PhD Seminar: HJM Forwards Price Models for Commodities

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Basic Definitions

Definition: Commodity Futures (electricity Future

Electricity Futures Obligation to buy/sell a specified amount of electricity during a delivery period, typically a month, quarter or year.

Definition: Futures Price

The futures price $f(t, T)$ is the delivery price which would make the obligation-contract have zero value at time t

Definition: Forward Curve

For a fixed t, the function : $T \to f\,(t, T)$ is called **forward curve**.

Spot-Forward Relationship I : REH

The rational expectation hypothesis (REH) states that the current futures price $f(t, T)$ for a commodity (interest rate) with delivery a time $T > t$ is the best estimator for the price $S(T)$ of the commodity.

$$f(t, T) = E_{\mathbb{Q}}[S(T)|\mathcal{F}_t] \tag{1}$$

where $\mathcal{F}_t$ represents the information available at time t.

Spot-Forward Relationship II

Under the no-arbitrage assumption we have for a given interest rates r^*:

$$f(t, T) = S(t)e^{r^*(T-t)} \tag{2}$$

$$\textit{with} \quad r^* = r - y$$

where S(t) is the spot price at the time t, r the interest rate at time t for maturity T and y the convenience yield.

Spot-Forward Relationship: Interpretation I

From the Spot-Forward relationships we can derive following results:

- The payoff of the Forward contract depends on the behavior of the interest rate r^* (A commodity forward is also a interest rate derivative) .
- The Spot-Forward relationship is linear since The expected value operator E in eq. (1) is linear and eq. (2) is a linear function with slope equal to $e^{r^*(T-t)}$, that means $f(t, T)$ and $S(t)$ have a perfect dependency structure.
- Spot and futures are comonotone. (Because of the perfect dependency structure)

Spot-Forward Relationship: Interpretation I

- according to eq. (2) two forward prices are necessary to compute one spot price. since we have two unknown in eq. (2) namely $S(t)$ and y.
- Knowledge of $S(t)-$ and r^*- processes allows us to construct the whole forward curve.
- Spot and futures are redundant (one can replace the other)
- If r^* is the expected return under probability $\mathbb{Q}$, the eq.(1) and eq(2) are equivalent. (We said in this case that $\mathbb{Q}$ is the risk-neutral probability)

Some Basic Concepts and Definition from Interest Rate Theory.

- The price of a risk-free zero-coupon bond at time t that pays one unit of currency at time T is denoted by:

$$p(t, T) = e^{-r(t,T)(T-t)} \tag{3}$$

- The short rate is defined as:

$$f(t, T) = -\frac{\partial log(p(t, T))}{\partial T} \tag{4}$$

- The instantaneous spot rate is given by:

$$r(t) = f(t, t) \tag{5}$$

- The money account is defined by

$$B(t) = exp\left\{\int_0^t r(s)\, ds\right\} \tag{6}$$

1. Spot model:
   Model the spot price dynamic. i.e

$$dS(t) = a(t)\,dt + b(t)\,dw(t) \tag{7}$$

2. Forward Models: Heath-Jarrow-Morton (HJM) model. Model the entire instantaneous forward rate (short rate) curve as an infinite system(infinite-dimensional) for SDEs (one for every maturity T) state variable. i.e

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW(t). \tag{8}$$

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HJM drift condition

Recall that according to the no-arbitrage theory, the prices process of interest rate derivatives discounted with the money account have to bemartingale.
To avoid arbitrage the drift and the volatility in a HJM framework (equation (8)) must satisfy the following condition:

$$\alpha(t,\ T) = \sigma(t,\ T) \int_t^T \sigma(t,\ s)\, ds \tag{9}$$

HJM Drift Condition: Proof

Commodity markets

Spot-Forward Relationship

Basic Concepts and definitions from Interest Rate Theory

Commodity Modeling

HJM Drift Condition: Proposition

From (4) we have :

$$p(t, T) = exp\left\{-\int_t^T f(t, z)\, dz\right\} \tag{10}$$

From (8) we have (Integral Form):

$$f(t, T) = f(0, T) + \int_0^t \alpha(s, T) ds + \int_0^t \sigma(s, T) dW(s)$$

and from (5) we have:

$$r(t) = f(t,\ t) = f(0, t) + \int_0^t \alpha(s, t) ds + \int_0^t \sigma(s, t) dW(s).$$

