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BILINEAR PROGRAMMING FORMULATIONS FOR WEBER PROBLEMS WITH CONTINUOUS AND NETWORK DISTANCES

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Abstract While travel cost in urban areas is generally best modeled using network distances, continuous distance measures are mostly used for location problems in rural environments. Combining both is not only interesting from a modeling perspective, but has also computational advantages since it combines reduced storage requirements for the network data with a high accuracy of the approximated distances. In this paper, Weber problems with combined distance measures are discussed. If continuous distances are measured using block norm distances, the resulting Weber problems can be represented by bilinear programming formulations. Theoretical properties of this model are discussed, and two possible solution strategies are suggested.

Keywords: Facility planning, Weber problem, mixed distances, barriers, embedded networks, mixed-integer bilinear programming

1. Introduction

The need for realistic representations of distance measures in location problems is reflected in the recent literature. Continuous models have been extended by various types of restrictions and constraints in order to better incorporate the geographic reality into the geometric representation. Location problems with *forbidden regions* have been extensively studied and can be considered relatively well-solved (for an overview, see Hamacher and Nickel [10]). On the other hand, problems involving physical barriers or congestions still give rise to many open questions that are caused by the non-convexity of the objective function (see Klamroth [13] for a detailed survey on location problems with barriers).

If on the other hand a road or transportation network serves as the basis for a (network or discrete) location model, a balance between the size of the network (and the resulting computational complexity) and the accuracy of the model has to be found (see, for example Drezner and Hamacher [6]). Moreover, the topology of the underlying network has a profound impact on the optimal facility locations. Changing this network may turn out to be more cost-effective than adding more facilities to improve some kind of service.

Very little has been done to include continuities and/or additional modeling parameters in network location models. Batta and Palekar [2] extended a network location model by adding so-called mega nodes which can be entered and left only at a finite set of access points. Erkut [8] added a finite candidate set for new locations outside a given transportation network. Travel distances are measured partly on the network, but - in order to model continuous propagations, for example, of polluted air from an industrial plant - continuous metrics are used in addition to the network metric in the objective function. Blanquero [4] further extended this model by defining a convex feasible region for new locations replacing the finite candidate set. A continuous location problem based on the superposition of a (polyhedral) gauge distance function and a finite set of so-called rapid transit lines modeling, for example, a high-speed transportation network, is suggested in Carrizosa and Rodriguez-Chia [5].

In this paper, bilinear programming formulations are derived that can be viewed as a unifying umbrella under which continuous location models *and* network location models can be described. We focus on the case that (continuous) distances in $\mathbb{R}^2$ are measured by a block norm and exploit the piecewise linear structure of the resulting distance measures. The implications resulting in the general case of arbitrary norm distances are discussed in Pfeiffer and Klamroth [15].

2. Weber Problems with Mixed Distances

Given a finite set of existing facilities $\mathcal{A} = \{a_1, \ldots, a_n\} \subset \mathbb{R}^2$ with positive weights $w_1, \ldots, w_n \in \mathbb{R}$, the classical, continuous *Weber problem* is to find one new facility $x = (x_1, x_2)^T \in \mathbb{R}^2$ such that the weighted sum of distances between x and the existing facilities at $a_1, \ldots, a_n$ is minimized:

$$\begin{array}{ll} \min & f(x) = \sum\limits_{m=1}^{n} w_m d(x, a_m) \\ \text{s.t.} & x \in \mathbb{R}^2. \end{array} \tag{2.1}$$

We replace the distance function d in this classical model by a combination of continuous *and* network distances by allowing switches between continuous and network travel at a finite set $\mathcal{I} := \{i_1, \ldots, i_k\} \subset \mathbb{R}^2$, $i_r = (i_{r1}, i_{r2})^T \in \mathbb{R}^2$, $r = 1, \ldots, k$, of *transit points* (also referred to as *transshipment points* or *intermediate points*). Defining binary variables y_{rm}, $r = 1, \ldots, k$, $m = 1, \ldots, n$, as

$$y_{rm} = \begin{cases} 1, & i_r \text{ is used as transit point on an } x\text{-}a_m\text{-path}, \\ 0, & i_r \text{ is not used as transit point on an } x\text{-}a_m\text{-path}, \end{cases} \qquad r = 1, \ldots, k, \ m = 1, \ldots, n,$$

a *Weber problem with mixed distances* can be formulated:

$$\begin{array}{ll} \min & \sum\limits_{m=1}^{n} w_m \left(\sum\limits_{r=1}^{k} y_{rm} \Big(d(x, i_r) + \alpha_{rm} \Big) \right) \\ \text{s.t.} & \sum\limits_{r=1}^{k} y_{rm} = 1, \quad m = 1, \ldots, n \\ & x \in X \\ & y_{rm} \in \{0, 1\}, \quad r = 1, \ldots, k, \ m = 1, \ldots, n. \end{array} \tag{2.2}$$

