Introduction to Financial Risk Measurement Einführung in die Bewertung von Finanzrisiken

M.Sc. Brice Hakwa

[1] Bergische Universität Wuppertal,
Fachbereich Angewandte Mathematik - Stochastik
Hakwa@math.uni-wuppertal.de

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Introduction

We begin with the introduction of a probabilistic framework for modeling financial risk from the perspective of an investor. For a given financial positions, let $\Omega \ = \{v_1, v_2, v_3, \quad, v_n\}$ be a finite set of possible future value of this positions, the uncertainty about the future can be represent by a probability space (Ω, F, P).
Then the random variables

$$X : \Omega \to \mathbb{R}, \quad v \to X_t(v)$$

denote the position value (pay off) of v at time t if the scenario $v \in \Omega$ is realized.

Monetary Measure of risk

In general, we can define a **financial risk** as the change in the position-values betwenn two dates t_0 and t_1(current date is $t_0 = 0$), or as the dispertion of unexpected outcomes due to uncertain events (scenarios v). We can quantify the risk of X by some number $\rho(X)$, where X belongs to a given linear function space $\mathcal{X}$, that contains the constant function. (note that $\mathcal{X}$ can be identified with $\mathbb{R}^n$, where $n = card(\Omega)$).

Definition: monetary measure

A function $\rho : \mathcal{X} \to \mathbb{R}$ is called a monetary measure of risk if it satisfies the following conditions for all $X, Y \in \mathcal{X}$

- Axiom M: Monotonicity
  if $X \leq Y$, *then* $\rho(X) \geq \rho(Y)$
- Axiom T: Translation invariance
  if $m \in \mathbb{R}$, *then* $\rho(X + m) = \rho(X) - m$

Imterpretation

- **Monotonicity**: The risk of a position is reduced if the payoff profile is increased in each state of the World. ($\rho(X)$ is a decreasing function)
- **Invariance**: The invariance property suggests that adding cash to a position reduces its risk by the amount of cash added. This is motivated by the idea that the risk measure can be used to determine capital requirements.

Exemple: capital requirement for risky position X

- if $\rho(X) > 0$ then regulatory authority requests additional capital to make this position acceptable.
- if $\rho(X) < 0$ then the position is is already acceptable.

As a consequence for the invariance of ρ we have

$$\rho(X + \rho(X)) = 0 \; and \; \rho(m) = \; \rho(0) - m \, \forall \; m \in \mathbb{R}.$$

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Convex Measure

Remark

m corresponds to the constant function

Definition: convexrisk measure

A monetary risk measure $\rho : \mathcal{X} \to \mathbb{R}$ is called a convex measure of risk if it satisfies:

- Axiom C: (Convexity)
  $\rho(\lambda X+(1-\lambda)Y) \leq \lambda\rho(X)+(1-\lambda)\rho(Y),$ *for* $0 \leq \lambda \leq 1.$

Interpretation: the convexity property states that the risk of a portfolio is not greater than the sum of the risks of its constituents, that means diversification in a given portfolio does not increase the risk

Definition: Coherent Risk Measure

A convex measure of risk ρ is called a coherent risk measure if it satisfies:

- Axiom PH: Positive Homogeneity
  If $\lambda \geq 0$ *then* $\rho(\lambda X) = \lambda\rho(X)\ \forall\ X \in \mathcal{X}$

If a monetary measure of risk ρ is positively homogeneous, then it is **normalized** ($\rho(0) = 0$), Under the assumption of positive homogeneity, convexity is equivalent to

- Axiom S: Subadditivity
  $\rho(X + Y) \leq \rho(X) + \rho(Y)$

Interpretation:

- The subadditivity axiom ensure that the risk of a diversified portfolio is no greater than the corresponding weighted average of the risks of the constituents.
- Capital requirement for holding company should never be larger than the sum of Capital requirement of all individual subs.
- subadditivity reflects the idea that risk can be reduced by diversification
- Subadditivity makes decentralization of risk-management systems possible.

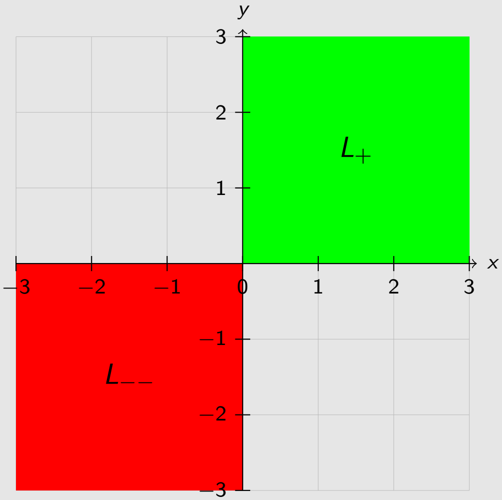
Acceptance set

One can separate the set $\mathcal{X}$ in different subsets :

- $L_+ = \{X \in \mathcal{X} | X(v) \geq 0 \; \forall \; v \in \Omega\}$ (the set of non-negative elements of $\mathcal{X}$)
- $L_- = \{X \in \mathcal{X} | \; \exists \; v \in \Omega \; s.t. \; X(v) < 0\}$ (the set of negative elements of $\mathcal{X}$)
- $L_{--} = \{X \in \mathcal{X} | X(v) < 0 \; \forall \; v \in \Omega\}$

For example in case n = 2 ($\Omega = \{v_1, v_2\}$) we can represent $\mathcal{X}$ in a 2-coordinate system. Where the x-axis represents the measurements of $x = X(v_1)$ and the y-axis represents the measures of $y = X(v_2)$.

