Norm-Based Approximation in Convex Multicriteria Programming[1]

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Abstract: Based on theoretical results on the applicability of specially structured block norms to multicriteria programs, an algorithm to find a piecewise linear approximation of the nondominated set of convex multicriteria programs is proposed. By automatically adapting to the problem's structure and scaling, the approximation is constructed objectively without interaction with the decision maker. Moreover, all generated extreme points of the approximation are nondominated.

1 Introduction

Decision making with respect to many conflicting criteria and constraints has become a central problem in management and technology. In the presence of multiple criteria, trade-off information plays a central role in decision making since it facilitates the comparison of nondominated outcomes (efficient alternatives). Among many methodological approaches to quantify trade-offs, approximation of the nondominated set is most attractive as it can visualize the outcomes for the decision maker and provide this information in a simple and understandable way.

In this paper we suggest to use cones and norms, two concepts well known in convex analysis, to construct piecewise linear approximations of the nondominated set of convex multicriteria programming problems. Both cones and norms have been used in multicriteria programming quite extensively (see e.g. Steuer (1986) and Kaliszewski (1987)) but, to our knowledge, Kaliszewski (1994) is the only other source to combine both concepts in order to describe and solve multicriteria programs. There have been quite a few approximation approaches developed for bicriteria convex as well as general problems, see, e.g., Fruhwirth et al. (1989) and Jahn and Merkel (1992). For general multicriteria optimization problems, approximation approaches were developed by Helbig (1991), Sobol and Levitan (1997), and others. For an overview of approximation algorithms for multicriteria programming problems we refer to Schandl (1999). The approach presented in this paper uses concepts employed by other authors but puts them in the new framework of norms. This results in approximation properties not yet present in the literature such as scale independence, weight independence and the generation of a problem dependent measure of the approximation quality.

In the next section, we state the multicriteria programming problem and give some general definitions and notations. A theoretical basis for the approximation algorithm is discussed in Section 3. An approximation approach for problems with $\mathbb{R}^n_{\geqq}$-convex sets of criterion vectors is presented in Section 4. The last section includes a short summary and some concluding remarks.

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2 Problem Formulation

To facilitate further discussions, we use the following notation: Let $u, w \in I\!R^n$ be two vectors. $u < w$ denotes $u_i < w_i$ for all $i = 1, \ldots, n$. $u \leq w$ denotes $u_i \leq w_i$ for all $i = 1, \ldots, n$, but $u \neq w$. $u \leqq w$ allows equality. The symbols $>, \geq, \geqq$ are used accordingly. Let $I\!R^n_{\geqq} := \{x \in I\!R^n : x \geqq 0\}$. If $S \subseteq I\!R^n$, then $S_{\geqq} := S \cap I\!R^n_{\geqq}$.

A set $C \in I\!R^n$ is called a *cone* if for all $u \in C$ and $\alpha > 0$ we also have $\alpha u \in C$. The origin may or may not belong to C.

We consider the following general multicriteria program

$$\begin{array}{ll} \min & \{z_1 = f_1(x)\} \\ & \vdots \\ \min & \{z_n = f_n(x)\} \\ \text{s.t.} & x \in X, \end{array} \tag{1}$$

where $X \subseteq I\!R^m$ is the *feasible set* and $f_i(x), i = 1, \ldots, n$, are real-valued functions.

We define the *set of all feasible criterion vectors* Z, the *set of all nondominated criterion vectors* N and the *set of all efficient points* E of (1) as follows

$$\begin{array}{ll} Z & = \{z \in I\!R^n : z = f(x), x \in X\} = f(X) \\ N & = \{z \in Z : \nexists \tilde{z} \in Z \text{s.t.} \tilde{z} \leq z\} \\ E & = \{x \in X : f(x) \in N\}, \end{array}$$

where $f(x) = (f_1(x), \ldots, f_n(x))^T$. We assume that the set Z is closed and that we can find $u \in I\!R^n$ so that $u + Z \subseteq I\!R^n_{\geqq}$.

The set of properly nondominated solutions is defined according to Geoffrion (1968): A point $\bar{z} \in N$ is called *properly nondominated*, if there exists $M > 0$ such that for each $i = 1, \ldots, n$ and each $z \in Z$ satisfying $z_i < \bar{z}_i$ there exists a $j \neq i$ with $z_j > \bar{z}_j$ and

$$\frac{z_i - \bar{z}_i}{\bar{z}_j - z_j} \leq M.$$

Otherwise, $\bar{z} \in N$ is called *improperly nondominated*.

