Introducing Oblique Norms into Multiple Criteria Programming[1]

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Abstract

We propose to use block norms to generate nondominated solutions of multiple criteria programs and introduce the new concept of the oblique norm that is specially tailored to handle general problems. We prove the equivalence of finding the properly nondominated solutions of a multiple criteria program and solving its scalarization by means of oblique norms.

Key Words. Multiple Criteria Programming, Oblique Norms, Block Norms, Properly Nondominated Solutions.

1 Introduction

Block norms, also referred to as polyhedral norms, play an important role in the measurement of distances. Not only can they be used to model real world situations (like measuring highway distances) more accurately than the generally used Euclidean norm, but they can also be used to approximate arbitrary norms since the set of block norms is dense in the set of

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all norms, see Ward and Wendell (1985). Due to their polyhedral structure, block norms imply piecewise linearity of the corresponding distance measure and thus often lead to efficiently solvable (piecewise linear) optimization problems.

In multiple criteria programming, norms have usually been used to identify the nondominated solutions that are the closest to some reference point, very often chosen as the utopia (ideal) point. The identification relies on measuring the distance between the nondominated solutions and the reference point in the objective space. In particular, the family of l_p norms has been extensively studied by many researchers, including Yu (1973), Zeleny (1973), Steuer and Choo (1983), Steuer (1986), Lewandowski and Wierzbicki (1988), and many others. Gearhart (1979) studied a family of norms including the l_p norms. The l_∞ norm and the augmented l_∞ norm turned out to be very useful in generating nondominated solutions of general continuous and discrete multiple criteria programs and led to the well known weighted (augmented) Tchebycheff scalarization and its variations. Kaliszewski (1987) introduced a modified l_∞ norm and showed its applicability in generating nondominated solutions. More recently, a new class of norms that contains the family of l_p norms was proposed in Carrizosa et al. (1997) to generate the set of points that have minimal distance to the utopia point with respect to at least one norm within this class of norms. This approach leads to solving linear programs while generating nondominated solutions.

The success of norm-based approaches in multiple criteria programming and decision making suggests the application of block norms that can be beneficial both for the determination of nondominated solutions and for the comparison of the resulting alternatives. Moreover, the choice of a suitable norm combined with the choice of a reference point can be used to express decision maker's preferences in the objective space while selecting the most preferred nondominated solution.

In this paper, we introduce the concept of the oblique norm into multiple criteria programming. This concept was first introduced and applied to bicriteria programs in Schandl et al. (2000). In general, oblique norms can be viewed as a generalization of the augmented l_∞ norm. They are designed to preserve capabilities of the l_∞ norm and the augmented l_∞ norm while allowing the decision maker more freedom in the choice of a distance measure. Therefore they accommodate more specific decision maker's preferences due to their more refined structure.

We intend to show that oblique norms are effective tools to generate nondominated solutions of general multiple criteria programs. In the next section we state the multiple criteria problem and give general definitions and notations. In Section 3, we define oblique norms and state some of their properties. Different ways to construct oblique norms are discussed in Section 4. In Section 5, we use oblique norms to find nondominated points. In particular, we examine the relationship between (properly) nondominated solutions of general multiple criteria programs and optimal solutions of their scalarization by means of an oblique norm. The last section includes conclusions and a short discussion of the implications of using oblique norms in multiple criteria decision making.

2 Problem Formulation

To facilitate further discussions, the following notation is used throughout the paper. Let $u, w \in \mathbb{R}^n$ be two vectors.

- We denote components of vectors by subscripts and enumerate vectors using superscripts.
- $u < w$ denotes $u_i < w_i$ for all $i = 1, \ldots, n$. $u \leq w$ denotes $u_i \leq w_i$ for all $i = 1, \ldots, n$, but $u \neq w$. $u \leqq w$ allows equality. The symbols $>, \geq, \geqq$ are used accordingly.
- Let $\mathbb{R}^n_{\geqq} := \{x \in \mathbb{R}^n : x \geqq 0\}$. If $U \subseteq \mathbb{R}^n$, then $U_{\geqq} := U \cap \mathbb{R}^n_{\geqq}$. The sets $\mathbb{R}^n_{\geq}$, $\mathbb{R}^n_{>}$, $U_{\geq}$ and $U_{>}$ are defined accordingly.
- $\langle u, w \rangle$ denotes the scalar product in $\mathbb{R}^n$: $\langle u, w \rangle = \sum_{i=1}^n u_i w_i$.
- $\text{conv}(U)$ denotes the convex hull of $U \subseteq \mathbb{R}^n$.
- $\text{int}(U)$ denotes the interior of $U \subseteq \mathbb{R}^n$.

A set $C \in \mathbb{R}^n$ is called a *cone* if for all $u \in C, \alpha > 0$ we also have $\alpha u \in C$. The origin may or may not belong to C. If $U = \{u^1, \ldots, u^k\} \subseteq \mathbb{R}^n$ is a set of vectors, then

$$\operatorname{cone}(U) := \left\{ v \in \mathbb{R}^n : v = \sum_{i=1}^{k} \alpha_i u^i, \alpha \geq 0, u^i \in U \right\}$$

is the cone generated by U.

