\partial \eta}{\partial z} & \frac{\partial \zeta}{\partial z} \end{pmatrix} + \begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial}{\partial t}\frac{\partial \eta}{\partial x} & \frac{\partial \zeta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial}{\partial t}\frac{\partial \eta}{\partial y} & \frac{\partial \zeta}{\partial y} \\ \frac{\partial \xi}{\partial z} & \frac{\partial}{\partial t}\frac{\partial \eta}{\partial z} & \frac{\partial \zeta}{\partial z} \end{pmatrix}$$

$$+ \begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} & \frac{\partial}{\partial t}\frac{\partial \zeta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} & \frac{\partial}{\partial t}\frac{\partial \zeta}{\partial y} \\ \frac{\partial \xi}{\partial z} & \frac{\partial \eta}{\partial z} & \frac{\partial}{\partial t}\frac{\partial \zeta}{\partial z} \end{pmatrix}$$

By definition of the velocity field

$$\frac{\partial}{\partial t}\varphi(x,t) = \mathbf{u}(\varphi(x,t),t)$$

Thus

$$\frac{\partial}{\partial t}\frac{\partial \xi}{\partial x} = \frac{\partial}{\partial x}\frac{\partial \xi}{\partial t} = \frac{\partial}{\partial x}u(\varphi(x,t),t)$$

$$\frac{\partial}{\partial t}\frac{\partial \xi}{\partial y} = \frac{\partial}{\partial y}\frac{\partial \xi}{\partial t} = \frac{\partial}{\partial y}u(\varphi(x,t),t)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{\partial}{\partial t}\frac{\partial \zeta}{\partial z} = \frac{\partial}{\partial z}\frac{\partial \zeta}{\partial t} = \frac{\partial}{\partial z}w(\varphi(x,t),t)$$

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....

$$\frac{\partial}{\partial t}\frac{\partial \zeta}{\partial z} = \frac{\partial}{\partial z}\frac{\partial \zeta}{\partial t} = \frac{\partial}{\partial z}w(\varphi(x,t),t)$$

Moreover

$$\frac{\partial}{\partial x}u(\varphi(x,t),t) = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial u}{\partial \zeta}\frac{\partial \zeta}{\partial x},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{\partial}{\partial z}w(\varphi(x,t),t) = \frac{\partial w}{\partial \xi}\frac{\partial \xi}{\partial z} + \frac{\partial w}{\partial \eta}\frac{\partial \eta}{\partial z} + \frac{\partial w}{\partial \zeta}\frac{\partial \zeta}{\partial z},$$

Now plug these expressions into $\frac{\partial}{\partial t}J$, we get

$$\frac{\partial}{\partial t}J = \begin{pmatrix} \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial u}{\partial \zeta}\frac{\partial \zeta}{\partial x} & \frac{\partial \eta}{\partial x} & \frac{\partial \zeta}{\partial x} \\ \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} + \frac{\partial u}{\partial \zeta}\frac{\partial \zeta}{\partial y} & \frac{\partial \eta}{\partial y} & \frac{\partial \zeta}{\partial y} \\ \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial z} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial z} + \frac{\partial u}{\partial \zeta}\frac{\partial \zeta}{\partial z} & \frac{\partial \eta}{\partial z} & \frac{\partial \zeta}{\partial z} \end{pmatrix} + \ldots\ldots$$

$$= \begin{pmatrix} \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} & \frac{\partial \zeta}{\partial x} \\ \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} & \frac{\partial \zeta}{\partial y} \\ \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial z} & \frac{\partial \eta}{\partial z} & \frac{\partial \zeta}{\partial z} \end{pmatrix} + \begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial v}{\partial \eta}\frac{\partial \eta}{\partial x} & \frac{\partial \zeta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial v}{\partial \eta}\frac{\partial \eta}{\partial y} & \frac{\partial \zeta}{\partial y} \\ \frac{\partial \xi}{\partial z} & \frac{\partial v}{\partial \eta}\frac{\partial \eta}{\partial z} & \frac{\partial \zeta}{\partial z} \end{pmatrix} + \begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} & \frac{\partial w}{\partial \zeta}\frac{\partial \zeta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} & \frac{\partial w}{\partial \zeta}\frac{\partial \zeta}{\partial y} \\ \frac{\partial \xi}{\partial z} & \frac{\partial \eta}{\partial z} & \frac{\partial w}{\partial \zeta}\frac{\partial \zeta}{\partial z} \end{pmatrix}$$

$$= \frac{\partial u}{\partial \xi}J + \frac{\partial v}{\partial \eta}J + \frac{\partial w}{\partial \zeta}J = [\mathrm{div}\mathbf{u}(\varphi(x,t),t)]\,J$$

The proof is complete.