3 Oblique Norms in Multicriteria Programming

The concept of oblique norms was introduced in Schandl (1999) and Schandl et al. (2000). They can be viewed as specific block norms that are suitable to generate nondominated solutions of multicriteria programs.

Let $u \in I\!R^n$. The *reflection set* of u is the set $R(u) := \{w \in I\!R^n : |w_i| = |u_i| \; \forall i = 1, \ldots, n\}$.

Definition 1 *A block norm γ with a unit ball B is called* oblique *if*

(i) $\gamma(w) = \gamma(u) \quad \forall w \in R(u),\ u \in I\!R^m$ *, and*

(ii) $(z - I\!R^n_{\geqq}) \cap I\!R^n_{\geqq} \cap \partial B = \{z\} \quad \forall z \in (\partial B)_{\geqq}$.

Observe that an oblique norm is a block norm where no facet of the unit ball is parallel to any coordinate axis. Moreover, the structure of the norm's unit ball is the same in each orthant of the coordinate system. An example of an oblique norm in $I\!R^2$ is given in Figure 1.

The following two theorems justify the application of oblique norms for the generation of nondominated solutions.

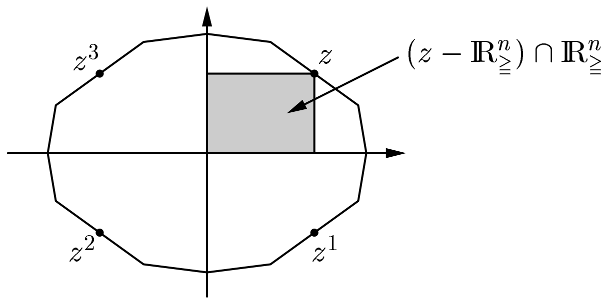

Figure 1: Example of the unit ball of an oblique norm with $R(z) = \{z, z^1, z^2, z^3\}$

Theorem 2 *Assume wlog that* $0 \in Z + \mathbb{R}^n_{\geqq}$. *Let* γ *be an oblique norm with the unit ball* B. *If* $\bar{z} \in \mathbb{R}^n$ *is a solution of*

$$\begin{array}{ll} \max & \gamma(z) \\ \text{s.t.} & z \in -\mathbb{R}^n_{\geqq} \cap Z \end{array} \tag{2}$$

then $\bar{z}$ *is nondominated.*

Proof. Assume $\bar{z} \notin N$, that is, there exists $\tilde{z} \in Z$ with $\tilde{z} \leq \bar{z}$. Since $\bar{z}$ is feasible for (2), we have $\tilde{z} \in -\mathbb{R}^n_{\geqq}$, and it follows that

$$\bar{z} \in -\mathbb{R}^n_{\geqq} \cap (\tilde{z} + \mathbb{R}^n_{\geqq}).$$

Since γ is oblique and therefore absolute, we can use the fact that an oblique norm γ with the unit ball B has the following properties:

$$(z - \mathbb{R}^n_{\geqq}) \cap \mathbb{R}^n_{\geqq} \cap \partial(\gamma(z)B) = \{z\} \quad \forall z \in \mathbb{R}^n_{\geqq}\ , \tag{3}$$

and

$$(z - \mathbb{R}^n_{\geqq}) \cap \mathbb{R}^n_{\geqq} \subseteq \gamma(z)B_{\geqq} \quad \forall z \in \mathbb{R}^n_{\geqq}\ , \tag{4}$$

see Schandl et al. (1999). Using (3) and (4) in $-\mathbb{R}^n_{\geqq}$ instead of $\mathbb{R}^n_{\geqq}$ we can infer that

$$\bar{z} \in \operatorname{int}((\gamma(\tilde{z})B)$$

which implies $\gamma(\bar{z}) < \gamma(\tilde{z})$, a contradiction to the optimality of $\bar{z}$.

□

Unfortunately, we cannot guarantee to find all nondominated points using an oblique norm with its unit ball's center in $Z + \mathbb{R}^n_{\geqq}$ in the general setting of Theorem 2. Therefore the next theorem applies only to problems with an $\mathbb{R}^n_{\geqq}$-convex feasible set Z.