The feasible region for new location $X \subseteq \mathbb{R}^2$, $X \neq \emptyset$, is a closed set and we will assume that X is given by a bounded polyhedron. The constants $\alpha_{rm} \in \mathbb{R}$, $r = 1 \ldots, k$, $m = 1, \ldots, n$, describe, for example, the fixed network or barrier distance from transit point i_r to existing facility a_m. In the case of networks the transit points i_r are chosen from the joint set of the vertices of the network and the existing facilities. In the case of location problems with barriers traveling and locating the new facility is prohibited in the interior of a set of given barrier regions as, for example, lakes, mountain ranges, military zones or national parks. If the barriers are polyhedral, then the Barrier Touching Property holds: This states that there always exists a d-shortest path between two given points in the feasible region which is a piecewise linear path with breaking points only in extreme points of barriers. Therefore the plane can be decomposed into cells, where for each cell the set of extreme points, that are d-visible from the cell (i.e., the distance to these points is not lengthened by the barriers) is constant. From this set of d-visible points the transit points i_r in the barrier problem have to be chosen. So for a point x located in a known cell the distance between x and an existing facility a_m decomposes into a continuous part describing the distance between x and a d-visible extreme point i_r plus the distance between this extreme point i_r and the existing facility a_m. The latter one equals the fixed distance α_{rm}.

Hence, the Weber problem with mixed distances (2.2) includes as special cases, for example, Weber problems with polyhedral barriers (see Klamroth [12]) and Weber problems with embedded networks (see Carrizosa and Rodriguez-Chia [5]). Therefore, it can be viewed as a *unified model* that combines features from continuous and network location models.

The focus of this paper is on the case that continuous distances are measured by a block norm $\|\cdot\|_S$ whose unit ball S is a polytope with extreme points $v_g = (v_{g1}, v_{g2})^T \in \mathbb{R}^2,\ g = 1, \ldots, s$. The class of block norms is a very general one and comprises for example the l_1-norm an the 1-∞-norm (see Figure 1), which is a hybrid version of the rectilinear norm an the Tchebycheff norm. The Manhattan norm is a very common norm to describe street distances in cities and the weighted 1-∞-norm is very useful to characterize distances in actual road networks (see Ward and Wendell [18]).

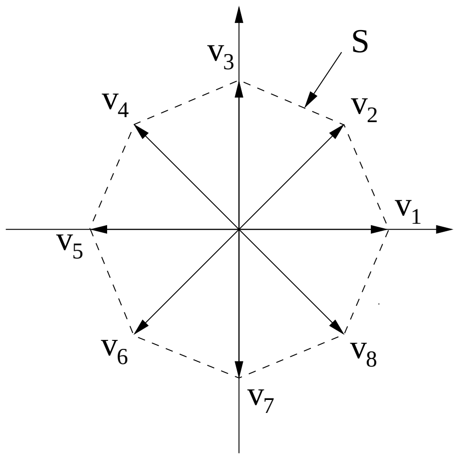

Figure 1: The 1-∞-norm with its unit ball S and its fundamental directions $v_g, g = 1, \ldots, 8$, as an example for a block norm.

Following the definition of Ward and Wendell [19], the block norm $\|\cdot\|_S$ is given by

$$\|v\|_S = \min\left\{ \sum_{g=1}^{s} \beta_g : v = \sum_{g=1}^{s} \beta_g v_g,\ \beta_g \geq 0 \right\}. \tag{2.3}$$

This characterization of block norms is one of their great advantages since this is a linear description which yields a tractable version of the objective function of the Weber problem with mixed distances (2.2). The corresponding block norm distances can be computed as

$$d(x, y) = \min\left\{ \sum_{g=1}^{s} \beta_{gr} : y - x = \sum_{g=1}^{s} \beta_{gr} v_g \right\}. \tag{2.4}$$

Using (2.4) in (2.2) leads to the following bilinear programming formulation for the problem:

$$\begin{array}{lll} \min & \sum\limits_{m=1}^{n} w_m \left(\sum\limits_{r=1}^{k} y_{rm} \sum\limits_{g=1}^{s} \beta_{gr} + \sum\limits_{r=1}^{k} y_{rm}\alpha_{rm} \right) & \\ \text{s.t.} & \sum\limits_{r=1}^{k} y_{rm} = 1, & m = 1,\dots,n \\ & x \in X & \\ & y_{rm} \in \{0,1\}, & r = 1,\dots,k,\ m = 1,\dots,n \\ & i_{rp} - x_p = \sum\limits_{g=1}^{s} \beta_{gr} v_{gp}, & r = 1,\dots,k,\ p = 1,2 \\ & \beta_{gr} \geq 0, & g = 1,\dots,s,\ r = 1,\dots,k. \end{array} \tag{2.5}$$

Note that problem (2.5) is a mixed integer programming problem with a bilinear objective function.

