Prof. Dr. Barbara Rüdiger
Bergische Universität Wuppertal, Exercises Wednsday 10.15 - 11.45, Thursday 12.30 -13.45

Exercise -Exam

Notation:

0) $\{\Omega, \mathcal{F}, \mu\}$ denotes a finite measure space, $\{\Omega, \mathcal{F}, P\}$ a probability space

a) μ_U denotes the uniform distribution on $[0, 1]$.

b) μ_c denotes the distribution which distribution function is given by the Cantor function.

Ex. I: Find an example of two real valued random variables X, Y on $(\Omega, \mathcal{F}, P)$ which are NOT stochastic independent and such that for each $x, y \in \mathbb{R}$ the sets $\{X = x\}$, $\{Y = y\}$ are stochastic independent.

Ex. II: Let $p \in (0, 1)$ be fixed. Let X take value 20 with probability p, and -10 with probability $1 - p$. An asset $\{S_n\}_{n\in\mathbb{N}}$ has value 100 Euro in the first month. It increases each month with a value X_n which is distributed like X. X_n, for $n \in \mathbb{N}$, are stochastic independent.

2) Discuss the convergence in Probability of $\frac{S_n}{n}$

3) Given $N \in \mathbb{N}$. Prove that $P(S_n - 100 < -N) \leq \frac{900(n-1)(1-p)p+(n-1)^2(30p-10)^2}{N^2}$

4) Compute $P(S_4 \geq 140/S_2 \geq 120)$

5) Give the characteristic function of S_n/n

Ex. III:
Let $X_n := 6^n 1_{[1-\frac{1}{3^n},1]}$ with $n \in \mathbb{N}$ be defined on $(\Omega, \mathcal{F}, P)$= $([0, 1], \mathcal{B}([0, 1], \mu_C)$

6) Analyze the convergence μ_C -a.s. of $\{X_n\}_{n\in\mathbb{N}}$, with $X_n := 6^n 1_{[1-\frac{1}{3^n},1]}$

7) Analyze the convergence in Probability μ_C of $\{X_n\}_{n\in\mathbb{N}}$, with $X_n := 6^n 1_{[1-\frac{1}{3^n},1]}$

Ex. IV:
8) Prove the Theorem of total probability

Ex. V:
9) Given $(\Omega, \mathcal{F}, P)$ and $A \in \mathcal{F}$, such that $P(A) = 1$. Prove that $P(A \cap B) = P(B)$ for all $B \in \mathcal{F}$.

Ex. VI:
10) $(\Omega, \mathcal{F}, P)$= $([0, 1], \mathcal{B}([0, 1], \mu_U)$. Find an example of a sequence $\{X_n\}_{n\in\mathbb{N}}$ of random variables which converge in probabilty to 0, but not in $L_4(\Omega, \mathcal{F}, P)$.

Ex VII: (two students)

11) Let X be a random variable on $(\Omega, \mathcal{F}, P)$ with values $x_1, ..., x_n \in \mathbb{R}$. Prove that for every measurable function $f : \mathbb{R} \to \mathbb{R}$

$$\mathbb{E}[f(X)] = \int f(x)\mu(dx)$$

where μ the is the distribution of X.

EX VIII: (two students)

12) Let $c_n = \frac{1}{n^3}$, $n \in N$. Find a constant c and a sequence $\{x_n\}$ of different real numbers such that

$\mu := c \sum_n c_n \delta_{x_n}$ is the distribution of a random variable with mean zero and the second moment is not finite.

Remark: all results must be motivated and proven