Theorem 3 *Let* $Z \subseteq \mathbb{R}^n$ *be* $\mathbb{R}^n_{\geqq}$*-convex and assume wlog that* $0 \in Z + \mathbb{R}^n_{\geqq}$. *Let* $\bar{z}$ *be properly nondominated with* $\bar{z} \in -\mathbb{R}^n_{\geqq} \cap Z$. *Then there exists an oblique norm* γ *so that* $\bar{z}$ *solves problem (2).*

Proof. From Geoffrion (1968), we know that there exists a weight vector $w \in \mathbb{R}^n_{>}$ with $\sum_{i=1}^n w_i = 1$ so that $\bar{z}$ solves

$$\min_{z \in Z} \sum_{i=1}^n w_i z_i.$$

Let H be the hyperplane with normal vector w and passing through the point $\bar{z}$, and let H^+ be the halfspace defined as

$$H^+ := \{z \in \mathbb{R}^n : \langle w, z \rangle \geq \langle w, \bar{z} \rangle\}.$$

Then the set $R(-\mathbb{R}^n_{\geqq} \cap H^+)$ is the unit ball of an oblique norm γ.

Since $\bar{z} \in H$, it follows that $\bar{z}$ is located on a facet of the unit ball and thus $\gamma(\bar{z}) = 1$. So there cannot exist $z \in Z$ with $\gamma(z) > 1$, because H is a tangent plane of Z. Therefore $\bar{z}$ solves (2).

□

4 Approximation of the Nondominated Set

For multicriteria programs with an $\mathbb{R}^n_{\geqq}$-convex feasible set Z, an approximation algorithm based on Theorems 2 and 3 can be designed that utilizes oblique norms for the generation of nondominated solutions. To keep explanations straight-forward, the general idea of this approach will be outlined using a bicriteria example problem.

Starting from a given reference point $z^0 \in Z + \mathbb{R}^2_{\geqq}$ (z^0 may be for example a currently implemented solution, or the nadir point in bicriteria problems), a first approximation is obtained by exploring the feasible set along $m \geq 2$ search directions $d^1, \ldots, d^m \in -\mathbb{R}^2_{\geqq}$, specified by the decision maker. To obtain nondominated points along these search direction, an adaptation of the *direction method* introduced in Pascoletti and Serafini (1984) can be used (see Schandl (1999)). In the example given in Figure 2(a), the search directions are chosen as the negative unit vectors in $\mathbb{R}^2$, $d^1 = (-1, 0)$ and $d^2 = (0, -1)$, yielding the points z^1 and z^2.