We consider the following general multiple criteria program

$$\begin{array}{ll} \min & \{z_1 = f_1(x)\} \\ & \vdots \\ \min & \{z_n = f_n(x)\} \\ \text{s. t.} & x \in S, \end{array} \tag{1}$$

where $S \subseteq \mathbb{R}^m$ is the *feasible set* and $f_i(x), i = 1, \ldots, n$, are real-valued functions. We define the *set of all feasible criterion vectors* Z and the *set of all nondominated criterion vectors* N of (1) as follows

$$\begin{aligned} Z &= \{z \in \mathbb{R}^n : z = f(x), x \in S\} = f(S) \\ N &= \{z \in Z : \nexists \tilde{z} \in Z \text{ s. t. } \tilde{z} \leq z\}, \end{aligned}$$

where $f(x) = \big(f_1(x), \ldots, f_n(x)\big)^T$. We assume that the set Z is closed and that we can find $u \in \mathbb{R}^n$ so that $u + Z \subseteq \mathbb{R}^n_{\geqq}$.

The point $z^* \in \mathbb{R}^n$ with

$$z_i^* = \min\{f_i(x) : x \in S\} - \epsilon_i \qquad i = 1, \ldots, n$$

is called the *ideal (utopia) criterion vector*, where the components of $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in \mathbb{R}^n$ are small positive numbers.[5] Without loss of generality we assume $z^* = 0$.

[5] Strictly speaking, $\epsilon = 0$ can be used throughout this paper as well. In applications though, $\epsilon > 0$ can help to avoid numerical problems.

We define the set of properly nondominated solutions according to Geoffrion (1968). A point $\bar{z} \in N$ is called *properly nondominated*, if there exists $M > 0$ such that for each $i = 1, \ldots, n$ and each $z \in Z$ satisfying $z_i < \bar{z}_i$ there exists a $j \neq i$ with $z_j > \bar{z}_j$ and

$$\frac{z_i - \bar{z}_i}{\bar{z}_j - z_j} \leq M.$$

Otherwise $\bar{z} \in N$ is called *improperly nondominated.* The set of all properly nondominated points is called N_p. Henig (1982) shows that properly nondominated solutions can also be characterized by means of a cone.

Lemma 2.1 (Henig, 1982) A vector $\bar{z}$ is properly nondominated iff there exists a convex cone $\tilde{C}$ with $\mathbb{R}^n_{\geq} \subseteq \operatorname{int}(\tilde{C})$ so that

$$(Z - \bar{z}) \cap (-\tilde{C}) = \{0\}. \tag{2}$$

Note that $0 \in \tilde{C}$, $\tilde{C} \neq \mathbb{R}^n$ and that Equation (2) can be rewritten as

$$(\bar{z} - \tilde{C}) \cap Z = \{\bar{z}\}.$$

3 Oblique Norms

In order to develop the new concept of oblique norms we first review some basic definitions about block norms. For a detailed introduction to norms and their properties we refer the reader to Rockafellar (1970) and Hiriart-Urruty and Lemaréchal (1993a,b). An overview of basic properties of block norms is also given in Schandl (1998).

Definition 3.1 A norm γ with a polyhedral unit ball $B \subset \mathbb{R}^n$ is called a *block norm.* The vectors defined by the extreme points of the unit ball are called *fundamental vectors* and are denoted by v^i. The fundamental vectors defined by the extreme points of a facet of B span a *fundamental cone.*

Definition 3.2 Let $u \in \mathbb{R}^n$ and $U \subseteq \mathbb{R}^n$. The *reflection sets* of u and U are defined as

$$\begin{aligned} R(u) &:= \{w \in \mathbb{R}^n : |w_i| = |u_i| \text{ for all } i = 1, \ldots, n\} \\ R(U) &:= \bigcup_{u \in U} R(u), \end{aligned}$$

respectively.

Using Definition 3.2 we define absolute norms analogously to Bauer et al. (1961).

Definition 3.3 A norm γ is said to be *absolute* if for any given $u \in \mathbb{R}^n$, all elements of $R(u)$ have the same distance from the origin with respect to γ, that is,

$$\gamma(w) = \gamma(u) \quad \forall w \in R(u).$$

Note that the unit ball of an absolute norm has the same structure in every orthant, which is convenient as well as sufficient for multiple criteria programs as all nondominated solutions are located in the cone $z^* + \mathbb{R}^n_{\geqq}$ and one does not need to search the entire space $\mathbb{R}^n$.

The following lemma will later be used to establish a property of oblique norms.

Lemma 3.4 Let $B \subseteq \mathbb{R}^n$ be the unit ball of an absolute norm γ and let $z, z^1, z^2 \in B$ have the following properties:

$$\begin{aligned} &z \neq z^1 \neq z^2 \neq z \\ &z = \lambda z^1 + (1 - \lambda) z^2 \text{ for some } \lambda \in (0, 1) \\ &z = (0, \ldots, 0, z_k, 0, \ldots, 0) \text{ for some } k = 1, \ldots, n \\ &z_k > 0. \end{aligned}$$

Then there exist $\tilde{z}^1, \tilde{z}^2 \in B$ with the following properties:

$$\begin{aligned}
& z \neq \tilde{z}^1 \neq \tilde{z}^2 \neq z \\
& z = \lambda \tilde{z}^1 + (1-\lambda)\tilde{z}^2 \text{ for some } \lambda \in (0,1) \\
& \tilde{z}^1 \in \mathbb{R}^n_{\geq} \\
& \tilde{z}^1_k > 0 \\
& \tilde{z}^1_k \geq \tilde{z}^2_k.
\end{aligned} \tag{3}$$

Proof. Without loss of generality $k = 1$ and $z^1_1 \geq z^2_1$. Since $z_1 > 0$ and $z_1 \in [z^2_1, z^1_1]$ it follows that $z^1_1 > 0$. Define $\tilde{z}^1$ and $\tilde{z}^2$ componentwise as follows:

$i = 1:$ $\tilde{z}^1_1 = z^1_1, \tilde{z}^2_1 = z^2_1$.