3. Properties of the Model

The main difficulty of the Weber problem with mixed distances (2.2) is the non-convexity of the objective function. To overcome this difficulty, this section is devoted to the derivation of general properties of the unified model (2.2) that facilitate the development of solution methods for both problems.

3.1. Relation to classical Weber problems

Theorem 3.1 *Any Weber problem with mixed distances (2.2) can be solved by solving a finite series of Weber problems (2.1) with a finite set of existing facilities* $\mathcal{A} \subseteq \{i_1, \dots, i_k\}$, $|\mathcal{A}| \leq k$, *and with the additional constraint that* x *is restricted to the feasible region for new location* X.

Proof. For any feasible assignment $\bar{y}$ of binary values to the variables y with $\sum_{r=1}^{m} \bar{y}_{rm} = 1$ $\forall m = 1,\dots,n$, the optimal values of x can be found by solving a Weber problem (2.1) with existing facilities at i_r, $r = 1,\dots,k$, and weights $\bar{w}_r := \sum_{m=1}^{n} \bar{y}_{rm} w_m$, and with the additional constraint $x \in X$. Since only finitely many feasible assignments for y exist, the result follows. □

Note that $\bar{w}_r$ may be zero for some values of r. Then the corresponding existing facilities i_r have no impact on the solution of the related Weber problem (2.1) and can be omitted.

Theorem 3.1 relates the Weber problem with mixed distances (2.2) to the classical Weber problem (2.1) with forbidden regions, see Hamacher and Nickel [10] and Nickel [14]. This relation will be used in the following sections to transfer properties of problem (2.1) to problem (2.2).

3.2. Convex hull properties

Particularly if general solution methods are applied to problem (2.5), a reduction of the set of optimal locations to a smaller subset of $\mathbb{R}^2$ can significantly improve the computational efficiency.

Theorem 3.2 *Let* $\operatorname{conv}(\mathcal{I}) \subseteq X$ *and let* d *be a metric induced by a block norm. Then at least one optimal solution of problem (2.5) is contained in the convex hull* $\operatorname{conv}(\mathcal{I})$ *of* $\mathcal{I}$.

Proof. According to Theorem 3.1, the solution of problem (2.5) can be reduced to the solution of a finite number of Weber problems (2.1) with the feasible set X and with existing facilities that form different subsets of the set $\mathcal{I}$. Since the desired property holds for all of these subproblems (see Juel and Love [11]; Wendell and Hurter [20]), the result follows. □

Using the fundamental directions $v_1, \ldots, v_s$ of the given block norm, it is also possible to characterize the complete set of optimal solutions of problem (2.5). The following theorem states a corresponding result for a frequently applied block norm, the l_1-distance function, where the description of the bounding set is particularly simple:

Theorem 3.3 *Let* $\mathrm{conv}(\mathcal{I}) \subseteq X$ *and let* $d = l_1$*. Then every optimal solution of problem (2.5) is contained in the* rectangular hull *of the set* $\mathcal{I}$*, i.e. in the smallest rectangle with sides parallel to the coordinate axes containing all facilities in the set* $\mathcal{I}$*.*

Proof. Analogous to Theorem 3.2. □

3.3. Integrality of the solution

Besides the non-convexity of the objective function, a further difficulty of problem (2.2) is imposed by the integrality constraints on the variables y_{rm}, $r = 1, \ldots, k$, $m = 1, \ldots, n$. We will show in this section that these integrality constraints $y_{rm} \in \{0, 1\}$ can be relaxed to $0 \leq y_{rm} \leq 1$ for all $r = 1, \ldots, k$, $m = 1, \ldots, n$.

Theorem 3.4 *If the set of optimal solutions of the continuous relaxation of (2.2) with* $0 \leq y_{rm} \leq 1$ *for all* $r = 1, \ldots, k$, $m = 1, \ldots, n$ *is nonempty, then there exists at least one optimal solution* x^*, y^* *of this continuous relaxation which satisfies* $y^*_{rm} \in \{0, 1\}$ $\forall r = 1, \ldots, k$, $m = 1, \ldots, n$.

