Unbiased Approximation in Multicriteria Optimization

Kathrin Klamroth[1] Jørgen Tind[1] Margaret M. Wiecek[2]

Abstract

Algorithms generating piecewise linear approximations of the nondominated set for general, convex and nonconvex, multicriteria programs are developed. Polyhedral distance functions are used to construct the approximation and evaluate its quality. The functions automatically adapt to the problem structure and scaling which makes the approximation process unbiased and self-driven. Decision makers preferences, if available, can be easily incorporated but are not required by the procedure.

Keywords: multicriteria programs, nondominated set, approximation, distance functions

1 Introduction

Multicriteria optimization problems have countless applications, for example, in engineering design, capital budgeting and location and layout planning. To support the decision making process, approximations of the nondominated set are an attractive tool since they visualize the alternatives for the decision maker and provide valuable trade-off information in a simple and understandable fashion.

[1]Department of Statistics and Operations Research, Insitute for Mathematical Sciences, University of Copenhagen, Denmark.

[2]Department of Mathematical Sciences, Clemson University, Clemson, SC, USA.

In this paper, algorithms generating piecewise linear approximations of the nondominated set for general, convex and nonconvex, multicriteria programs are developed. Although this type of approximation of the nondominated set has already been introduced in the literature, the proposed algorithms have some distinct properties that make them different from other approaches. To justify this statement, we first review the literature and then discuss the properties of the new algorithms.

The approximation of the nondominated set has been of interest to researchers at least since the nineteen seventies. Most approaches and algorithms discussed in the literature focus on bicriteria problems (see, for example, Schandl *et al.*, 2001b, for an overview) or multiple criteria convex problems while comparatively fewer methods are available for general, possibly nonconvex and/or discrete, multicriteria problems on which this review is focused.

The methods can be classified with respect to different evaluation criteria and properties. Of particular importance are the following aspects: the conceptual approach to approximation, the method used for the generation of nondominated points, the measure used to evaluate the approximation quality, and the interface to the decision making stage of MCDM.

Convolution-based approximation methods were proposed as early as 1982 by Popov (1982) and then researched by Belov and Shafranskii (1991) and Smirnov (1996). The methods involve constructing a parametric family of problems that produce the approximation of the related multicriteria problem. Nefedov (1984) studies the approximation by a finite set of elements using a convolution method and a method of regular points.

In Helbig (1991), a direction method is used to compute a discrete approximation of the nondominated set for multicriteria optimization problems with convex cones.

Lemaire (1992) presents results on the approximation of the efficient set of a limit multicriteria problem by the efficient sets of a sequence of related multicriteria problems.

An approximation method based on the Tchebycheff approach is developed in Kaliszewski (1994). Using a modified weighted Tchebycheff norm, several nondominated points are generated and combined into an approximation by intersecting cones that correspond to the employed norms and that are pointed at the generated nondominated points.

Kostreva *et al.* (1995) minimize the weighted Tchebycheff distance to the utopia point to find points in the nondominated set that are later used to construct approximating simplices. The method is applicable to problems with discontinuous criteria and/or disconnected feasible set. Statnikov and Matusov (1996) and Sobol' and Levitan (1997) develop approximation methods that are based on the parameter space investigation producing a discrete representation of the nondominated set.

Benson and Sayin (1997) propose a global shooting procedure to find a global representation of the nondominated set for problems with compact sets of feasible criterion vectors. Das (1999) briefly discusses an approach based on the normal-boundary intersection technique. Using the hyperplane defined by the individual minimizers of the criteria, the nondominated points with maximal distance from this hyperplane in some specified directions are determined and included in a piecewise linear approximation.

A very different approach is offered by Galperin and Wiecek (1999) who propose the balance set as an approximation tool and demonstrate its derivation for problems with no more than three criteria.

Recently, approximation algorithms based on metaheuristics, and, in particular, evolutionary algorithms, have had some success in the effective generation of well-diversified approximations of the nondominated set for combinatorial multicriteria problems. For a survey of the application of evolutionary algorithms we refer to Deb (2001) and Zitzler *et al.* (2001). Simulated annealing based algorithms are studied, among others, by Czyzak and Jaszkiewicz (1998) and Ulungu *et al.* (1999), and Gandibleux *et al.* (1997) propose a metaheuristic based on tabu search.

There are two general conceptual approaches in the methods reviewed above: the use of a family (series) of auxiliary problems whose solution sets approximate the nondominated set, and the generation of points in the nondominated set that become a final discrete approximation or are fitted into an approximating set (e.g., simplex, polyhedral set).

The approximation algorithms proposed in this paper follow upon an earlier research effort initiated by Schandl (1999) and continued by Schandl *et al.* (2001c). The approximation comes in the form of a polyhedral distance measure that is being constructed successively during the execution of the algorithm. The measure is being utilized both to evaluate the quality of the approximation and to generate additional nondominated solutions. The authors are not aware of another approximation technique with all these properties.

For convex problems, the approximating measure is defined as a polyhedral gauge. Although the concept of a gauge cannot be carried over to the nonconvex case due to lack of convexity, it serves as an inspiration to define a nonconvex distance function in that case. In effect, the use of these distance measures guarantees that given an initial approximation in the form of a distance function, the algorithms automatically construct successive approximations (functions) emulating the shape of the nondominated set. In each step, the approximation is independent of scalings of the objective functions.

