Prof. Dr. Barbara Rüdiger
Bergische Universität Wuppertal, Exercises wednsday 10.15 - 11.45, thursday 12.30 -13.45

Exercise Sheet II -Probability

Notation:

0) $\{\Omega, \mathcal{F}, \mu\}$ denotes a measure space (finite or σ- finite measure), $\{\Omega, \mathcal{F}, P\}$ a probability space

a) $g \in \tau(\{\Omega, \mathcal{F}\})$, if g is a real valued function and $g(s) = \sum_{k=0}^{n-1} g_k \mathbf{1}_{A_k}(s)$, $A_k \in \mathcal{F}$

b) $g \in \Sigma_\infty(\{\Omega, \mathcal{F}\})$, if g is a real valued function and $g(s) = \sum_{k\in\mathbb{N}} g_k \mathbf{1}_{A_k}(s)$, $A_k \in \mathcal{F}$

c) Let $p \geq 1$, $\|\cdot\|_p$ is the norm in $\mathcal{L}^p(\Omega, \mathcal{F}, \mu)$

d) $\mathcal{B}(\mathbb{R}) := \sigma(\{(a, b] : a \leq b\}$

Ex. I:
1) Define in two ways the "standard form" of $g \in \tau(\{\Omega, \mathcal{F}\})$ and prove that these are equivalent.

Ex. II:
Let $p \geq 1$ be fixed.

2) Prove that $\tau(\{\Omega, \mathcal{F}\}))$ is dense in $\Sigma_\infty(\{\Omega, \mathcal{F}\})) \cap \mathcal{L}^p(\Omega, \mathcal{F}, \mu)$ w.r.t the norm $\|\cdot\|_p$.

3) Prove that $\tau(\{\Omega, \mathcal{F}\}))$ is dense in $\mathcal{L}^p(\Omega, \mathcal{F}, \mu)$ w.r.t the norm $\|\cdot\|_p$.

Ex. III:
4) Let $0 \leq p < 1$ be fixed. Compute $\mathbb{E}[exp(X)]$ for a random Variable X which is $B(n, p)$ distributed (minimizing the effort and passages. Find a smart way!)

Ex. IV: Given a probability space $(\Omega, \mathcal{F}, P)$.
5) Given $A \in \mathcal{F}$ such that $P(A) = 1$, or $P(A) = 0$. Prove that any $B \in \mathcal{F}$ is stochastic independent of A.

Ex. V: (two students)
6) Suppose you plan to continue to throw a dice and stop only when you have got 10 times consecutively the number 6. Prove that with probability 1 you will stop in a finite time.

Ex. VI:
7) Find a two dimensional random variable (X, Y) such that its marginals X and Y are normal Gauss distributed, but such that it is not Gauss distributed (i.e. the two dimensional dsitribution of (X, Y) is not Gaussian)

Ex. VII: Let $p \in (0, 1)$ be fixed. Let X take value 20 with probability p, and -10 with probability $1 - p$.

8) Give all sets of the Product σ -algebra $\sigma(X) \otimes \sigma(X)$. In particular prove that in this case $\sigma(X) \otimes \sigma(X) = \sigma(X) \times \sigma(X)$, i.e. the Product σ -algebra contains only the product sets.

9) Give an example of a random variable Y where $\sigma(Y) \otimes \sigma(Y)$ is not equal to $\sigma(Y) \times \sigma(Y)$

Ex. VIII: An asset $\{S_n\}_{n \in \mathbb{N}}$ has value 100 Euro in the first month. It increases each month with a value X_n which is distributed like X defined in EX VII. X_n for $n \in \mathbb{N}$ are stochastic independent.

10) Write the distribution of the asset $S_n = \sum_{k=1}^{n} X_k$ for each $n \in \mathbb{N}$ as a combination of Delta distributions.

11) Write the distribution function of S_4 and sketch a picture of it.

12) Prove that $\sigma(X_1, ...X_n) = \sigma(S_1, ...S_n)$

13) Compute the probability that the asset increases infinitly often.

14) Given $N \in \mathbb{N}$. Compute the probability of the event that there exists a month n , such that for all $k > n$ the asset has value $S_k > N$

Ex. VIII:

15) Prove that $\mathcal{B}(\mathbb{R}) = \sigma(\{[a, b] : a \leq b\}$

16) Prove that the $\mathcal{B}(\mathbb{R})$ coincides with the sigma algebra generated by the open sets on $\mathbb{R}$ defined by the usual topology.

Remark: all results must be motivated and proven