$i > 1:$ If $z^1_i \geq 0$: $\tilde{z}^1_i = z^1_i, \tilde{z}^2_i = z^2_i$. If $z^1_i < 0$: $\tilde{z}^1_i = -z^1_i, \tilde{z}^2_i = -z^2_i$.

Since γ is absolute we know that $\tilde{z}^1, \tilde{z}^2 \in B$. Since $\tilde{z}^1_1 > 0$ it follows that $\tilde{z}^1 \in \mathbb{R}^n_{\geq}$. Now we are left with checking Equation (3) componentwise.

$i = 1:$ Simple calculations yield

$$\lambda \tilde{z}^1_1 + (1-\lambda)\tilde{z}^2_1 = \lambda z^1_1 + (1-\lambda) z^2_1 = z_1.$$

$i > 1:$ For $z^1_i \geq 0$ we get

$$\lambda \tilde{z}^1_i + (1-\lambda)\tilde{z}^2_i = \lambda z^1_i + (1-\lambda) z^2_i = z_i$$

and $z^1_i < 0$ yields

$$\begin{aligned}
\lambda \tilde{z}^1_i + (1-\lambda)\tilde{z}^2_i &= \lambda(-z^1_i) + (1-\lambda)(-z^2_i) \\
&= -\left(\lambda z^1_i + (1-\lambda) z^2_i\right) = -z_i = 0 = z_i.
\end{aligned}$$

□

Definition 3.5 A block norm γ with the unit ball B is called *oblique* if it has the following properties:

(i) γ is absolute.

(ii) $(z - \mathbb{R}^n_{\geqq}) \cap \mathbb{R}^n_{\geqq} \cap \partial B = \{z\} \quad \forall z \in (\partial B)_{\geq}$.

Figure 1 depicts a simple oblique norm. Observe that no facet is orthogonal or parallel to any coordinate axis which is a general property of oblique norms that we show in Lemma 3.9. Furthermore, an oblique norm is strictly $\mathbb{R}^n_{\geq}$-monotone as defined in Gerth and Weidner (1990).

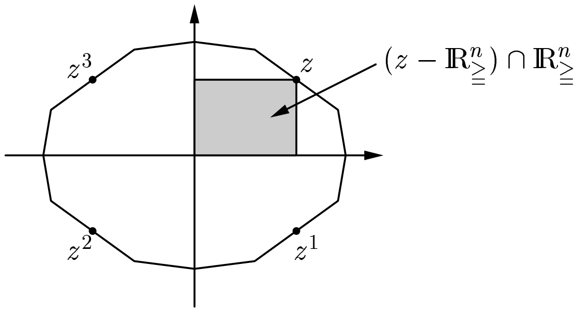

Figure 1: Example of the unit ball of an oblique norm
with $R(z) = \{z, z^1, z^2, z^3\}$

The following corollary immediately results from Definitions 3.3 and 3.5.

Corollary 3.6 If γ with the unit ball B is an oblique (absolute) norm, then $\tilde{\gamma}$ with the unit ball $\alpha B, \alpha > 0$ is also an oblique (absolute) norm.

In the following, we prove some further properties of oblique norms. These properties will be used in Section 5 to prove the applicability of oblique norms for the generation of nondominated solutions.

Lemma 3.7 An absolute norm γ with the unit ball B has the following property:

$$(z - \mathbb{R}^n_{\geqq}) \cap \mathbb{R}^n_{\geqq} \subseteq \gamma(z) B_{\geq} \quad \forall z \in \mathbb{R}^n_{\geq}. \tag{4}$$

Proof. For $z = 0$, the property obviously holds. So consider only $z \neq 0$.
Consider first $z \in (\partial B)_{\geqq}$. It follows that $\gamma(z) = 1$. Since γ is absolute, all points in $R(z)$ are in B. Because of the convexity of B, we have $\operatorname{conv}\bigl(R(z)\bigr) \subseteq B$. But $(z - \mathbb{R}^n_{\geqq}) \cap \mathbb{R}^n_{\geqq}$ is a subset of $\operatorname{conv}\bigl(R(z)\bigr)$ and therefore also of $B_{\geqq}$.
The general case $z \in \mathbb{R}^n_{\geqq}$ follows from Corollary 3.6. □

Note that Lemma 3.7 guarantees that all normals $\mathfrak{n}$ of facets in $\mathbb{R}^n_{\geq}$ are nonzero and have only nonnegative components, that is, $\mathfrak{n} \geq 0$. Otherwise (4) would be wrong for (at least) every point in the interior of a facet with a normal $\mathfrak{n} \not\geq 0$.

Lemma 3.8 An oblique norm γ with the unit ball B has the following properties:

(i) $(z - \mathbb{R}^n_{\geqq}) \cap \mathbb{R}^n_{\geqq} \cap \partial\bigl(\gamma(z)B\bigr) = \{z\} \quad \forall z \in \mathbb{R}^n_{\geqq}$.

(ii) There exist fundamental vectors of B along each coordinate axis.

Proof.

(i) For $z = 0$, the lemma obviously holds. Otherwise, since $z \in \partial\bigl(\gamma(z)B\bigr)$ and $\gamma(z) > 0$, the statement follows directly from Definition 3.5 (ii) and Corollary 3.6.