The algorithms require that the decision maker provide an initial approximation (a starting "point"), and two termination parameters to be used jointly or separately. The parameters are related to the desired accuracy of the approximation and the maximum number of steps to be performed. Upon the initialization, the algorithms are performed without any interaction with the decision maker. As the resulting approximation is induced by the problem and adapted to its structure, the approximation itself entirely controls the algorithmic process. Therefore the process is unbiased and once started, it is naturally self-driven.

In the next section the multicriteria program is stated and some general definitions and notations are given. Section 3 discusses inner as well as outer approximation algorithms for problems with an $\mathbb{R}^n_{\leqq}$ - convex set of feasible criterion vectors along with some convergence results for the bicriteria case.

Algorithms for inner and outer approximation in the nonconvex case are presented in Section 4. The paper is concluded with a short summary in Section 5.

2 Problem Formulation

To facilitate further discussions, the following notation is used throughout the paper.

Let $u, w \in \mathbb{R}^n$ be two vectors. We denote components of vectors by subscripts and enumerate vectors by superscripts. $u > w$ denotes $u_i > w_i$ for all $i = 1, \ldots, n$. $u \geq w$ denotes $u_i \geq w_i$ for all $i = 1, \ldots, n$, but $u \neq w$. $u \geqq w$ allows equality. The symbols $<, \leq, \leqq$ are used accordingly. Let $\mathbb{R}^n_{\leqq} := \{x \in \mathbb{R}^n : x \leqq 0\}$. The set $\mathbb{R}^n_{\geqq}$ is defined accordingly and the set $u + \mathbb{R}^n_{\geqq}$, where $u \in \mathbb{R}^n$, is referred to as a dominating cone.

We consider the following general multicriteria program

$$\begin{array}{ll} \max & \{z_1 = f_1(x)\} \\ & \vdots \\ \max & \{z_n = f_n(x)\} \\ \text{s. t.} & x \in X, \end{array} \tag{1}$$

where $X \subseteq \mathbb{R}^m$ is the *feasible set* and $f_i(x), i = 1, \ldots, n$, are real-valued functions. We define the *set of all feasible criterion vectors* Z, the *set of all (globally) nondominated criterion vectors* N and the *set of all efficient points* E of (1) as follows

$$\begin{aligned} Z &= \{z \in \mathbb{R}^n : z = f(x), x \in X\} = f(X) \\ N &= \{z \in Z : \nexists \tilde{z} \in Z \text{ s. t. } \tilde{z} \geq z\} \\ E &= \{x \in X : f(x) \in N\}, \end{aligned}$$

where $f(x) = \big(f_1(x), \ldots, f_n(x)\big)^T$. We assume that the set Z is $\mathbb{R}^n_{\leqq}$-closed, i.e., the set $Z + \mathbb{R}^n_{\leqq}$ is closed.

The set of properly nondominated solutions is defined according to Geoffrion (1968): A point $\bar{z} \in N$ is called *properly nondominated*, if there exists $M > 0$ such that for each $i = 1, \ldots, n$ and each $z \in Z$ satisfying $z_i > \bar{z}_i$ there exists a $j \neq i$ with $z_j < \bar{z}_j$ and

$$\frac{\bar{z}_i - z_i}{z_j - \bar{z}_j} \leq M.$$

Otherwise $\bar{z} \in N$ is called *improperly nondominated.* The set of all properly nondominated points is denoted by N_p.

Moreover, a point $\bar{z} \in Z$ is called *weakly nondominated* if there does not exist $z \in Z$ with $z < \bar{z}$, and the set of all weakly nondominated points is denoted by N_w.

The point $z^* \in \mathbb{R}^n$ with

$$z_i^* = \max\{f_i(x) : x \in X\} + \epsilon_i \qquad i = 1, \ldots, n$$

is called the *ideal (utopia) criterion vector*, where the components of $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in \mathbb{R}^n$ are small positive numbers. We assume that we can find $u \in \mathbb{R}^n$ such that $u + Z \subseteq \mathbb{R}^n_{\leqq}$ and thus an ideal criterion vector exists. Without loss of generality let $z^* = 0$. For bicriteria problems, the point $z^\times \in \mathbb{R}^2$ with

$$z_i^\times = \max\Big\{f_i(\bar{x}) : f_j(\bar{x}) = \max_{x \in X} f_j(x), j \neq i\Big\} \qquad i = 1, 2$$

is called the *nadir point.* Note that this definition cannot be directly generalized to multicriteria problems.

We define polyhedral gauges according to Minkowski (1911):

Definition 2.1 Let B be a polytope in $\mathbb{R}^n$ containing the origin in its interior and let $z \in \mathbb{R}^n$.

(1) The *polyhedral gauge* $\gamma : \mathbb{R}^n \to \mathbb{R}$ of z is defined as

$$\gamma(z) := \min\{\lambda \geq 0 \,:\, z \in \lambda B\}.$$

(2) If B is symmetric with respect to the origin, then γ is called a *block norm*.

(3) The vectors defined by the extreme points of the unit ball B of γ are called *fundamental vectors* and are denoted by v^i. The fundamental vectors defined by the extreme points of a facet of B span a *fundamental cone*.