Proof. Let x^*, y^* be an optimal solution of the continuous relaxation of (2.2). Suppose that y^* is not integer, i.e., there exists $t \in \{1, \ldots, n\}$ and $j, l \in \{1, \ldots, k\}$ such that $0 < y^*_{jt} < 1$, $0 < y^*_{lt} < 1$ and $y^*_{jt} + y^*_{lt} \leq 1$. Hence the objective value of x^*, y^* can be computed as

$$y^*_{jt} w_t \left(d(x^*, i_j) + \alpha_{jt} \right) + y^*_{lt} w_t \left(d(x^*, i_l) + \alpha_{lt} \right) + \underbrace{\sum_{\substack{m=1 \\ m \neq t}}^{n} \sum_{r=1}^{k} y^*_{rm} w_m \left(d(x^*, i_r) + \alpha_{rm} \right) + \sum_{\substack{r=1 \\ r \neq j,l}}^{k} y^*_{rt} w_t \left(d(x^*, i_r) + \alpha_{rt} \right)}_{=:C}$$

Case 1: One of the paths from x^* to a_t through the intermediate points i_j and i_l, respectively, is shorter/cheaper than the other. Wlog suppose that $d(x^*, i_j) + \alpha_{jm_t} < d(x^*, i_l) + \alpha_{lm_t}$. Inserting this inequality into the objective function leads to

$$y^*_{jt} w_t \left(d(x^*, i_j) + \alpha_{jt} \right) + y^*_{lt} w_t \left(d(x^*, i_l) + \alpha_{lt} \right) + C \;>\; (y^*_{jt} + y^*_{lt}) w_t \left(d(x^*, i_j) + \alpha_{jt} \right) + C.$$

Since the solution $\bar{x}, \bar{y}$ with $\bar{x} := x^*$, $\bar{y}_{rm} := y^*_{rm}$ $\forall (r, m) \notin \{(j, t), (l, t)\}$, $\bar{y}_{jt} := y^*_{jt} + y^*_{lt}$ and $\bar{y}_{lt} := 0$ is feasible for the continuous relaxation of (2.2), this contradicts the optimality of x^*, y^*.

Case 2: Both paths from x^* to a_t through the intermediate points i_j and i_t, respectively, have the same length, i.e., $d(x^*, i_j) + \alpha_{jt} = d(x^*, i_l) + \alpha_{lt}$. Define a new solution $\bar{x}, \bar{y}$ of the continuous relaxation of (2.2) as $\bar{x} := x^*$, $\bar{y}_{rm} := y^*_{rm}$ $\forall (r, m) \notin \{(j, t), (l, t)\}$, $\bar{y}_{jt} := y^*_{jt} + y^*_{lt}$ and $\bar{y}_{lt} := 0$. The objective value of $\bar{x}, \bar{y}$ is the same as of x^*, y^*, and $\bar{y}$ has at least one additional integer component. After finitely many iterations either case 1 or an integer optimal solution is obtained. □

According to Theorem 3.4, the binary constraints on y can be omitted such that (2.2) can be transformed into a bilinear programming problem. Problem (2.2) can therefore be solved by applying general methods for bilinear programming problems.

3.4. Reformulation linearization technique

As shown in Sherali and Adams [17], mixed integer bilinear programming problems can be transformed into mixed integer linear programming problems using a so-called *reformulation linearization technique* (RLT). In this section the application of the RLT to problem (2.5) will be discussed, implying an exact solution method for (2.5).

A central assumption for the applicability of the RLT is that the continuous variables are constrained to a feasible set which is given by a bounded polyhedron. According to the convex hull results in Section 3.2, this assumption does not significantly restrict the generality of (2.5). We assumed that the feasible set for the new location X is given by a bounded polyhedron. This implies that the coefficients β_{gr} can be bounded as well by a sufficiently large constant $M \in \mathbb{R}$ since they are used to represent finite distances.

In a first step, problem (2.5) has to be rewritten in standard form such that continuous and binary variables are easily distinguished. For this purpose, suppose that the constraints $x \in X$ are given by a finite set of linear inequalities, and that, wlog, $X \subseteq \mathbb{R}^2_+$. Transformation into standard form requires the introduction of a finite number l of slack variables. We define a vector u composed of all continuous variables x and β_{gr} as

$$u := (x_1, x_2, x_3, \dots, x_{2+l}, \beta_{11}, \dots, \beta_{s1}, \beta_{12}, \dots, \beta_{s2}, \dots, \beta_{1k}, \dots, \beta_{sk})^T \in \mathbb{R}_+^{2+l+s\cdot k},$$

and a vector y composed of the remaining binary variables

$$y = (y_1, \dots, y_{k\cdot n})^T := (y_{11}, \dots, y_{k1}, y_{12}, \dots, y_{k2}, \dots, y_{1n}, \dots, y_{kn})^T \in \{0,1\}^{k\cdot n}.$$