Representation of $\mathcal{X}$ for n =2



Acceptance set

Depending on its risk policy, the regulator can separate the set $\mathcal{X}$ into two distinct subset:

1. Acceptable set
2. Uncceptable set

Definition: Acceptance set

The acceptance set A is defined as the set of financial positions that are acceptable (from the regulator's point of view) without any additional capital requirement.

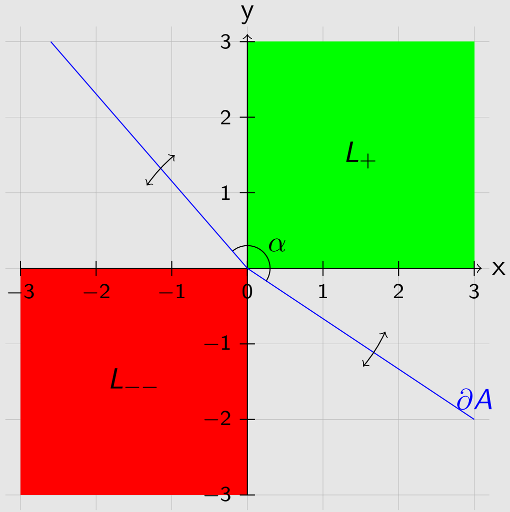
Acceptance set their Axioms

As in [Artzner99] we state here some axioms that an acceptance set $\mathcal{A}$ must satisfy:

- Axiom A1: The acceptance set $\mathcal{A}$ contains $L_+ \subset \mathcal{X}$
- Axiom A2: The acceptance set $\mathcal{A}$ does not intersect L_{--}
- Axiom A3: The acceptance set $\mathcal{A}$ is convex

The separation of acceptance set A from unacceptable set can be materialized by the characterization of the boundary of A (of ∂A).

Characterization of ∂A for n =2. I



Characterization of ∂A for n =2. II

In the case n=2, we can deduce from the axioms of acceptanceset some geometrical proprieties of the boundary of A (∂A).
Consider the angle α in the previous picture, then we get according to the axioms of the acceptance set the following relationships:

- Axiom A1 $\Rightarrow \alpha \geq 90°$
- Axiom A2 $\Rightarrow \alpha \leq 270°$

Relation to Risk Measures I

Definition: Acceptance set associated to a risk measure

The acceptance set associated to a risk measure ρ is the set denoted by $\mathcal{A}_\rho$ and defined by

$$A_\rho = \{X \in \mathcal{X} : \rho(X) \leq 0\} \quad (*)$$

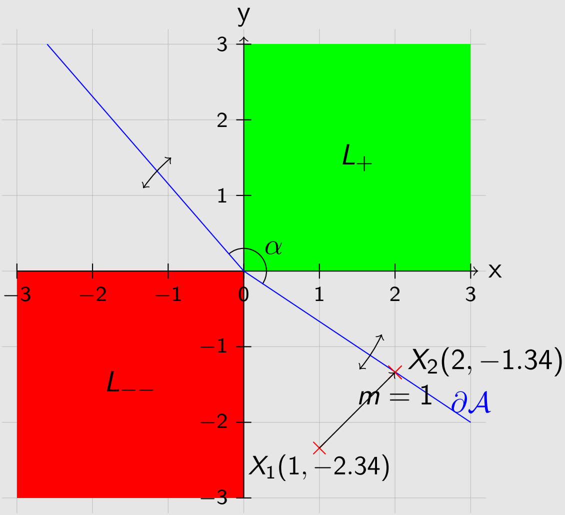
Definition: Risk measure associated to an acceptance set

The risk measure associated to the acceptance set $\mathcal{A}$ is the mapping from $\mathcal{X}$ to $\mathbb{R}$ denoted by $\rho_\mathcal{A}$ and defined by

$$\rho_\mathcal{A}(X) = \inf\{m \in \mathbb{R} | X + m \in \mathcal{A}\} \quad (**)$$

The amount m may be interpreted as the required regulatory capital to cover the position's risk

Relation to Risk Measures II



Relation to Risk Measures III

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The previous graphic illustrate the relationsheap betwenn the acceptence set $\mathcal{A}$ (namely the boundary $\partial\mathcal{A}$ of $\mathcal{A}$) and measure of the risk in the case n=2. In this example we can see that initialy $X_1(1, -2.34)$ does not belong to the acceptance set, it is also unacceptable from the perspective of the regulator, but we can make it acceptable, according to the definiton $(**)$ by Adding some cash m = 1 (or hihger than 1). The new position $X_2 = X(1, -2.34) + m(1, 1) = (2, -1.34)$ is on the boundary $\partial\mathcal{A}$ of the acceptance set and therefore acceptable.