(4) A block norm γ with a unit ball B is called *oblique* (Schandl *et al.*, 2001a) if it has the following properties:

   (i) γ is absolute, i.e., $\gamma(w) = \gamma(u)\ \forall w \in R(u) := \{w \in \mathbb{R}^n : |w_i| = |u_i|\,\forall i = 1, \dots, n\}$,

   (ii) $(z - \mathbb{R}^n_{\geqq}) \cap \mathbb{R}^n_{\geqq} \cap \partial B = \{z\} \quad \forall z \in (\partial B \cap \mathbb{R}^n_{\geqq})$.

If z is in a fundamental cone C of a polyhedral gauge γ then one needs to consider only the fundamental vectors generating this cone to calculate the gauge of z.

Lemma 2.2 (Schandl *et al.* (2001a)) Let γ be a polyhedral gauge with the unit ball $B \subseteq \mathbb{R}^n$. Let $\bar{z} \in C$ where C is the fundamental cone generated by the fundamental vectors $v^1, \dots, v^k$, $k \geq n$. Let $\bar{z} = \sum_{i=1}^k \lambda_i v^i$ be a representation of $\bar{z}$ in terms of $v^1, \dots, v^k$. Then $\gamma(\bar{z}) = \sum_{i=1}^k \lambda_i$.

3 Approximation in the $\mathbb{R}^n_{\leqq}$ - Convex Case

Let $Z \subseteq \mathbb{R}^n$ be $\mathbb{R}^n_{\leqq}$ - convex, i.e. $Z + \mathbb{R}^n_{\leqq}$ is convex, with $\mathrm{int} Z \neq \emptyset$, and assume without loss of generality that $0 \in Z_{\leqq} := Z + \mathbb{R}^n_{\leqq}$.

3.1 Inner Approximation

For a polyhedral gauge γ, consider the problem

$$\begin{array}{ll} \max & \gamma(z) \\ \text{s.t.} & z \in \mathbb{R}^n_{\geqq} \cap Z. \end{array} \tag{2}$$

Theorem 3.1 (Schandl *et al.* (2001c)) If γ is an oblique norm then the solution of (2) is nondominated. Conversely, for any properly nondominated solution $\bar{z}$ there exists an oblique norm γ such that $\bar{z}$ solves (2).

In the following we consider the more general case that γ is an arbitrary polyhedral gauge and discuss alternative formulations of (2) as generalizations of Theorem 3.1.

Let $d^1, \dots, d^s \in \mathbb{R}^n$ be the normal vectors of the facets of the unit ball B of a polyhedral gauge γ such that $\{z \geqq 0 : d^i z \leq 1, i = 1, \dots, s\} = B \cap \mathbb{R}^n_{\geqq}$ and

$$\left\{z \geqq 0 \;:\; d^i z \leq 1,\; i = 1, \dots, s\right\} \subseteq Z.$$

Then problem (2) can be formulated as the following disjunctive programming problem:

$$\begin{array}{ll} \max & \lambda \\ \text{s.t.} & \bigvee_{i=1}^{s} \left(d^i z^i \geq \lambda \wedge z^i \in Z\right) \\ & \lambda \in \mathbb{R}. \end{array} \tag{3}$$

Figure 1 shows an example with two facets represented by the normal vectors d^1 and d^2. The point $\bar{z}$ corresponds to an optimal λ in (3).

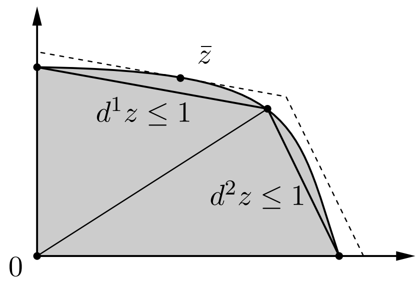


Figure 1: Inner approximation

Problem (3) can be reformulated as a linear programming problem in the case that the set Z has a linear programming representation, i.e., $Z = \{Cx : Ax \leqq b, x \geqq 0, x \in \mathbb{R}^m\}$, where C is an $n \times m$ - matrix with $Cx = f(x)$ and $X = \{x \in \mathbb{R}^m : Ax \leqq b, x \geqq 0\}$ is a bounded polyhedron. Then (3) can be written as

$$
\begin{array}{ll}
\max & \lambda \\
\text{s. t.} & \bigvee_{i=1}^{s} \left(\lambda - d^i C x^i \leq 0 \wedge A x^i \leqq b \wedge x^i \geqq 0\right) \\
& \lambda \in \mathbb{R}.
\end{array} \tag{4}
$$

As shown by Balas (1985), an equivalent linear programming representation of (4) is given by

$$
\begin{array}{lll}
\max & \sum_{i=1}^{s} \lambda_i & \\
\text{s. t.} & \lambda_i - d^i C x^i \leq 0 & \forall i = 1, \dots, s \\
& A x^i \leqq p_i b & \forall i = 1, \dots, s \\
& \sum_{i=1}^{s} p_i = 1 & \\
& p_i \geq 0,\ x^i \geqq 0,\ \lambda_i \in \mathbb{R} & \forall i = 1, \dots, s.
\end{array} \tag{5}
$$