(ii) Assume there does not exist a fundamental vector along some coordinate axis, say $(1, 0, \dots, 0)$. Without loss of generality $z \in \partial B$, $z = (z_1, 0, \dots, 0)$, $z_1 > 0$ is not an extreme point, that is, there exist $z^1, z^2 \in B$ and $\lambda \in (0, 1)$ with

$$z \neq z^1 \neq z^2 \neq z$$
$$z = \lambda z^1 + (1 - \lambda) z^2.$$

Due to Lemma 3.4 we can assume that $z^1 \in \mathbb{R}^n_{\geq}$ and $z^1_1 \geq z^2_1$. It follows that $z^1_1 \geq z_1$ and therefore $z \in z^1 - \mathbb{R}^n_{\geqq}$. Since we know that $z \in \mathbb{R}^n_{\geqq} \cap \partial B$ it is enough to show $z^1 \in \partial B$ in order to get a contradiction to Definition 3.5 (ii).

Assume that there exists $\tilde{z} \in B$ with $\tilde{z}_1 > z_1$. Since γ is absolute it follows that $\hat{z} := (\tilde{z}_1, -\tilde{z}_2, \ldots, -\tilde{z}_n) \in B$ and since B is convex $\frac{1}{2}\tilde{z} + \frac{1}{2}\hat{z} = (\tilde{z}_1, 0, \ldots, 0) \in B$. But $\tilde{z}_1 > z_1$ and therefore z would not be in ∂B, a contradiction. Thus $\forall \tilde{z} \in B : \tilde{z}_1 \leq z_1$. But since we know that $z_1^1 \geq z_1$ it follows $z_1^1 = z_1$. This implies that $z^1 \in \partial B$ because B is closed and there does not exist $\tilde{z} \in B$ with $\tilde{z}_1 > z_1 = z_1^1$. $\square$

Lemma 3.9 Let γ be an absolute block norm with the unit ball B. Let $\mathcal{N}$ denote the set of outer normal vectors of all the facets of B, let $\mathfrak{n} \in \mathcal{N}$ be the normal vector of a facet of B, and let e^j be the j^{th} unit vector, $j = 1, \ldots, n$. Then the following two statements are equivalent:

(i) $(z - \mathbb{R}^n_{\geqq}) \cap \mathbb{R}^n_{\geqq} \cap \partial B = \{z\} \qquad \forall z \in (\partial B)_{\geqq}$.

(ii) $\langle \mathfrak{n}, e^j \rangle \neq 0 \qquad \forall j = 1, \ldots, n$ and $\forall \mathfrak{n} \in \mathcal{N}$.

Proof.

(i) $\Rightarrow$ (ii) Let F be a facet of B with the normal vector $\mathfrak{n} \in \mathcal{N}$. Assume $\langle \mathfrak{n}, e^j \rangle = 0$ for some j. Then there exists a point $z \in F$ with $z_j \neq 0$ (otherwise F would not be a facet). Since γ is absolute, we can assume without loss of generality that $z \in \mathbb{R}^n_{\geqq}$. Define a point $\tilde{z}$ as follows:

$$\begin{aligned} \tilde{z}_k &= z_k \quad \forall k \neq j \\ \tilde{z}_j &= \tfrac{1}{2} z_j. \end{aligned}$$

Then $\tilde{z}$ is in $F \subseteq \partial B$, because γ is absolute. But we also have that

$$\tilde{z} \in (z - \mathbb{R}^n_{\geqq}) \cap \mathbb{R}^n_{\geqq} \cap \partial B,$$

which is a contradiction to (i).

(ii) $\Rightarrow$ (i) Let $z \in (\partial B)_{\geqq}$ and assume there exists $\tilde{z} \neq z$ with

$$\tilde{z} \in (z - \mathbb{R}^n_{\geq}) \cap \mathbb{R}^n_{\geqq} \cap \partial B. \qquad (5)$$

Because of Lemma 3.7 and its following remark we have $\mathfrak{n} \geq 0$ for all normals of facets in $\mathbb{R}^n_{\geqq}$. Together with $\left\langle \mathfrak{n}, e^j \right\rangle \neq 0$ for all j we even know that $\mathfrak{n} > 0$ for these same normals. Since we assumed that both z and $\tilde{z}$ are in ∂B, they are either on the same or on two different facets.

Assume first that z and $\tilde{z}$ are on the same facet F with normal $\mathfrak{n}$. Consequently $\langle z - \tilde{z}, \mathfrak{n} \rangle = 0$, but since $z - \tilde{z} \geq 0$ and $\mathfrak{n} > 0$ it follows that $z = \tilde{z}$, a contradiction to our assumption.

Assume now that z and $\tilde{z}$ are on different facets, say F and $\tilde{F}$ with normals $\mathfrak{n}$ and $\tilde{\mathfrak{n}}$, respectively. Since $z \in B$ and $\tilde{z} \in \tilde{F}$, the definition of the outer normal yields $\langle z - \tilde{z}, \tilde{\mathfrak{n}} \rangle \leq 0$. But since $z - \tilde{z} \geq 0$ and $\tilde{\mathfrak{n}} > 0$ it follows again that $z = \tilde{z}$, a contradiction.

Thus $\tilde{z} \notin \partial B$ and assumption (5) was wrong. $\square$

4 Construction of Oblique Norms

In this section, we demonstrate two approaches to constructing oblique norms. We first present an algebraic construction by giving linear inequalities for the unit ball of the norm. Then a set-theoretic construction based on polyhedral cones is given. The latter is used in Section 5 to construct oblique norms for the generation of nondominated solutions.

We can define a block norm in terms of its unit ball using a set of linear inequalities, see, for example, Anderson and Osborne (1977). Let $z \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$ and $e := (1, \ldots, 1)^T \in \mathbb{R}^m$, where $m > n$ and the set $B := \{x : Az \leqq e\}$ has the following properties:

(i) B is bounded.