Relation to Risk Measures IV

- $\rho_{\mathcal{A}}$ is a convex risk measure if and only if $\mathcal{A}$ is convex.
- $\rho_{\mathcal{A}}$ is positively homogeneous if and only if $\mathcal{A}$ is a cone. In particular, $\rho_{\mathcal{A}}$ is coherent if and only if $\mathcal{A}$ is a convex cone.
- $\mathcal{A}_\rho$ is clearly convex if ρ is a convex measure of risk

Assume that $\mathcal{A}_\rho$ is a non-empty subset of $\mathcal{X}$ which satisfies $\rho_{\mathcal{A}} > -\infty$ and $X \in \mathcal{A}, Y \in \mathcal{X}, Y \geq X \Rightarrow Y \in \mathcal{A}$ Then:

- $\rho_{\mathcal{A}}$ is a monetary measure of risk.
- If $\mathcal{A}_\rho$ is a convex set, then $\rho_{\mathcal{A}}$ is a convex measure of risk.
- $\rho_{\mathcal{A}}$ is a coherent measure of risk if $\mathcal{A}_\rho$ is a convex cone.

Risk Measurement

- Let X be (X is define as a random variable $X : \Omega \to \mathbb{R}$) a value of a financial position, the set $\mathcal{X}$ of all possible financial position is finite (since Ω assumed to be finite) and can be separated into acceptable set and non-acceptable sep (Regulatory perspective)
- We want to quantify the amount that, added to a non-acceptable position X, make its acceptable to the regulator.

Desirable Properties for Monetary Risk Measure

- Therefore we need a appropriate monetary risk measure for the unacceptable position (downsid risk).
- Such a measure ρ should have the following properties.
  - ρ is a decreasing function of X
  - ρ is stated in the same units as X
  - ρ is positive if X is non-acceptable
  - ρ is a coherent risk measure

Exemple I : Worst-Case Risk Measure

Definition

The worst-case risk measure ρ_{max} defined by

$$\rho_{max}(X) = -\inf_{v \in \Omega} X(v) \; \forall X \in \mathcal{X}$$

- The value of ρ_{max} (Capital requirements) is the least upper bound for the potential loss which can occur in any scenario.
- ρ_{max} is a coherent measure of risk
- Any normalized monetary risk measure ρ on $\mathcal{X}$ satisfies
  $$\rho(X) \leq \rho_{max}(X)$$
  It is also the most conservative measure of risk.
- PML (Probable maximum loss) could be seen as equivalent in insurance

Example II: Value at Risk

Suppose that we have a probability measure P on $(\Omega, \mathcal{F})$. In this context, a position X is often considered to be acceptable if the **probability of a loss** is bounded by a given level $\lambda \in (0,1)$ that means:

$$\mathrm{P}[X < 0] \leq \lambda.$$

The corresponding monetary risk measure (necessary capital) is called Value at Risk (VaR)

Definition: Value at Risk

$$VaR_{\lambda}A(X) = \inf\left\{m \in \mathbb{R} | P\left[m + X < 0\right] \leq \lambda\right\}$$

- Value-at-Risk refers to a quantile of the loss distribution
- Value-at-Risk is positively homogeneous, but in general it is not convex

VaR and Regulatory approach to Bank sector

VaR is the standard regulatory MMR for Bank sector for market-risk(MR). The internal model (IM) proposes the following formula to calculate the market-risk-capital at day t,

$$RC_{IM}^{t}\left(MR\right) = max\left\{ VaR_{0.99,10}^{t}\ ;\ \frac{k}{60}\sum_{i=1}^{60} VaR_{0.99,\ 10}^{t-i+1}\right\} + RC_{SR}^{t}$$

where

- $VaR_{0.99,\ 10}^{t}$ stands for a 10-day VaR at the 99% confidence level,calculated on day t.
- RC$=$ Risk Capital
- MR $=$ Market Risk and SR $=$ Specific Risk
- $k \in [3,\ 5]$ Stress Factor

VaR Regulatory approach to insurance sector

The Solvency Capital Requirement for every individual risk i, $SCR_\alpha(i)$, is defined the The **German Standard Model** of GDV and BaFin as the difference between the Value at Risk and expected value (premium income),

$$SCR_\alpha(i) = VaR_\alpha(i) - \mu_i$$

References

- Artzner, Phillippe Delbaen, Freddy Eber, Jean-Marc Heath, David (1999): *Coherent Measures of Risk.*, Mathematical Finance, 9. Jg. (1999), S. 203-228.
- McNeil, A.J., Frey, R., and Embrechts, P. (2005) *Quantitative Risk Management: Concepts, Techniques and Tools* Princeton University Press
- Föllmer, H., Schied, A., (2004) *Stochastic Finance, An Introduction in Discrete Time. Second Edition.*Edition. de Gruyter Studies in Mathematics 27