From an algorithmic point of view we would like to decompose problem (2) (or problem (3), respectively) into subproblems whose structure is as simple as possible. For this purpose, let B be the unit ball of γ and denote by $C_1, \dots, C_s$ and $v^1, \dots, v^t$ the fundamental cones and the fundamental vectors of $B \cap \mathbb{R}^n_{\geqq}$, respectively. If we denote by I_j the index set of those fundamental vectors generating the cone C_j, $j = 1, \dots, s$, then Lemma 2.2 implies that (2) can be decomposed into s subproblems (P^j_{inner}), $j = 1, \dots, s$, of the form

$$
\begin{array}{rlll}
\delta_j = & \max & \sum\limits_{i \in I_j} \lambda_i & \\
& \text{s. t.} & \sum\limits_{i \in I_j} \lambda_i v^i \leqq z & \\
& & \lambda_i \geq 0 & \forall i \in I_j \\
& & z \in Z. &
\end{array} \tag{6}
$$

Note that each subproblem (6) has a very simple linear objective function and only linear inequality constraints in addition to the problem dependent constraint $z \in Z$.

We will show in the following that, under some assumptions of nondegeneracy, each subproblem (6) generates a nondominated solution $\bar{z}$.

Theorem 3.2 Let Z be strictly $\mathrm{int}\mathbb{R}^n_{\leqq}$ - convex, i.e. $Z + \mathrm{int}\mathbb{R}^n_{\leqq}$ is strictly convex, and let C_j be a fundamental cone of a polyhedral gauge γ. Then the optimal solution of problem (6) is properly nondominated.

Proof. Let $\bar{z}$ be an optimal solution of (6). Then there exist optimal dual multipliers $\bar{u} \geq 0$ of (6) such that $\bar{z}$ solves

$$\begin{array}{ll} \max & \sum\limits_{i\in I_j} \lambda_i - \bar{u}\left(\sum_{i\in I_j} \lambda_i v^i - z\right) \\ \text{s. t.} & z \in Z,\ \lambda_i \geq 0 \qquad \forall i \in I_j, \end{array} \tag{7}$$

(see, for example, Rockafellar, 1970). We rewrite the objective function of (7) as

$$\sum_{i\in I_j} \lambda_i - \bar{u}\left(\sum_{i\in I_j} \lambda_i v^i - z\right) \;=\; \sum_{i\in I_j} \lambda_i(1-\bar{u}v^i) + \bar{u}z.$$

Since the problem is bounded it follows that $(1-\bar{u}v^i) \leq 0$ for all $i \in I_j$ (otherwise, increasing λ_i would result in an unbounded objective value). Hence an optimal solution of (7) satisfies $\lambda_i = 0$ whenever $(1-\bar{u}v^i) \neq 0$, $i \in I_j$. Therefore $\sum_{i\in I_j} \lambda_i(1-\bar{u}v^i) = 0$ at optimality, and (7) can be replaced by

$$\begin{array}{ll} \max & \bar{u}z \\ \text{s. t.} & z \in Z \end{array} \tag{8}$$

with $\bar{u} \geq 0$. Under the assumption that Z is strictly $\mathrm{int}\mathbb{R}^n_{\leqq}$ - convex this implies that $\bar{z}$ is indeed a properly nondominated solution. $\square$

Note that if Z is not strictly $\mathrm{int}\mathbb{R}^n_{\leqq}$ - convex, problem (6) may generate weakly nondominated solutions since the optimal dual multiplier $\bar{u}$ used in the proof of Theorem 3.2 may have zero components. However, in the nondegenerate case that $\bar{u} > 0$ the result of Theorem 3.2 applies also to $\mathbb{R}^n_{\leqq}$ - convex problems.

Corollary 3.3 Under the assumptions of Theorem 3.2, the optimal solution of (2) (or (3), respectively) is properly nondominated.

Based on the above results, an inner approximation of the nondominated set can be constructed by iteratively solving a problem (2) (or (3), respectively). The generated solution (which is nondominated at least in the strictly int$\mathbb{R}^n_{\leqq}$ - convex case) is then added to the current approximation by including it into the convex hull of the unit ball of the polyhedral gauge γ, and a new iteration is performed with the updated γ.

Schandl *et al.* (2001c) showed that this procedure can be implemented very efficiently based on the representation (6) of (2): Starting with an initial approximation as shown in Figure 2(b), problem (6) is solved in the single cone determined by this approximation. In each of the consecutive iterations, the optimal solution vector $\bar{z}$ is added to the convex hull of the approximation, splitting the corresponding cone into at most $O(|I_j|^{\lfloor \frac{n-1}{2} \rfloor})$ subcones, see Edelsbrunner (1987). Note that at most 2 subcones are obtained in the bicriteria case, see Figures 2(c) - 2(f). In each iteration problem (6) has to be solved only for the newly generated cones since the optimal values $\bar{\delta}_j$ remain optimal in the unmodified cones of the approximation.

Summarizing the discussion above, Figure 3 gives an outline of the inner approximation algorithm for $\mathbb{R}^n_{\leqq}$ - convex sets (see also Schandl *et al.*, 2001c).

