Introduction to Mathematical Fluid Dynamics-I

Conservation of Mass

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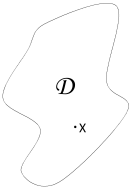
Fluid in a domain

We consider flows inside a domain $\mathcal{D} \subset \mathbb{R}^3$. $x = (x, y, z)$ is a point in $\mathcal{D}$.



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$$\mathbf{u}(x, t) \rightarrow \textit{velocity field of the fluid}$$

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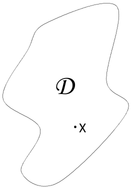

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Laws of conservation

We assume ρ and $\mathbf{u}$ are smooth enough. To what extent the regularity is needed will be seen in later lectures.

Here are three principles to derive the equations of motions:

- Mass is neither created nor destroyed.
- The rate of change of momentum of a portion of the fluid equals the force applied to it. (Newton's second law)
- Energy is neither created nor destroyed.

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Conservation of mass

Let $W \subset \mathcal{D}$ be a fixed region, the total mass of fluid inside W is given by

$$m(W,t) = \int_W \rho(x,t)dV$$

Here dV is the volume element.

The rate of change of mass in W is thus

$$\frac{d}{dt}m(W,t) = \frac{d}{dt}\int_W \rho(x,t)dV = \int_W \frac{\partial \rho}{\partial t}(x,t)dV$$

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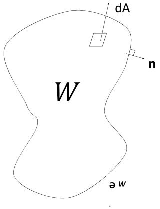
Flow through the boundary

Denote the boundary of W as ∂W, the unit normal outward vector as $\mathbf{n}$ and the area element as dA.



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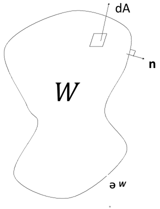

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The rate of increase of mass in W equals the rate at which mass is crossing ∂W in the inward direction.

Conservation of Mass(Integral Form)

By the mass conservation principle, we have

$$\frac{d}{dt}\int_W \rho dV = -\int_{\partial W} \rho \mathbf{u}\cdot\mathbf{n} dA \quad (1)$$

There is a negative sign on the right hand side because we assume mass is moving inward to W.

Integral form of mass conservation

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Divergence theorem

To derive a differential form for the mass conservation, we need the following divergence theorem to transform the surface integral in (1) into a volume integral.

Divergence Theorem

Let $Q \subset \mathbb{R}^3$ be a region bounded by a closed surface ∂Q and let $\mathbf{n}$ be the unit outward normal to ∂Q. If F is a vector function that has continuous first partial derivatives in Q, then

$$\int\int_{\partial Q} F \cdot \mathbf{n} ds = \int\int\int_{Q} \nabla \cdot F dV$$

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Proof of divergence theorem

Suppose

$$F(x,y,z) = M(x,y,z)i + N(x,y,z)j + P(x,y,z)k$$

then the divergence theorem can be stated as

$$\int\int_{\partial Q} F \cdot \mathbf{n} ds = \int\int_{\partial Q} M(x,y,z)i \cdot \mathbf{n} ds + \int\int_{\partial Q} N(x,y,z)j \cdot \mathbf{n} ds + \int\int_{\partial Q} P(x,y,z)k \cdot \mathbf{n} ds$$

$$= \int\int\int_Q \frac{\partial M}{\partial x} dV + \int\int\int_Q \frac{\partial N}{\partial y} dV + \int\int\int_Q \frac{\partial P}{\partial z} dV$$

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$$\begin{aligned}\int\int_{\partial Q} F\cdot \mathbf{n}ds &= \int\int_{\partial Q} M(x,y,z)i\cdot \mathbf{n}ds + \int\int_{\partial Q} N(x,y,z)j\cdot \mathbf{n}ds \\ &\quad + \int\int_{\partial Q} P(x,y,z)k\cdot \mathbf{n}ds \\ &= \int\int\int_Q \frac{\partial M}{\partial x}dV + \int\int\int_Q \frac{\partial N}{\partial y}dV + \int\int\int_Q \frac{\partial P}{\partial z}dV \\ &= \int\int\int_Q \nabla\cdot F(x,y,z)dV\end{aligned}$$

The divergence theorem is proved if we can show that

$$\int\int_{\partial Q} M(x,y,z)i \cdot \mathbf{n} ds = \int\int\int_{Q} \frac{\partial M}{\partial x} dV$$

$$\int\int_{\partial Q} N(x,y,z)j \cdot \mathbf{n} ds = \int\int\int_{Q} \frac{\partial N}{\partial y} dV$$

$$\int\int_{\partial Q} P(x,y,z)i \cdot \mathbf{n} ds = \int\int\int_{Q} \frac{\partial P}{\partial z} dV$$

Proofs of above equalities are similar so we only focus on the third one.