(ii) $\text{int}(B) \neq \varnothing$.

(iii) $z \in B \iff -z \in B$.

Then the block norm specified by A is

$$\gamma_A(z) = \min\{\alpha : Az \leqq \alpha e\}.$$

Note that the number of facets of the unit ball is smaller than or equal to m and that, due to Property (ii), the rank of the matrix A is n.

To describe an oblique norm in a similar way, we have to pose additional restrictions on the constraint matrix A. Consider the properties of an oblique norm given in Definition 3.5 and Lemma 3.9: γ is absolute and none of the normals of its unit ball's facets are orthogonal or parallel to any unit vector e^i. The latter implies that all entries of A are nonzero. Together with the former this also implies that we can subdivide the rows of A into 2^n blocks, one for each orthant. We need to consider only the first block defining the facets in $\mathbb{R}^n_{\geqq}$; all others can be obtained by changing the signs of the entries of this block appropriately. Note that such a subdivision into blocks is in general not possible for absolute norms.

Summarizing the discussion above, we can define an oblique norm by choosing $k \geq 1$ positive vectors which are the normals of the unit ball's facets in the first orthant so that the n columns (of length $2^n k$) of A are linearly independent. The remaining $2^{n-1}k$ rows of A can be found by changing the signs of the entries of the first k rows appropriately.

By following only these rules we might define many hyperplanes that do not affect the shape of the unit ball. To avoid unnecessary hyperplanes we could require that all the hyperplanes have the same distance from the origin. Note that this additional restriction makes the norm "more symmetrical" than required by the definition of oblique norms. A hyperplane defined by the equation $\langle \mathfrak{n}, x \rangle = d$ where $\mathfrak{n}$ is the hyperplane's normal has an l_2 distance of $\frac{|d|}{\|\mathfrak{n}\|_2}$ from the origin, see, for example, Tuy (1998, p. 4). Since the right hand side of each inequality in $Ax \leqq e$ is one, all hyperplanes have the same distance from the origin if the corresponding row vectors have the same l_2 norm.

We now present an example satisfying all the conditions mentioned above. We define the first block of A consisting of $k = n + 1$ rows as follows:

$$a_{ij} = \begin{cases} 1 & \text{if } i = 1, \ldots, n \text{ and } j = i \\ \frac{1}{n} & \text{if } i = 1, \ldots, n \text{ and } j \neq i \\ \sqrt{\frac{n^2+n-1}{n^3}} & \text{if } i = n+1 \text{ and } j = 1, \ldots, n. \end{cases}$$

All entries of the first k rows of A are positive, the columns of A are linearly independent and the l_2 norm of each row of A is equal to $\sqrt{\frac{n^2+n-1}{n^2}}$.

Another example where the rows do *not* have the same l_2 norm is the following. Again we give the first $k = n+1$ rows:

$$a_{ij} = \begin{cases} 1 & \text{if } i = 1, \ldots, n \text{ and } j = i \\ \frac{1}{10} & \text{if } i = 1, \ldots, n \text{ and } j \neq i \\ \frac{7}{10} & \text{if } i = n+1 \text{ and } j = 1, \ldots, n. \end{cases} \tag{6}$$

The norm is depicted in Figure 2 for two dimensions. The unit ball is then defined as

$$\begin{pmatrix} 1 & \frac{1}{10} \\ \frac{1}{10} & 1 \\ \frac{7}{10} & \frac{7}{10} \\ & \vdots \end{pmatrix} x \leqq \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{pmatrix}.$$

The four extreme points in the first quadrant are $z^1 = (0, 1)$, $z^2 = \left(\frac{10}{21}, \frac{20}{21}\right)$, $z^3 = \left(\frac{20}{21}, \frac{10}{21}\right)$, and $z^4 = (1, 0)$. An easy calculation shows that none of the hyperplanes in (6) can be omitted without changing the unit ball.

We now give an alternative set-theoretic construction of an oblique norm. First, we need the following lemma that states some properties of gauges and was proved in Minkowski (1967).

Lemma 4.1 Let $B \subset \mathbb{R}^n$ be a convex compact set containing the origin in its interior and γ be the gauge with the unit ball B, that is,

$$\gamma(z) := \min\{\lambda \geq 0 : z \in \lambda B\} \qquad z \in \mathbb{R}^n.$$

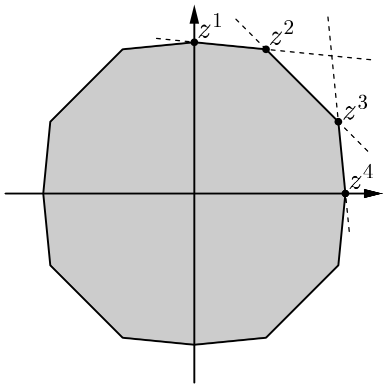

Figure 2: An oblique norm in two dimensions

Then γ satisfies

$$
\begin{aligned}
&\gamma(z) \geq 0 && \forall z \in \mathbb{R}^n \\
&\gamma(z) = 0 \iff z = 0 && \\
&\gamma(\alpha z) = \alpha\gamma(z) && \forall z \in \mathbb{R}^n, \forall \alpha \geq 0 \\
&\gamma(z^1 + z^2) \leq \gamma(z^1) + \gamma(z^2) && \forall z^1, z^2 \in \mathbb{R}^n.
\end{aligned}
$$

If γ additionally satisfies $\gamma(z) = \gamma(-z)$ for all $z \in \mathbb{R}^n$ then γ is a norm.