Two different stopping criteria are implemented in the above procedure that can be either jointly used or that can be specified separately, according to decision maker's suggestions. Here, the value of $\epsilon > 0$ specifies a bound on the required accuracy of the approximation, where the approximation error is measured as $|\gamma(\bar{z}) - 1|$ ($\bar{z}$ denotes the next point being added in the current iteration). Thus the error is measured in a problem dependent way and using the current approximation in the evaluation. Alternatively, the total number of subproblems (6) solved during the algorithm can be bounded by specifying the maximum number of cones `maxConeNo` to be generated during the algorithm. Note that then `maxConeNo` is also an upper

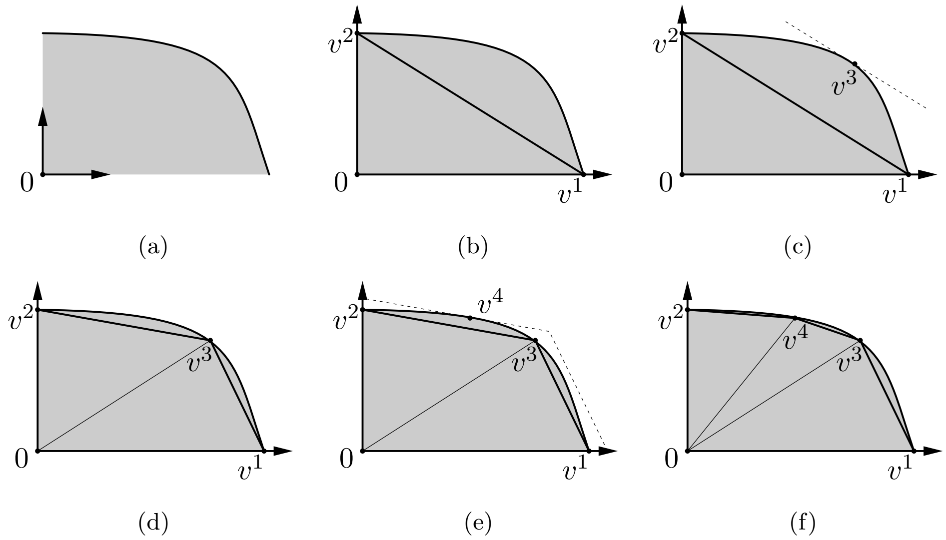


Figure 2: Inner approximation algorithm

PROCEDURE: INNER APPROXIMATION

Read/generate stopping criteria: $\epsilon > 0$, `maxConeNo`;
Read/generate an initial inner approximation represented by a polyhedral gauge γ with unit ball B;
Construct cones using the facets of B in $\mathbb{R}^n_{\geqq}$
for all cones **do**
  Solve (6) to find $\bar{z}$ and $\gamma(\bar{z})$
end for
while #cones < `maxConeNo` and $|\gamma(\text{next point}) - 1| \geq \epsilon$ **do**
  Add next point using the Beneath-Beyond technique;
  Identify new and modified cones
  for all new or modified cones **do**
    Solve (6) to find $\bar{z}$ and $\gamma(\bar{z})$
  end for
end while
Output inner approximation

Figure 3: Pseudo code of the inner approximation algorithm for an $\mathbb{R}^n_{\leqq}$ - convex problem

bound on the number of nondominated points generated and added to the approximation.

The addition of new points to the convex hull of the previous approximation is implemented using one iteration of the Beneath-Beyond Algorithm (see Edelsbrunner, 1987). This algorithm is among the most efficient convex hull algorithms (given a set S of k points in $\mathbb{R}^n$, the convex hull of S is computed in $O(k \log k + k^{\lfloor (n+1)/2 \rfloor})$ time) and particularly well-suited for an incorporation into the above procedure.

Summarizing the discussion above, the complexity of the inner approximation algorithm can be bounded by $O(k \log k + k^{\lfloor (n+1)/2 \rfloor} + \texttt{maxConeNo} \cdot T)$ where $k \leq \texttt{maxConeNo}$ denotes the total number of nondominated solutions generating the approximation and $O(T)$ is the complexity of solving (6) which particularly depends on the structure of the set Z.

3.2 Outer Approximation

Let B be the unit ball of a polyhedral gauge γ such that the fundamental vectors $v^1, \dots, v^t$ of $B \cap \mathbb{R}^n_{\geqq}$ satisfy

$$(Z \cap \mathbb{R}^n_{\geqq}) \subseteq \left\{ z \geqq 0 \ : \ z \leqq \sum_{i=1}^{t} \lambda_i v^i, \ \sum_{i=1}^{t} \lambda_i = 1, \ \lambda \geq 0 \right\}$$

and consider the problem

$$\begin{array}{lll} \max & \lambda & \\ \text{s.t.} & \lambda v^i \leqq z^i & \forall i = 1, \dots, t \\ & \lambda \geq 0 & \\ & z^i \in Z. & \end{array} \tag{9}$$

Figure 4 illustrates an example problem with three fundamental vectors v^1, v^2, v^3 where the point $\bar{z}$ represents an optimal solution of (9).

Theorem 3.4 Let Z be strictly $\mathrm{int}\mathbb{R}^n_{\geqq}$ - convex. Then the optimal solution of (9) is properly nondominated.