Replacing the original variables in (2.5) by the new variables u and y we obtain

$$\begin{array}{lll}
\min & q^T y + u^T Q y & \\
\text{s.t.} & \sum_{t=1}^{2+l} a_{et} u_t = c_e, & e = 1, \dots, E \qquad (*) \\
& u_p + \sum_{g=1}^{s} u_{2+l+(r-1)s+g} \cdot v_{gp} = i_{rp}, & r = 1, \dots, k, \ p = 1, 2 \\
& u_t \le M, & t = 2+l+1, \dots, 2+l+sk \\
& u \ge 0 & \\
& \sum_{h=1}^{k} y_{(m-1)k+h} = 1, & m = 1, \dots, n \\
& 0 \le y_j \le 1, & j = 1, \dots, k \cdot n \\
& y_j \in \{0,1\}, & j = 1, \dots, k \cdot n,
\end{array} \tag{3.1}$$

where Q is a $(2 + l + s \cdot k) \times (k \cdot n)$ - matrix containing the weights w_m in the appropriate positions, and q is a vector of length $(k \cdot n)$ given by

$$q := (w_1\alpha_{11}, w_1\alpha_{21}, \dots, w_1\alpha_{k1}, w_2\alpha_{12}, \dots, w_2\alpha_{k2}, \dots, w_n\alpha_{1n}, \dots, w_n\alpha_{kn})^T.$$

The constraints $(*)$ are the transformed constraints $x \in X$. Note that (3.1) is indeed a mixed-integer bilinear programming problem, i.e., for fixed y it reduces to a linear programming problem in u, and for fixed u it reduces to a binary linear programming problem in y.

Using the RLT, problem (3.1) can be transformed into an equivalent mixed-integer linear programming problem. Therefore slack variables are appended to the continuous vector u yielding an extended vector $u \in \mathbb{R}_+^{2+l+2sk}$. The transformation requires the definition of additional continuous variables $z_{tj} := u_t \cdot y_j$, $t = 1, \ldots, 2 + l + 2 \cdot s \cdot k$, $j = 1, \ldots, k \cdot n$, and corresponding constraints; see Sherali and Adams [17] for the details of the transformation. The resulting equivalent mixed-integer linear programming problem can be written as

$$
\begin{array}{lll}
\min & \sum\limits_{j=1}^{kn} q_j y_j + \sum\limits_{t=1}^{2+l+sk} \sum\limits_{j=1}^{kn} Q_{tj} z_{tj} & \\
\text{s.t.} & \sum\limits_{t=1}^{2+l} a_{et} z_{tj} = c_e y_j, & e = 1, \ldots, E,\ j = 1, \ldots, kn \\
& z_{pj} + \sum\limits_{g=1}^{s} z_{2+l+(r-1)s+g,j} v_{gp} = i_{rp} y_j, & r = 1, \ldots, k,\ j = 1, \ldots, kn,\ p = 1, 2 \\
& z_{tj} + z_{t+sk,j} = M y_j, & t = 2+l+1, \ldots, 2+l+sk,\ j = 1, \ldots, kn \\
& \sum\limits_{h=1}^{k} z_{t,(m-1)k+h} = u_t, & m = 1, \ldots, n,\ t = 1, \ldots, 2+l+2sk \quad \text{(i)} \\
& z_{tj} \leq u_t, & t = 1, \ldots, 2+l+2sk,\ j = 1, \ldots, kn \quad \text{(ii)} \\
& \sum\limits_{t=1}^{2+l} a_{et} u_t = c_e, & e = 1, \ldots, E \\
& u_p + \sum\limits_{g=1}^{s} u_{2+l+(r-1)s+g} v_{gp} = i_{rp}, & r = 1, \ldots, k,\ p = 1, 2 \\
& u_t + u_{t+sk} = M, & t = 2+l+1, \ldots, 2+l+sk \\
& \sum\limits_{h=1}^{k} y_{(m-1)k+h} = 1, & m = 1, \ldots, n \\
& 0 \leq y_j \leq 1, & j = 1, \ldots, kn \\
& y_j \in \{0, 1\}, & j = 1, \ldots, kn \\
& u_t \geq 0, & t = 1, \ldots, 2+l+2sk \\
& z_{tj} \geq 0, & t = 1, \ldots, 2+l+2sk,\ j = 1, \ldots, kn. \quad \text{(iii)}
\end{array}
\tag{3.2}
$$

Problems (3.1) and (3.2) are equivalent in the following sense: Given any feasible solution (u, y) of the bilinear problem (3.1), there exists z such that (u, y, z) is a feasible solution of (3.2) with the same objective value. Conversely, given any feasible solution (u, y, z) of (3.2), the solution (u, y) is a feasible solution of (3.1) (see Adams and Sherali [1]).