Second lemma

Lemma 2

Given a scalar or vector function $f(x,t)$, we have

$$\frac{d}{dt}\int_{W_t} f(x,t)dV = \int W_t \left[\frac{\partial f}{\partial t} + div(f\mathbf{u})\right] dV \qquad (2)$$

A similar result can be proved and is called the transport theorem.

Transport Theorem

$$\frac{d}{dt}\int_{W_t} \rho\mathbf{u}dV = \int_{W_t} \rho\frac{D\mathbf{u}}{Dt}dV \qquad (3)$$

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Proof of Lemma 2

Let us prove (2) first.

By change of variables formula and the first lemma

$$
\begin{aligned}
LHS &= \frac{d}{dt}\int_W f(\varphi(x,t),t)J(x,t)dV \\
&= \int_W \left[\frac{df}{dt}(\varphi(x,t),t)J + f(\varphi(x,t),t)\frac{\partial J}{\partial t}\right] dV \\
&= \int_W \left[\frac{Df}{Dt}(\varphi(x,t),t) + \mathrm{div}\mathbf{u}f\right] JdV
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\end{aligned}
$$

$$= \int_{W_t} \left[\frac{Df}{Dt} + \text{div}\mathbf{u} f \right] dV$$

$$= \int_{W_t} \left[\frac{\partial f}{\partial t} + \mathbf{u} f + \text{div}\mathbf{u} f \right] dV$$

$$= \int_{W_t} \left[\frac{\partial f}{\partial t} + \text{div}(f\mathbf{u}) \right] dV$$

Thus (2) is proved.

To prove (3), we first observe that

$$\frac{d}{dt}(\rho\mathbf{u})(\varphi(x,t),t) = \frac{D}{Dt}(\rho\mathbf{u})(\varphi(x,t),t)$$

This is because the time derivative takes into account the fact that the fluid is moving and that the positions of fluid particles change with time. So, if $f(x,y,z,t)$ is any function of position and time, then by the chain rule

$$\begin{aligned}&\frac{d}{dt}f(x(t),y(t),z(t),t)\\ &= \partial_t f + \mathbf{u}\cdot\nabla f\\ &= \frac{Df}{Dt}(x(t),y(t),z(t),t)\end{aligned}$$

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Using Lemma 1, we have

$$\frac{d}{dt}\int_{W_t}\rho\mathbf{u}dV = \frac{d}{dt}\int_W(\rho\mathbf{u})JdV = \int_W \frac{d}{dt}[(\rho\mathbf{u})J]dV$$
$$= \int_W \frac{D}{Dt}(\rho\mathbf{u})(\varphi(x,t),t)J + (\rho\mathbf{u})(\varphi(x,t),t)\frac{\partial}{\partial t}J(x,t)dV$$
$$= \int_W \left[\frac{D}{Dt}(\rho\mathbf{u}) + (\rho\mathrm{div}\mathbf{u})\mathbf{u}\right]JdV$$

By the conservation of mass

$$\frac{D\rho}{Dt} + \rho\mathrm{div}\mathbf{u} = \frac{\partial\rho}{\partial t} + \mathrm{div}(\rho\mathbf{u}) = 0$$

Using Lemma 1, we have

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Thus

$$\frac{d}{dt}\int_{W_t}\rho\mathbf{u}dV = \int_{W_t}\rho\frac{D\mathbf{u}}{Dt}dV$$

Incompressible fluid

Definition

We call a flow incompressible if for any fluid subregion W,

$$\text{volume}(W_t) = \int_{W_t} dV = \text{constant} \quad \text{in} \quad t$$

From the first lemma, we know

$$0 = \frac{d}{dt}\int_{W_t} dV = \frac{d}{dt}\int_W J dV$$
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The following statements are equivalent:

- the fluid is incompressible.
- $\text{div}\mathbf{u} = 0$
- $J \equiv 1$

Previous slide shows that the first and second statements are equivalent. To show $J \equiv 1$ for incompressible fluid, recall the first lemma and divergence free condition,

$$\int_{W_t} dV = C = \int_W J dV = J \int_W dV$$

Since the volume of W_t remains the same, we get

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Continuity equation for incompressible fluid

Recall the continuity equation

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