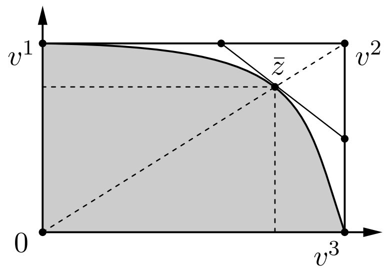


Figure 4: Outer approximation

Proof. Suppose that the j^{th} constraint of (9), $j \in \{1, \dots, t\}$, is binding at optimality, and let $\bar{u} \geq 0$ be the related optimal dual multiplier. Then (9) is equivalent to

$$\begin{array}{ll} \max & \lambda - \bar{u}\left(\lambda v^j - z\right) \\ \text{s.t.} & \lambda \geq 0,\ z \in Z. \end{array} \tag{10}$$

Rewriting the objective function of (10) we obtain

$$\begin{array}{ll} \max & \left(1 - \bar{u}v^j\right)\lambda + \bar{u}z \\ \text{s.t.} & \lambda \geq 0,\ z \in Z. \end{array} \tag{11}$$

Using the fact that (11) is bounded we can conclude that $(1 - \bar{u}v^j) \leq 0$. Since $\lambda \geq 0$ can be selected independently of $z \in Z$ in (11), it follows that $(1 - \bar{u}v^j)\lambda = 0$ at optimality and thus the hyperplane $H = \{z \in \mathbb{R}^n : \bar{u}z = \bar{u}\bar{z}\}$ supports Z at the optimal $\bar{z}$. The assumption that Z is strictly $\mathrm{int}\mathbb{R}^n_{\leqq}$ - convex together with the fact that $\bar{u} \geq 0$ implies that $\bar{z}$ is properly nondominated. $\square$

Based on Theorem 3.4 we can develop an outer approximation algorithm that can be viewed as a dual approach with regard to the inner approximation algorithm described in Section 3.1:

Starting with an initial approximation as shown in Figure 5(b) we iteratively solve a problem of type (9). While we have used the generated nondominated solution $\bar{z}$ to update γ during the inner approximation procedure, we now wish to incorporate an additional facet supporting Z at $\bar{z}$

into the boundary of the current approximation (i.e., into the unit ball of the corresponding polyhedral gauge γ). This facet is defined by the optimal dual multipliers $\bar{u}$ and the corresponding optimal solution $\bar{z}$ of (9), i.e., by the hyperplane given by

$$H = \{z \in \mathbb{R}^n \,:\, \bar{u}z = \bar{u}\bar{z}\}.$$

H is incorporated into the approximation by computing the intersection of all halfspaces defined by the hyperplanes generating the current approximation (and containing the origin in their interior) and the newly generated halfspace defined by H, see Figure 5(d). Since the intersection of a set of k halfspaces in $\mathbb{R}^n$ is dual to the convex hull of a set of k points in $\mathbb{R}^n$, generated by a suitable geometric transform (see Edelsbrunner, 1987, for details about this duality transform), the intersection of halfspaces can be computed using the Beneath-Beyond Algorithm as in the convex case, c.f. Section 3.1.

The procedure is then iterated with the new approximation, see Figures 5(e) and 5(f).

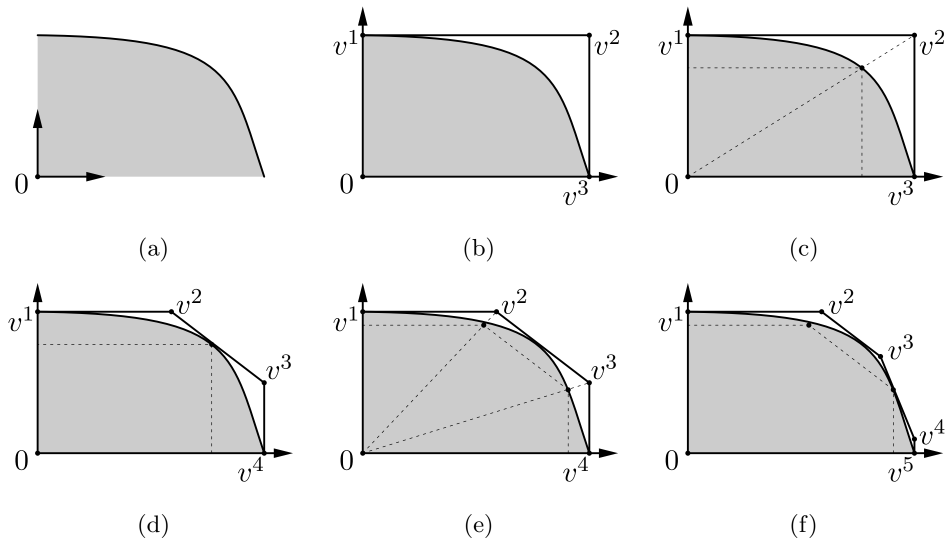


Figure 5: Outer approximation algorithm

Instead of repeatedly solving problems of the type (9) with a growing number of constraints we may again use a decomposition of the problem into subproblems (P^j_{outer}), $j = 1, \ldots, t$, of the type

$$\begin{array}{rcll} \delta_j & = & \max & \lambda \\ & & \text{s.t.} & \lambda v^j \leqq z \\ & & & \lambda \geq 0,\ z \in Z. \end{array} \tag{12}$$

Obviously the optimal solution of (9) can be obtained by minimizing δ_j over all the subproblems (P^j_{outer}), $j = 1, \ldots, t$, and (10) is dual to (12) for the optimal j. Since only a subset of the fundamental vectors v^j changes in each iteration of the procedure (note that in the bicriteria case at most two new fundamental vectors are generated in each iteration), problem (12) has to be solved only for the newly generated fundamental vectors while the values of δ_j can be stored and reused for all unchanged vectors v^j.