Since the binary constraints on $y_j, j = 1, \ldots, kn$, play a central role in the proof of the equivalence of (3.1) and (3.2), they cannot be omitted in (3.2) even though this was the case in (3.1), see Theorem 3.4. An example problem where the optimal solution of the linear programming relaxation of (3.2) has non-integral components is given in Section 3.6. However, constraints (ii) can be omitted without changing the solution set of the problem:

Lemma 3.5 *Constraints (i) and (iii) of (3.2) imply constraints (ii) which can be omitted.*

Proof: Let (u, y, z) satisfy (i) and (iii). Wlog let $m = 1$ and $t = 1$, and suppose that there exists an index $l \in \{1, \ldots, k\}$ such that $z_{1,l} > u_1$. Then

$$
\sum_{h=1}^{k} z_{1,h} = \sum_{\substack{h=1 \\ h \neq l}}^{k} z_{1,h} + z_{1,l} > \sum_{\substack{h=1 \\ h \neq l}}^{k} z_{1,h} + u_1 \overset{\text{(iii)}}{\geq} u_1,
$$

which is a contradiction to (i). Hence, the assumption is false. $\square$

According to Lemma 3.5 we can relax constraint (ii) of (3.2). The resulting problem is a mixed-integer linear programming problem with $O(kn)$ binary variables, $O(k^2 ns)$ continuous variables and $O(k^2 ns)$ linear constraints.

3.5. Discretization results

A well-known property of the unconstrained Weber problem (2.1) with block norms is that the fundamental directions of $\|\cdot\|_S$ rooted at the existing facilities $a_m \in \mathcal{A}$, the so-called *construction lines*, define a grid tessalation of the plane such that the set of optimal locations is a cell, a line connecting two adjacent grid points of a cell or a single grid point (see Durier and Michelot [7]). If none of these optimal locations is feasible for the constrained Weber problem with convex forbidden regions and block norms, then Nickel [14] showed that it is sufficient to consider only the intersection points of construction lines and the boundary ∂X of the feasible set X. This result is based on the fact that the objective function is convex and linear in each cell. However, if the grid is extended by construction lines rooted at all points in $\mathcal{I}$, a slightly weaker result can also be proven for the Weber problem with mixed distances (2.5):

Theorem 3.6 *Let d be a metric induced by a block norm, and let the feasible set for new location X be convex. Then there exists at least one optimal solution of (2.5) that is an intersection point of the construction line grid obtained from rooting all fundamental directions at all points in $\mathcal{I}$, or an intersection point of the construction line grid with the boundary ∂X of the feasible set X.*

Proof. According to Theorem 3.1, any problem of type (2.5) can be reduced to a finite number of Weber problems (2.1) with the feasible set X. Since Nickel [14] showed that for all of these subproblems at least one optimal solution with the desired property exists, the result follows. □

Theorem 3.6 immediately implies that problem (2.5) can be solved by enumerating the finite candidate set induced by the construction line grid and evaluating the respective objective function values. If $X = \mathbb{R}^2$, then the size of the candidate set can be bounded by $O(k^2 s^2)$.

3.6. Example

We consider a location problem in Japan where five cities are considered as existing facilities, and their approximate number of inhabitants define the respective demands, see Table 1. One new location according to the Weber objective has to be found. We assume that travel is possible by car and by the Shinkansen, a bullet train, which provides an alternative mode of transportation. The resulting problem can be viewed as a Weber problem with embedded networks and can therefore be represented by model (2.2).

The Shinkansen network of the example problem shown in Figure 2 consists of the lines and the main stations (marked by $\bullet$) of the Shinkansen train with the actual transportation time as edge lengths (solid lines). Transportation times outside the network are measured by moving along the fundamental directions of the l_1-metric with a speed of 80 km/h. The resulting mixed distances measure the shortest travel time according to an optimized combination of network and continuous travel.

To model the problem using formulation (2.5), an extended network is defined by adding the five existing facilities (marked by $\circ$) and appropriate edges (dotted lines) representing shortest travel times. The optimal solution x_{opt} is near the existing facility at Kofu and was obtained by solving problem (3.2) with XPress-MP 1.2.4., see Figure 2. The dashed lines represent the optimal paths connecting x_{opt} to the extended network.