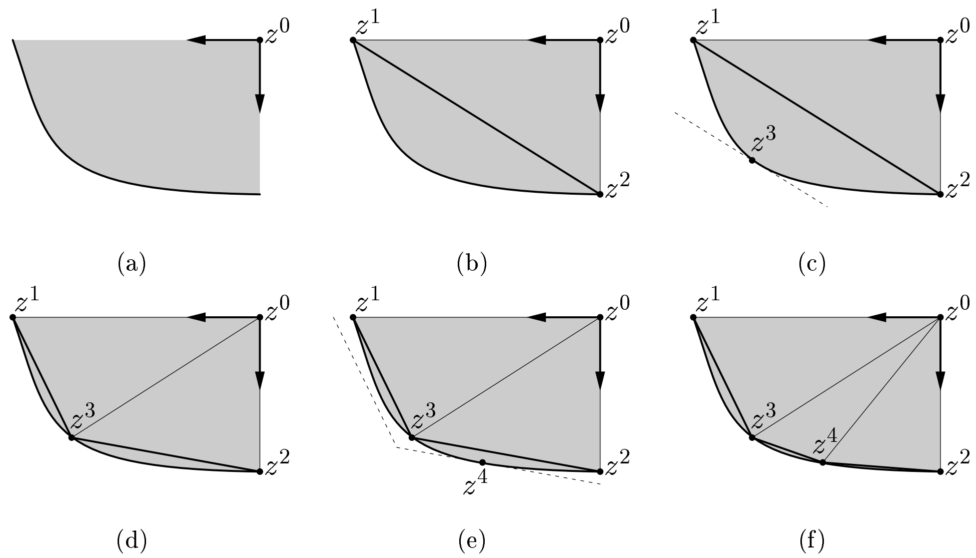

Figure 2: The steps of the approximation algorithm

These two points together with the reference point z^0 are used to define a cone and a first approximation, see Figure 2(b). Interpreting this approximation as the lower left part of the unit ball of an oblique norm γ (or, more general, of an oblique gauge) with z^0 as its center, this norm is then maximized in $Z \cap (z^0 - \mathbb{R}^2_{\geqq})$. Consequently the next point (z^3 in the example problem) is found as a solution of problem (2), where γ is an oblique norm (gauge), see Figure 2(c).

The point z^3 is added to the approximation by building the convex hull of the points generated so far and thus updating the approximation and the underlying norm (gauge) simultaneously as shown in Figure 2(d). Continuing this process, we get a finer approximation of the nondominated set while generating nondominated points and updating the unit ball of the oblique norm (gauge), see Figures 2(e) and 2(f). In each iteration, the point of maximal norm (gauge) is added. Since this point is "farthest away" from the approximation with respect to the current oblique norm (gauge), we always add the point of worst approximation with respect to this norm (gauge).

Observe that in each iteration the maximization problem (2) has to be solved only in those cones whose facets were newly generated due to the addition of the last point. This includes new and modified cones. By updating the convex hull, the resulting approximation is always $\mathbb{R}^n_{\geqq}$-convex.

The following theorem shows that the quality of the approximation improves with each new point if we assume that Z is $\mathbb{R}^n_{\geqq}$-convex.

Theorem 4 *Let $Z \subseteq \mathbb{R}^n$ be $\mathbb{R}^n_{\geqq}$-convex and γ^k be an approximating oblique norm (oblique gauge) constructed from k nondominated points, or points on the boundary of Z. Let $\bar{z}$ be the solution of*

$$\begin{array}{ll} \max & \gamma^k(z) \\ \text{s.t.} & z \in Z \cap (z^0 - \mathbb{R}^n_{\geqq}). \end{array} \tag{5}$$

Let γ^{k+1} be the updated norm (gauge) including the new point $\bar{z}$. Then

$$\gamma^{k+1}(z) \leq \gamma^k(z) \qquad \forall z \in Z \cap (z^0 - \mathbb{R}^n_{\geqq}).$$

Proof. Let B^k and B^{k+1} be the unit balls of γ^k and γ^{k+1}, respectively. Since Z is $\mathbb{R}^n_{\geqq}$-convex, it follows that $\gamma^k(\bar{z}) \geq 1$ and therefore $B^k \subseteq B^{k+1}$. Thus for every $z \in Z \cap (z^0 - \mathbb{R}^n_{\geqq})$ we have

$$\gamma^{k+1}(z) = \min\{\lambda \geq 0 : z \in \lambda B^{k+1}\} \leq \min\{\lambda \geq 0 : z \in \lambda B^k\} = \gamma^k(z).$$

□

5 Conclusions

In this paper we developed an approximation algorithm for convex multicriteria programs generalizing the ideas for the bicriteria case as given in Schandl et al. (2000).

The described approximation algorithm combines several desirable properties which have been confirmed by computational results, see Schandl et al. (1999). The most important and notable are:

- The approximation is always improved in the area where "it is needed most" because in each iteration, the point of worst approximation is added.
- Using the approximation or a norm induced by it to improve the approximation releases the decision maker from specifying preferences (in the form of weights, norms, or directions) to evaluate the quality of the approximation. Such preferences can be used in the initialization step (specifying the reference point z^0 and the initial search directions) but apart from that the approximation is carried out in a neutral manner without decision maker's involvement.

The approximation algorithm described above can be generalized to nonconvex and discrete problems. However, in these cases a more detailed analysis is needed and the maximization problem (2) has to be combined with methods particularly designed to handle nonconvexity and/or discrete variables.

The algorithm yields a piecewise linear approximation of the nondominated set which can easily be visualized if not more than three criteria are present. For more criteria, plots of selected criteria against each other can be created. Such plots and the approximation in general should help the decision maker find a preferred solution within the nondominated set. While the approximation is carried out in an objective manner, the subjective preferences must be (and should be) applied to single out one (or several) final result(s).

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