Suppose Q can be described as

$$Q = \{(x, y, z) | g(x, y) \leq z \leq h(x, y), \quad \textit{for} \quad x, y \in R\}$$

where R is the region in the xy-plane.

Think of Q as being bounded by three surface S_1(top), S_2(bottom) and S_3(side).

On surface S_3 the unit outward normal is parallel to the xy-plane and thus

$$\int\int\int_Q P(x, y, z)k \cdot \mathbf{n} ds = \int\int_{\partial Q} 0 ds = 0$$

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Now we calculate the surface integral over S_1

$$S_1 = \{(x, y, z) | z - h(x, y) = 0, \quad \text{for } (x, y) \in R\}$$

The unit outward normal can be calculated as

$$\mathbf{n} = \frac{\nabla(z - h(x, y))}{||\nabla(z - h(x, y))||}$$
$$= \frac{-h_x(x, y)i - h_y(x, y)j + k}{\sqrt{[-h_x(x, y)]^2 + [-h_y(x, y)]^2 + 1}}$$

Thus

$$k \cdot \mathbf{n} = \frac{1}{\sqrt{[h_x(x, y)]^2 + [h_y(x, y)]^2 + 1}}$$

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We have

$$\int\int_{S_1} P(x,y,z)k \cdot \mathbf{n} ds = \int\int_{S_1} \frac{P(x,y,z)}{\sqrt{[h_x(x,y)]^2 + [h_y(x,y)]^2 + 1}}$$
$$= \int\int_R P(x,y,h(x,y))dA$$

In a similar way we can show that the surface integral over S_2 is

$$\int\int_{S_2} P(x,y,z)k \cdot \mathbf{n} ds = -\int\int_R P(x,y,g(x,y))dA$$

with a negative sign on the right hand side. This is because the outward unit normal of S_2 is pointing opposite to the direction of k.

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Finally

$$\begin{aligned}
&\int\int_{\partial Q} P(x,y,z)k\cdot \mathbf{n}ds \\
&= \int\int_{S_1} P(x,y,z)k\cdot \mathbf{n}ds + \int\int_{S_2} P(x,y,z)k\cdot \mathbf{n}ds \\
&\qquad + \int\int_{S_3} P(x,y,z)k\cdot \mathbf{n}ds \\
&= \int\int_{R} P(x,y,h(x,y))dA - \int\int_{R} P(x,y,g(x,y))dA \\
&= \int\int_{R} P(x,y,z)|_{z=g(x,y)}^{z=h(x,y)} dA \\
&= \int\int_{R}\int_{g(x,y)}^{h(x,y)} \frac{\partial P}{\partial z} dz dA = \int\int\int_{Q} \frac{\partial P}{\partial z} dV
\end{aligned}$$

and the proof is complete.

Differential form of mass conservation

Recall the integral form of mass conservation

$$\frac{d}{dt}\int_W \rho dV = -\int_{\partial W} \rho \mathbf{u} \cdot \mathbf{n} dA$$

Using the divergence theorem, one can show that

$$\int_{\partial W} \rho \mathbf{u} \cdot \mathbf{n} dA = \int_W \nabla \cdot (\rho \mathbf{u}) dV$$

Thus by putting the time derivative inside of the integral, we get

$$\int_W \left[\frac{\partial \rho}{\partial t} + div(\rho \mathbf{u})\right] dV = 0$$

for any $W \subset \mathcal{D}$.

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Differential form of mass conservation

The integrand must be equal to zero for the above integral to vanish, we end up with

Conservation of Mass(Differential Form)

By the mass conservation principle and the divergence theorem, we have

$$\frac{\partial \rho}{\partial t} + div(\rho \mathbf{u}) = 0 \tag{2}$$

Equation (2) is also called the continuity equation in fluid dynamics.

Remark: If ρ and $\mathbf{u}$ are not smooth enough, then the integral form is the one to use.

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