Observe that if the set B in Lemma 4.1 was not bounded, a functional γ could be similarly defined replacing min by inf. However, the second of the four properties given in Lemma 4.1 holds if an only if B is compact, see Hiriart-Urruty and Lemaréchal (1993a).

With the help of the following lemma we will be able to construct oblique norms by means of a convex, closed, polyhedral cone satisfying two additional conditions.

Lemma 4.2 Let $C \subsetneq \mathbb{R}^n$ be a convex, closed, polyhedral cone and $\mathbb{R}^n_{\geq} \subseteq \operatorname{int}(C)$. Let $\bar{z} \in \mathbb{R}^n_{>}$ be some vector in the first orthant. Then

$$B := R\big((\bar{z} - C) \cap \mathbb{R}^n_{\geqq}\big)$$

is the unit ball of an oblique norm.

Proof. The proof is subdivided into four parts. First we show the auxiliary statement that $B \subseteq \bar{z} - C$ which is then used to prove the convexity of B. This is followed by a proof of the norm properties of the gauge γ defined by B and finally we show that γ is in fact an oblique norm.

1. B is a subset of $\bar{z} - C$.
   We show $B \subseteq \bar{z}-C$ by contradiction. Assume there exists $z \in B \setminus (\bar{z}-C)$. Define $\tilde{z}$ componentwise as $\tilde{z}_i = |z_i|$. Clearly $\tilde{z} \in \mathbb{R}^n_{\geqq}$. Since B is defined using a reflection set, $\tilde{z} \in B$. Since
   $$B \cap \mathbb{R}^n_{\geqq} = (\bar{z} - C) \cap \mathbb{R}^n_{\geqq}, \tag{7}$$
   we also have that $\tilde{z} \in \bar{z} - C$. It follows
   $$z \in \tilde{z} - \mathbb{R}^n_{\geqq} \subseteq \bar{z} - C,$$
   which is a contradiction to the assumption that $z \in B \setminus (\bar{z} - C)$. Thus $B \subseteq \bar{z} - C$.

2. Convexity of B.
   Now we show by contradiction that B is convex. Assume that we can find $z^1, z^2 \in B$ and $\lambda \in [0, 1]$ so that
   $$z := \lambda z^1 + (1 - \lambda) z^2 \notin B. \tag{8}$$
   Since B is defined by means of a reflection set, we can multiply vector components in (8) by -1 and we still have $z^1, z^2 \in B$ and $z \notin B$. Therefore we can assume without loss of generality that $z \in \mathbb{R}^n_{\geqq}$. Since $z \notin B$ it follows from (7) that $z \notin \bar{z} - C$. Because C is a convex cone, $\bar{z} - C$ is also convex and therefore either $z^1 \notin \bar{z} - C$ or $z^2 \notin \bar{z} - C$ (or both). But since $B \subseteq \bar{z} - C$ it follows that $z^1 \notin B$ or $z^2 \notin B$ (or both), a contradiction to our assumption. Thus B is convex.

3. Norm properties of γ defined by B.
   Since $\bar{z} > 0$ it follows that $0 \in \operatorname{int}(B)$. B is closed because C is closed.

Because of the definition of B by means of a reflection set we know that

$$\gamma(z) = \gamma(-z) \qquad \forall z \in \mathbb{R}^n.$$

We are left with showing that B is bounded. Assuming that B is unbounded we can find $d \in C \setminus \{0\}$ so that

$$\bar{z} - \alpha d \in \mathbb{R}^n_{\geq} \quad \forall \alpha > 0.$$

Consequently $d \leq 0$. But then $d \in C$ and $\mathbb{R}^n_{\geq} \subseteq \operatorname{int}(C)$ implies $C = \mathbb{R}^n$, a contradiction to the assumption that $C \subsetneq \mathbb{R}^n$. Thus B is bounded and, using Lemma 4.1, γ defines a norm. Since C is a polyhedral cone, B is a polyhedral set and therefore γ is a block norm.

4. γ defined by B is an oblique norm.

   Now we check the two properties of an oblique norm given in Definition 3.5. By definition, γ is absolute and Property (i) is satisfied.

   To check Property (ii) we observe that since $\operatorname{int}(C)$ is a convex cone we have

$$z - \operatorname{int}(C) \subseteq \bar{z} - \operatorname{int}(C) \qquad \forall z \in \bar{z} - C. \tag{9}$$

Let $z \in (\partial B)_{\geqq} \subseteq \bar{z} - C$, so (9) holds. It follows that

$$z - \mathbb{R}^n_{\geq} \subseteq z - \operatorname{int}(C) \subseteq \bar{z} - \operatorname{int}(C),$$

which, using (7), implies

$$(z - \mathbb{R}^n_{\geq}) \cap \mathbb{R}^n_{\geqq} \subseteq \big(\bar{z} - \operatorname{int}(C)\big) \cap \mathbb{R}^n_{\geqq} = \operatorname{int}(B) \cap \mathbb{R}^n_{\geqq},$$

and consequently

$$(z - \mathbb{R}^n_{\geq}) \cap \mathbb{R}^n_{\geqq} \cap \partial B = \varnothing.$$

Substituting $\mathbb{R}^n_{\geqq}$ for $\mathbb{R}^n_{\geq}$, we add only z which proves property (ii). $\square$

Given a cone C, depending on the point $\bar{z}$, the unit ball of the oblique norm can look very different, see Figure 3. Note that, given one cone, we can only construct oblique norms with at most one extreme point per orthant.