Figure 6 outlines the outer approximation algorithm for $\mathbb{R}^n_{\leqq}$ - convex problems.

Note that the stopping criteria used in the above algorithm correspond exactly to those used in the inner approximation algorithm discussed in Section 3.1. Moreover, the complexity of the outer approximation algorithm corresponds exactly to that of the inner approximation algorithm.

3.3 Simultaneous Inner and Outer Approximation

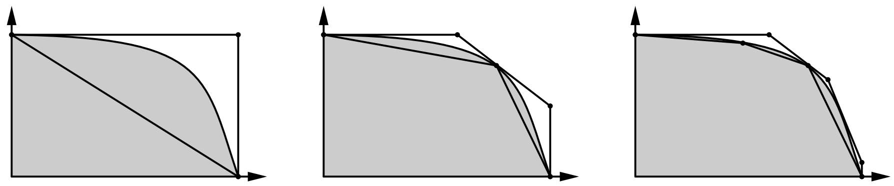
Figure 7 shows the progression of a *sandwich approximation* that in each iteration applies one step of the inner and of the outer approximation algorithm.

Since the nondominated set is enclosed by the two polyhedral unit balls, this approach not only allows a nice visualization of the achieved approximation accuracy but also generates a set of nondominated points whose overall distribution on the nondominated set is in general better than in each of the two approximations separately.

Procedure: Outer Approximation

Read/generate stopping criteria: $\epsilon > 0$, `maxVecNo`;
Read/generate an initial outer approximation represented by a polyhedral gauge γ with unit ball B;
Construct fundamental vectors using the extreme points of B in $\mathbb{R}^n_{\geqq}$
for all vectors **do**
    Solve (12) to find $\bar{z}$ and $\gamma(\bar{z})$
end for
while #vectors $<$ `maxVecNo` and $|\gamma(\text{next point})-1| \geq \epsilon$ **do**
    Add next facet using a duality transform and the Beneath-Beyond technique;
    Identify new and modified fundamental vectors
    for all new or modified vectors **do**
        Solve (12) to find $\bar{z}$ and $\gamma(\bar{z})$
    end for
end while
Output outer approximation

Figure 6: Pseudo code of the outer approximation algorithm for an $\mathbb{R}^n_{\leqq}$ - convex problem



(a) (b) (c)

Figure 7: Simultaneous application of the inner and the outer approximation algorithms

3.4 Convergence Rate of the Approximation for Bicriteria Problems

The problem of generating a piecewise linear approximation of a nondominated set is closely related to the problem of approximating a convex set by an inscribed or a circumscribed polyhedron. Since the literature on polyhedral approximations of convex sets is relatively rich (see, for example, Gruber, 1992, for an overview), in this section we use this connection to derive convergence results for the two algorithms described in Sections 3.1 and 3.2, concentrating on the special case of bicriteria problems (i.e., $n = 2$).

As we have indicated, the outer approximation algorithm is closely related to the inner approximation algorithm by the concept of geometric duality, see Edelsbrunner (1987). Consequently, the discussion will mainly focus on the inner approximation algorithm while transferring results to the case of the outer approximation algorithm whenever it is convenient.

Wlog let the unit ball B of the current approximating gauge γ be given by the reflection set of $B \cap \mathbb{R}^n_{\geqq}$, i.e., B is symmetric with respect to the origin and satisfies

$$B = R(B \cap \mathbb{R}^n_{\geqq}) := \{b \in \mathbb{R}^n \,:\, |b_i| = |\bar{b}_i|,\, \bar{b} \in (B \cap \mathbb{R}^n_{\geqq})\}.$$

Moreover, let $\bar{Z}$ be the reflection set of $Z \cap \mathbb{R}^n_{\geqq}$, i.e.,

$$\bar{Z} = R(Z \cap \mathbb{R}^n_{\geqq}) := \{z \in \mathbb{R}^n \,:\, |z_i| = |\bar{z}_i|,\, \bar{z} \in (Z \cap \mathbb{R}^n_{\geqq})\}.$$

Then the Hausdorff distance, $d_H(B, \bar{Z})$, between the convex set $\bar{Z}$ and its polyhedral approximation B is given by

$$d_H(B, \bar{Z}) \;=\; \sup_{b \in B} \inf_{z \in \bar{Z}} \|z - b\|_2\,,$$

where $\|z - b\|_2$ denotes the Euclidean distance between the two points b and z.

Let C be a circular ball centered at the origin that is completely contained in B and let r be the radius of C, see Figure 8 for an example. If we denote the norm with unit ball C by $\|\bullet\|_C$, we obviously have $\|u\|_2 = r \cdot \|u\|_C$ for all $u \in \mathbb{R}^n$. Moreover, $\|u\|_C \geq \gamma(u)$ for all $u \in \mathbb{R}^n$ since $C \subseteq B$. Hence,

$$\begin{aligned} d_H(B, \bar{Z}) &= r \cdot \sup_{b \in B} \inf_{z \in \bar{Z}} \|z - b\|_C \\ &\geq r \cdot \sup_{b \in B} \inf_{z \in \bar{Z}} \gamma(z - b) \\ &= r \cdot |\gamma(\bar{z}) - 1|, \end{aligned}$$

where $\bar{z}$ is an optimal solution of (2) (or (9), respectively). Observe that the above relations are true for the approximating gauge γ and unit ball B at every iteration of the inner as well as the outer approximation algorithm.