HJM Drift Condition: Proof

Hence

$$\frac{p(t,\ T)}{B(t)} = exp\left\{\frac{-\int_t^T f(t,z)dz}{\int_0^t r(s)ds}\right\}$$
$$= exp\left\{-\int_t^T f(t,z)dz - \int_0^t r(s)ds\right\}$$

Set

$$\frac{p(t,\ T)}{B(t)} = exp\left\{X(t)\right\}$$

HJM Drift Condition: Proof

Then

$$
\begin{aligned}
X(t) &= \left\{ -\int_t^T f(t,z)dz - \int_0^t r(s)ds \right\} \\
&= -\int_t^T f_0(z)dz - \int_t^T \int_0^t \alpha(t,z)dsdz - \int_t^T \int_0^t \sigma(s,z)dW(s)\,dz \\
&\quad - \int_0^t f_0(z)dz - \int_0^t \int_0^u \alpha(s,u)dsdu - \int_0^t \int_0^u \sigma(s,u)dW(s)\,du
\end{aligned}
$$

Applying the classical Fubinis theorem and the Fubinis theorem for stochastic integrals we have ([1] Thm 6.2)

$$
\begin{aligned}
&= -\int_t^T f_0(z)dz - \int_0^t \int_t^T \alpha(t,z)dzds - \int_0^t \int_t^T \sigma(s,z)dzdW(s) \\
&\quad - \int_0^t f_0(z)dz - \int_0^t \int_s^t \alpha(s,u)duds - \int_0^t \int_s^t \sigma(s,u)dW(s)\,du \\
&= -\int_0^T f_0(z)dz - \int_0^t \int_s^T \alpha(t,z)dzds - \int_0^t \int_s^T \sigma(s,z)dzdW(s) \\
&= X(0) + \int_0^t A(s)ds + \sum(s)\,dW(s)\,. \qquad (11)
\end{aligned}
$$

with

$$
X(0) = -\int_0^T f_0(z)dz, \quad A(s) = -\int_s^T \alpha(t,z)dz \text{ and } \sum(s) = -\int_s^T \sigma(s,z)dz
$$

HJM Drift Condition: Proof

Rewrite (11) in Differential form.

$$dX(t) = A(t)\,dt + \sum dW(t)\,.$$

Recall that, under the Money account probability $\mathbb{Q}^M$, the prices process of interest rate derivatives discounted with the money account $\frac{p(t,\ T)}{B(t)} = exp\left\{X(t)\right\}$ is a martingales. if his drift term is equal to zero.
Applying Itoo-Lemma to $exp\left\{X(t)\right\}$ we obtain.

$$dexp\left\{X(t)\right\} = X(t)\,dX(t) + \frac{1}{2}X(t)\,dX(t)\,dX(t)$$

Hence

$$dexp\left\{X(t)\right\} = X(t)\left[A(s) + \frac{1}{2}\left(\sum(s)\,dt\right)^2\right]dt + \sum(t)\,dW(t)$$

this means that

$$A(s) + \frac{1}{2}\left(\sum(s)\,dt\right)^2 \tag{12}$$

should be equal to zero. Hence

$$-A(s) = \frac{1}{2}\left(\sum(s)\,dt\right)^2 \tag{13}$$

$$\Leftrightarrow \int_s^T \alpha(t,z)dz = \frac{1}{2}\left(\int_s^T \sigma(s,z)dz\right)^2 \tag{14}$$

HJM Drift Condition: Proof

After differentiation of (14) with T we obtain the HJM drift-condition

$$\alpha\,(t,\,T) = \sigma(t,\,T)\int_t^T \sigma\,(t,z)\,dz \tag{15}$$

HJM Drift Condition: Interpretation

The moral of the HJM Drift Condition is that when we specify the forward rate dynamics (under Q) we may freely specify the volatility structure. The drift parameters are then uniquely determined.

- Damir Filipovidc, *Term-Structure Models: A Graduate Course* , Springer-Verlag Berlin Heidelberg 2009.
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- Thomas bj´ork , *Arbitrage Theory in Continuous Time, third Edition.* .