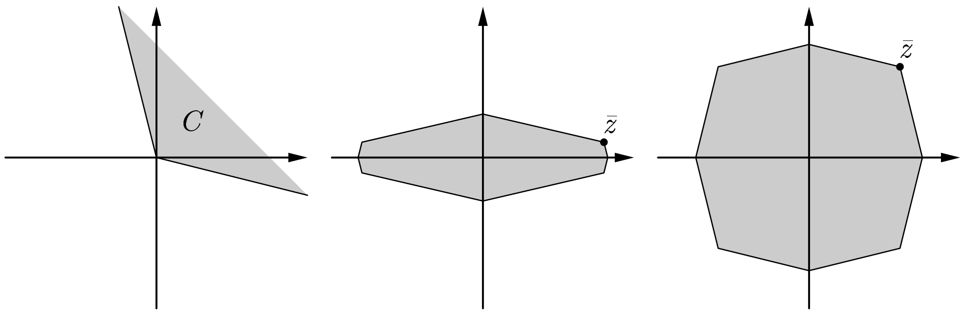

Figure 3: A cone C and two unit balls based on C with different points $\bar{z}$

5 Generating Nondominated Points

According to Lemma 2.1, one can find a convex cone satisfying (2) for each properly nondominated point. Due to the fact that an oblique norm can be constructed by means of a convex, closed, polyhedral cone as shown in Lemma 4.2, it is helpful to find a relationship, similar to that given in Lemma 2.1, between a properly nondominated solution and a convex, closed, polyhedral cone. The following lemma leads to this relationship.

Lemma 5.1 Let $\tilde{C} \subseteq \mathbb{R}^n$ be a convex cone containing the origin with $\mathbb{R}^n_{\geq} \subseteq \operatorname{int}(\tilde{C})$. Then there exists a convex, closed, polyhedral cone $C \subseteq \tilde{C}$ with $\mathbb{R}^n_{\geq} \subseteq \operatorname{int}(C)$.

Proof. Let $e^i \in \mathbb{R}^n$ and $c^i \in \mathbb{R}^n$, $i = 1, \ldots, n$, be the vectors with the following components:

$$e^i_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \qquad\qquad c^i_j = \begin{cases} 1 & \text{if } i = j \\ -\delta & \text{if } i \neq j \end{cases}$$

where $\delta > 0$. Note that $\mathbb{R}^n_{\geq} = \operatorname{cone}\big(\{e^i, i = 1, \ldots, n\}\big)$. According to Tuy

(1998, p. 37), the set $C := \operatorname{cone}\big(\{c^i, i = 1, \ldots, n\}\big)$ is a closed, convex, polyhedral cone.

Since $\operatorname{int}(\tilde{C})$ is open, we can find $\delta > 0$ so that $\forall i : c^i \in \operatorname{int}(\tilde{C})$. Since $\tilde{C}$ is a convex cone, we have $C \subseteq \tilde{C}$.

Since $\{e^i, i = 1, \ldots, n\} \subseteq \operatorname{int}(C)$ and $\operatorname{int}(C)$ is a convex cone, we have

$$\mathbb{R}^n_{\geq} = \operatorname{cone}\big(\{e^i, i = 1, \ldots, n\}\big) \subseteq \operatorname{int}(C). \qquad \square$$

We can now reformulate Lemma 2.1 using a closed, polyhedral cone.

Theorem 5.2 A vector $\bar{z}$ is properly nondominated iff there exists a convex, closed, polyhedral cone C with $\mathbb{R}^n_{\geq} \subseteq \operatorname{int}(C)$ so that

$$(\bar{z} - C) \cap Z = \{\bar{z}\}.$$

Proof. Use Lemmas 2.1 and 5.1. $\square$

Note that again $0 \in C$.

We now show that for every nondominated point there exists a block norm so that this point is a unique minimizer of the related block-norm-scalarization. To show the existence of the desired block norm we use the l_∞ norm, and thus not an oblique norm. The result gives another interpretation of the results on the weighted Tchebycheff approach in Steuer and Choo (1983) and Steuer (1986) and illustrates the idea of introducing block norms into multiple criteria programming.

Theorem 5.3 Let $\bar{z} \in N$. Then there exists a block norm γ so that $\bar{z}$ uniquely minimizes

$$\min_{z \in Z} \gamma(z) = \min_{x \in S} \gamma\big(f(x)\big).$$

Proof. Recall that we assumed without loss of generality $z^* = 0$. Define the unit ball B of a block norm γ as $B = \operatorname{conv}\big(R(\bar{z})\big)$. Assume there is a $\tilde{z} \in Z, \tilde{z} \neq \bar{z}$ with $\gamma(\tilde{z}) \leq \gamma(\bar{z})$. From the construction of γ we have that $\tilde{z} \leqq \bar{z}$. Since $\tilde{z} \neq \bar{z}$, we have $\tilde{z}_i < \bar{z}_i$ for some i, which is a contradiction to $\bar{z} \in N$. Thus $\gamma(z) > \gamma(\bar{z})$ for all $z \in Z$. $\square$

We now focus on oblique norms and show that any optimal solution of an oblique-norm-scalarization of (1) is a properly nondominated solution of (1). Note that an oblique norm cannot be used to generate improperly nondominated points.

Theorem 5.4 Let γ be an oblique norm and let $\bar{z}$ be a solution of

$$\min_{z \in Z} \gamma(z) = \min_{x \in S} \gamma\big(f(x)\big). \tag{10}$$

Then $\bar{z} \in N_p$.

