A Note on the Equivalence of Balance Points and Pareto Solutions in Multiple Objective Programming

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Abstract

It is shown that the concept of balance points introduced by Galperin [Gal92] is equivalent to the concept of pareto optimality.

We consider the general multiple objective programming problem (MOP)

$$\min f(x)$$
$$\text{s.t. } x \in X$$

where $X \subseteq I\!R^n$ and $f = (f_1, \ldots, f_m) : I\!R^n \to I\!R^m$. In [Gal92] the concept of balance points and balance sets was introduced as a new approach to nonscalarized solution of (MOP). In this note we show that the approach is equivalent to the approach of pareto optimality.
Let us denote the global minima of functions $f_i, i = 1, \ldots, m$ over X by c_i^0, respectively, and let $c^0 := (c_1^0, \ldots, c_m^0)$.

Definition 1 *Given $\eta = (\eta_1, \ldots, \eta_m) \in I\!R^m$ with $\eta_i \geq 0$, let $X_{\eta_i}^0 = \{x \in X | f_i(x) - c_i^0 \leq \eta_i\}$. η is a* balance point *if $X_\eta^0 := \cap_{i=1}^m X_{\eta_i}^0 \neq \emptyset$ and for every $\eta' \in I\!R^m$ such that $0 \leq \eta_i' \leq \eta_i, i = 1, \ldots, m$ and $\eta_j' < \eta_j$ for at least one $j \in \{1, \ldots, m\}$ $X_{\eta'}^0 = \emptyset$. The set of all balance points is called the* balance set, *denoted by Υ.*

Definition 2 *$x \in X$ is a* pareto solution *of (MOP) if there does not exist $x' \in X$ such that $f_i(x') \leq f_i(x), i = 1, \ldots, m$ with strict inequality for at least one i. $y = f(x)$ is then called* efficient point. *The set of pareto solutions is denoted by X_{par}, the set of efficient points by Y_{eff}.*

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To determine the relation between balance points and efficient points we will follow [HN93] and introduce level sets $L^i_{\leq}(y_i) := \{x \in X | f_i(x) \leq y_i\}$ where $y = (y_1, \dots, y_m) \in \mathbb{R}^m$, and level curves $L^i_{=}(y_i) := \{x \in X | f_i(x) = y_i\}$. The following Theorem 1, which was first shown for location problems in [HN93], shows that pareto solutions can be completely characterized using level sets and level curves.

Theorem 1 *Let $x \in X, y_i = f_i(x), i = 1, \dots, m$. Then x is a pareto solution for (MOP) iff*

$$\cap_{i=1}^m L^i_{\leq}(y_i) = \cap_{i=1}^m L^i_{=}(y_i)$$

Proof:

x is a pareto solution

iff $\nexists x' \in X$ s.t. $\quad (f_i(x') \leq f_i(x)\ i = 1, \dots, m \wedge \exists j \in \{1, \dots, m\}:\ f_j(x') < f_j(x))$

iff $\nexists x' \in X$ s.t. $\quad \left(x' \in \cap_{i=1}^m L^i_{\leq}(f_i(x)) \wedge \exists j \in \{1, \dots, m\}:\ x' \in L^j_{\leq}(f_j(x)) \setminus L^j_{=}(f_j(x))\right)$

iff $\cap_{i=1}^m L^i_{\leq}(y_i) = \cap_{i=1}^m L^i_{=}(y_i)$.

$\square$

The relation to balance points follows from the remark in [Gal92].

Remark 1 *If η is a balance point and $x \in X^0_\eta$ then $f_i(x) - c^0_i = \eta_i$.*

Now let η be a balance point. It follows that

$$\begin{aligned} X^0_\eta &= \cap_{i=1}^m \{x \in X | f_i(x) - c^0_i \leq \eta_i\} \\ &= \cap_{i=1}^m \{x \in X | f_i(x) \leq c^0_i + \eta_i\} \\ &= \cap_{i=1}^m \{x \in X | f_i(x) = c^0_i + \eta_i\} \end{aligned}$$

The last equation follows from Remark 1. But the last two set-intersections are nothing but $\cap_{i=1}^m L^i_{\leq}(c^0_i + \eta_i)$ and $\cap_{i=1}^m L^i_{=}(c^0_i + \eta_i)$, respectively, which hence coincide if η is a balance point. Theorem 1 then implies that each $x \in X^0_\eta$ is a pareto solution of (MOP).

On the other hand if x is a pareto solution we define η by $\eta_i := f_i(x) - c^0_i$. By Theorem 1

$$\begin{aligned} \cap_{i=1}^m L^i_{=}(\eta_i + c^0_i) &= \cap_{i=1}^m L^i_{\leq}(\eta_i + c^0_i) && (1) \\ &= \cap_{i=1}^m \{x \in X | f_i(x) - c^0_i \leq \eta_i\} && (2) \\ &= X^0_\eta && (3) \end{aligned}$$

Hence X^0_η is nonempty and (1) implies that decreasing one component of η would result in $X^0_\eta = \emptyset$. Thus η is a balance point and we have proved

Theorem 2

$$X_{par} = \cup_{\eta \in \Upsilon} X^0_\eta$$

Theorem 2 confirms our claim that the concepts of balance set and pareto optimality are equivalent. We can also reformulate Theorem 2 in the objective space:

Corollary 1

$$Y_{eff} = \Upsilon + c^0 := \{\eta + c^0 | \eta \in \Upsilon\}$$

This can either be seen by repeating the argumentation of the proof of Theorem 2 for objective function values or by directly applying Theorem 2 and Remark 1 to $Y_{eff} = f(X_{par})$.

References

[Gal92] E.A. Galperin. Nonscalarized multiobjective global optimization. *Journal of Optimization Theory and Applications*, 75(1):69–85, 1992.

[HN93] H.W. Hamacher and S. Nickel. Multicriteria planar location problems. Technical Report 243, Universität Kaiserslautern, Department of Mathematics, 1993. accepted in European Journal of Operational Research.