A second, modified example problem with seven existing facilities is shown in Figure 3. This second example illustrates that despite the fact that the binary constraints in the unified model (2.5) can be relaxed without changing the optimal solution value (see Theorem 3.4), this is no longer true for the linearized problem (3.2): Suppose that (3.2) is relaxed by

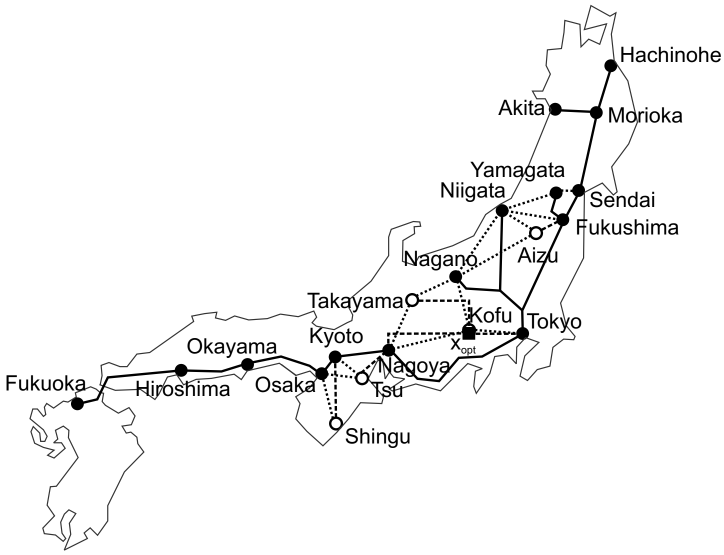

Figure 2: Example for a Weber problem with mixed distances in Japan.

replacing the binary constraints $y_j \in \{0, 1\}$, $j = 1, \ldots, kn$, by the continuous constraints $0 \leq y_j \leq 1$, $j = 1, \ldots, kn$. The relaxed problem is a linear programming problem which is again solved with XPress-MP 1.2.4. Its solution is not integer in the y-variables, and its objective value is smaller which implies that there cannot exist a feasible optimal solution of the relaxed problem with the same objective value, but with binary values for the y-variables. The objective value and the assignments to the access nodes are displayed in Table 1 both for the binary problem (3.2) and for its continuous relaxation.

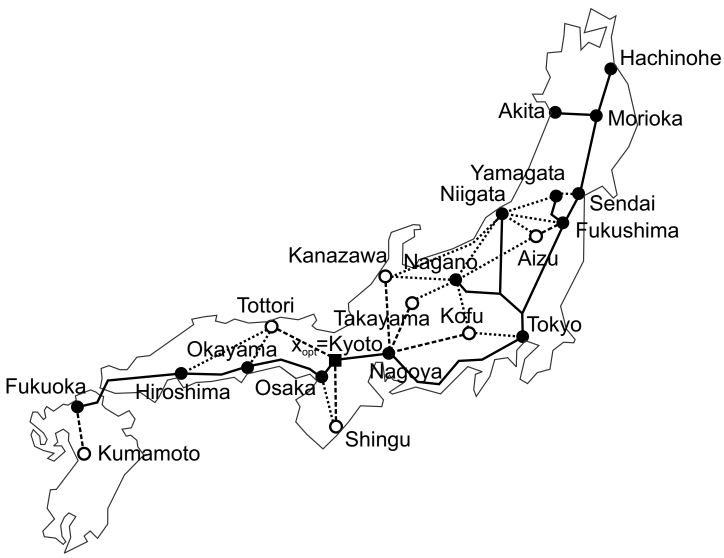

Figure 3: Modification of the example introduced in Figure 2.

4. Conclusions and Future Research

In this paper we have developed a unified model for Weber problems with distance measures that combine continuous block norm distances and network distances in a very general way. Two special generalized Weber problems arising from completely different questions, the

Table 1: Solution of (3.2) and of its continuous relaxation for the example problem introduced in Figure 3. Only those values of y are displayed that are different from 0. The optimal objective values are 3374.4 for the binary problem and 2821 for the relaxed problem. Note that in case of the continuous relaxation the solution does not represent feasible paths.

		y binary		y continuous	
City	Number of inhabitants in 10^6	Access node	y	Access node	y
Kumamoto	6.5	Kyoto	1	Tokyo Hachinohe Kumamoto	0.116535 0.292935 0.59053
Tottori	1.5	Kyoto	1	Tokyo Tottori	0.0779037 0.922096
Shingu	0.3	Kyoto	1	Hiroshima Shingu	0.301218 0.698782
Kanazawa	4.4	Kyoto	1	Fukuoka Osaka Kanazawa	0.275025 0.150257 0.574717
Takayama	0.7	Kyoto	1	Fukuoka Takayama	0.361572 0.638428
Kofu	1.9	Kyoto	1	Fukuoka Kofu	0.454335 0.545665
Aizu	1.2	Kyoto	1	Fukuoka Aizu	0.533994 0.466006
Tsu	1.6	Used only for the example shown in Figure 2.			