Proof. Since $\bar{z}$ solves (10), there is a $\bar{\lambda} > 0$ such that $\gamma(\bar{z}) = \bar{\lambda}$. Similar to the proof of Lemma 3.8 (ii), we can show that there is an $\epsilon_i > 0$ such that

$$\tilde{z}^i := (0, \ldots, \bar{z}_i + \epsilon_i, \ldots, 0) \in \bar{\lambda} B \qquad \forall i = 1, \ldots, n.$$

Define a cone $\hat{C} := -\operatorname{cone}\big(\{\tilde{z}^1 - \bar{z}, \ldots, \tilde{z}^n - \bar{z}\}\big)$. Obviously, $\hat{C}$ is convex and $\mathbb{R}^n_{\geq} \subseteq \operatorname{int}(\hat{C})$. Since $(\bar{z} - \hat{C}) \cap \mathbb{R}^n_{\geqq} \subseteq \bar{\lambda} B$ and $\bar{z}$ minimizes (10), we know that

$$(\bar{z} - \hat{C}) \cap Z = \{\bar{z}\}.$$

Thus Lemma 2.1 shows that $\bar{z} \in N_p$. □

In the following theorem we show that all properly nondominated solutions can be found by solving suitable scalarizations involving oblique norms.

Theorem 5.5 Let $\bar{z} \in N_p$. Then there exists an oblique norm γ so that $\bar{z}$ uniquely minimizes

$$\min_{z \in Z} \gamma(z) = \min_{x \in S} \gamma\big(f(x)\big). \tag{11}$$

Proof. We assumed that the utopia point $z^* = 0$, so it follows that $\bar{z} \in Z \subseteq \mathbb{R}^n_{>}$. Since $\bar{z} \in N_p$ and because of Theorem 5.2 there exists a closed, polyhedral, convex cone C with $0 \in C$ and $\mathbb{R}^n_{\geq} \subseteq \operatorname{int}(C)$ so that

$$(\bar{z} - C) \cap Z = \{\bar{z}\}.$$

With the help of Lemma 4.2 we can construct an oblique norm γ with the unit ball $B = R\big((\bar{z} - C) \cap \mathbb{R}^n_{\geqq}\big)$. Then

$$B \cap \mathbb{R}^n_{>} = R\big((\bar{z} - C) \cap \mathbb{R}^n_{\geqq}\big) \cap \mathbb{R}^n_{>} = (\bar{z} - C) \cap \mathbb{R}^n_{>}.$$

Therefore

$$B \cap \mathbb{R}^n_{>} \cap Z = (\bar{z} - C) \cap \mathbb{R}^n_{>} \cap Z = \{\bar{z}\}.$$

Thus $\bar{z}$ uniquely minimizes (11). □

Considering problems with only properly nondominated points, i.e. for which $N = N_p$, Theorems 5.4 and 5.5 show the equivalence between the multiple criteria program and its scalarization by means of an oblique norm. We state this in the following corollary.

Corollary 5.6 Let $\bar{z} \in \mathbb{R}^n$. The following two statements are equivalent:

(i) $\bar{z} \in N_p$.

(ii) There exists an oblique norm γ so that $\bar{z}$ uniquely minimizes

$$\min_{z \in Z} \gamma(z) = \min_{x \in S} \gamma(f(x)).$$

6 Conclusions

In this paper we introduce oblique norms into multiple criteria programming. Oblique norms are block norms with none of their facets being orthogonal or parallel to any coordinate axis. We prove that a properly nondominated solution of the general multiple criteria program is also a solution of a scalarization of the program by means of an oblique norm and vice versa. The definition of oblique norms opens up possibilities for constructing a variety of such norms in order to not only generate nondominated solutions but to effectively support the decision making process of choosing a preferred nondominated solution. The main difference between l_∞ norms and oblique norms is that the former have only one fundamental direction in $\mathbb{R}^n_{>}$ while the latter may feature any finite number of those directions and therefore can be more useful in accommodating decision maker's preferences and more precise in measuring the distance in the objective space. Moreover, the unit ball of an oblique norm yields a piecewise linear utility function and partitions the objective space into cones generated by the extreme points of the facets of the unit ball. In each of these cones, different trade-offs can be used by changing the length of the fundamental vectors. This has two important consequences. On one hand, it is easy to incorporate decision maker's preferences by defining an appropriate norm. On the other hand, the more detailed trade-off information provided by oblique norms can be used to evaluate and/or compare nondominated points.

Norms and cones have been used extensively in multiple criteria programming but, to our knowledge, Kaliszewski (1994) is perhaps the only other source to use both concepts *simultaneously* in order to analyze and solve multiple criteria programs. In Schandl (1999) and Schandl et al. (1999), we show that the combination of norms and cones is a powerful tool to gain important information concerning the structure of the nondominated set and the trade-offs between the criteria in different regions of the nondominated set. In particular, we develop a norm-based methodology to evaluate and approximate nondominated points and present applications in engineering design and capital budgeting. The usage of oblique norms instead of weighted (augmented or modified) l_∞ norms (which are special cases of oblique norms) generalizes and enhances the method by providing more versatility.

Solutions of the oblique-norm scalarization could be studied in a broader sense and compared to other classes of nondominated solutions proposed in the literature. For example, in Ester (1986), Dubov (1981), and Sawaragi et al. (1985) various classes of such solutions are defined. In particular, oblique norms are not suited for properly nondominated solutions in the sense of Germeier (1971) or Schönfeld (1970). However, the class of ϵ-uniform nondominated solutions studied in Dubov (1981) seems to be closely related to the nondominated solutions generated with oblique norms.

The scalarization proposed in this paper lays down a foundation for the methodology applicable to decision making problems with multiple criteria encountered in many areas of human activity including engineering, business and management problems as well as location theory, scheduling and others.

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