Weber problem with polyhedral barriers assigned to the class of continuous problems and the Weber problem with embedded networks related to network location problems, are covered by this formulation. Since many practical location problems comprise continuous aspects and network features, this problem formulation opens up new possibilities for more realistic and concise model development. We derive theoretical properties of the unified model and suggest algorithmic approaches.

A challenge of this model consists in the size of the resulting MIPs. The solution methods presented in this paper are exact solution methods and therefore applicable only to relatively small problem instances. Future research should focus on heuristic approaches as, for example, iterative location-allocation heuristics (see Fleischmann [9]), decomposition methods (see Plastria [16]) or evolutionary algorithms (see Bischoff [3]).

Different transformations of the ideas presented in this paper to objective functions other than the Weber objective suggest themselves. One example are multi-facility location problems as discussed in the case of Weber problems with embedded networks in Carrizosa and Rodriguez-Chia [5]. Other examples include the center objective as well as ordered Weber functions and multi-criteria models.

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References

[1] W.P. Adams and H.D. Sherali: Mixed-integer bilinear programming problems. *Mathematical Programming*, **59** (1993), 279–305.

[2] R. Batta and U.S. Palekar: Mixed planar/network facility location problems. *Computers and Operations Research*, **15** (1988), 61–67.

[3] M. Bischoff: *Anwendung Genetischer Algorithmen zur effizienten Lösung von Standortproblemen mit Barrieren.* Diplomarbeit, Institute of Applied Mathematics, University Erlangen-Nuremberg, Germany (2004).

[4] R. Blanquero, E. Carrizosa and R. Infante: Locating a facility outside the transportation network. Localization results. *Studies in Locational Analysis*, **14** (2000), 1–21.

[5] E. Carrizosa and A.M. Rodriguez-Chia: Weber problems with alternative transportation systems. *EJOR*, **97**, (1997) 87–93.

[6] Z. Drezner and H.W. Hamacher (editors): *Facility location: applications and theory.* (Springer-Verlag, New York, 2002).

[7] R. Durier and C. Michelot: Geometrical properties of the Fermat-Weber Problem. *Journal of Mathematical Analysis and Applications*, **117** (1985), 506–528.

[8] E. Erkut: A mixed-norm location problem for an undesirable facility. Technical report No. 92-1, University of Alberta (1992).

[9] T. Fleischmann: *The uncapacitated multi facility location-allocation problem with barriers.* Diplomarbeit, Institute of Applied Mathematics, University Erlangen-Nuremberg, Germany (2004).

[10] H.W. Hamacher and S. Nickel: Restricted planar location problems and applications. *Naval Research Logistics*, **42** (1995), 967–992.

[11] H. Juel and R.F. Love: Hull properties in location problems. *European Journal of Operational Research*, **12** (1983), 262–265.

[12] K. Klamroth: A reduction result for location problems with polyhedral barriers. *European Journal of Operational Research*, **130** (2001), 486–497.

[13] K. Klamroth: *Single-Facility Location Problems with Barriers.* Springer Series in Operations Research, (Springer-Verlag, Berlin, 2002).

[14] S. Nickel: *Discretization of Planar Location Problems.* (Shaker Verlag, Aachen, 1995).

[15] B. Pfeiffer and K. Klamroth: A unified model for Weber problems with continuous and network distance measures. Technical report No. 301, Applied Mathematics, University of Erlangen-Nuremberg, Germany (2004), submitted for publication.

[16] F. Plastria: GBSSS, the Generalized Big Square Small Square method for planar single facility location. *European Journal of Operational Research*, **62** (1992), 163–174.

[17] H.D. Sherali and W.P. Adams: Reformulation-linearization techniques for discrete optimizaton problems. In D.-Z. Du and P.M. Pardalos (editors):*Handbook of Combinatorial Optimization*, pages 1–54. (Kluwer Academic Publishers, Dordrecht, 1998).

[18] J.E. Ward and R.E. Wendell: A new norm for measuring distance which yields linear location problems. *Operations Research*, **28** (1980), 836–844.

[19] J.E. Ward and R.E. Wendell: Using block norms for location modeling. *Operations Research*, **33** (1985), 1074–1090.

[20] R.E. Wendell and A.P. Hurter: Location theory, dominance and convexity. *Operations Research*, **21** (1973), 314–320.

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