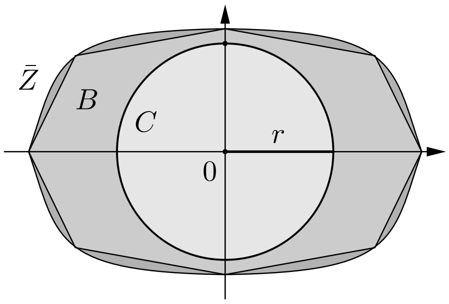


Figure 8: Comparing the problem dependent gauge distance to the Hausdorff distance

Rote (1992) showed that if a so-called sandwich algorithm is applied to approximate a convex set $\bar{Z}$ in $\mathbb{R}^2$ by an inscribed and a circumscribed polyhedron P_{inner} and P_{outer}, having k extreme points each ($k \geq 4$), and using the *chord rule* or the *maximum error rule* to generate the next point in each iteration of the algorithm, then the Hausdorff distance between the two approximating polyhedra can be bounded by

$$d_H(P_{\text{inner}}, P_{\text{outer}}) \leq \frac{8D}{(k-2)^2},$$

where D is the circumference of $\bar{Z}$. If k approaches infinity, the value of the multiplicative constant 8 can be reduced arbitrarily close to 2π, see Rote (1992).

Since the chord rule applied in the sandwich algorithm generates the same points that are also found by solving problem (2), this result can be immediately transferred to the inner approximation algorithm as described in Section 3.1. Moreover, the maximum error rule of the sandwich algorithm applied to convex functions generates the same points as problem (9) in the outer approximation algorithm if we assume that the boundary of $\bar{Z}$ is decomposed into infinitely small curve segments.

If we additionally use the fact that $d_H(P_{\text{inner}}, P_{\text{outer}}) \geq d_H(P_{\text{inner}}, \bar{Z})$ (and $d_H(P_{\text{inner}}, P_{\text{outer}}) \geq d_H(\bar{Z}, P_{\text{outer}})$, respectively) and that in our case an approximation of $\bar{Z}$ is generated only in the nonnegative orthant of the coordinate system (the other parts of the approximating polyhedra follow by symmetry), the approximation error of the inner and the outer approximation after k iterations can be bounded by

$$\begin{aligned} |\gamma(\bar{z}) - 1| &\leq \frac{1}{r} \cdot d_H(B, \bar{Z}) \\ &= \frac{1}{r} \cdot d_H(P_{\text{inner}}, \bar{Z}) \\ &\leq \frac{2\,D}{r\,k^2}; \qquad k \geq 4. \end{aligned} \tag{13}$$

(Note that the approximation either consists of $k+2$ nondominated points after k iterations, or it is exact.) Since the above bound is inversely proportional to the radius r of the circular ball C inscribed into the final approximation, we can try to choose C as large as possible to obtain a sharper bound. Yet any circular ball inscribed into the initial inner approximation yields a constant r that can be used in the above inequality and hence the inner approximation algorithm has a quadratic convergence rate. Note that any ball C inscribed into the initial inner approximation can be used as a ball inscribed into the initial outer approximation so that the arguments and conclusion above are also valid for the outer approximation algorithm.

Theorem 3.5 The approximation error after k iterations of the inner approximation algorithm or the outer approximation algorithm, respectively, measured by the approximating gauge γ, decreases by the order of $O(\frac{1}{k^2})$

which is optimal.

Proof. The convergence rate of $O(\frac{1}{k^2})$ follows directly from (13). That a quadratic convergence rate is best possible for approximating a convex set in $\mathbb{R}^2$ by inscribed or by circumscribed polyhedra is a well known result which can be easily verified by considering the example of a circle (see, for example, Gruber, 1992). $\square$

It may be conjectured that the convergence rate of the two approximation algorithms if applied to problems in $\mathbb{R}^n$, $n \geq 3$, is of the order $O(\frac{1}{k^{2/(n-1)}})$ which would be also best possible. However, corresponding results for algorithms approximating convex sets in $\mathbb{R}^n$ are - to the best knowledge of the authors - not yet available in the literature, and further research is needed in this direction.

4 Approximation in the $\mathbb{R}^n_{\leqq}$ - Nonconvex Case

Let $Z \subseteq \mathbb{R}^n$ be $\mathbb{R}^n_{\leqq}$ - closed with $\mathrm{int}(Z) \neq \emptyset$, and assume without loss of generality that $0 \in Z_{\leqq} = Z + \mathbb{R}^n_{\leqq}$.

Since the nondominated set N may be nonconnected in general, a piecewise linear approximation should aim at approximating the set

$$N_c := \{z \in Z_{\leqq} \,:\, \nexists\, \tilde{z} \in Z_{\leqq} \text{ s.t. } \tilde{z} \geq z\},$$

see Figure 9 for an example. Consequently, we will replace the convex unit ball of a distance measuring gauge (or norm) γ by a nonconvex "unit ball" B containing the origin in its interior and being constructed from the intersection or union of dominating cones. This unit ball is then used to define a new distance measuring function γ as

$$\gamma(z) := \min\{\lambda \,:\, z \in \lambda B\}. \tag{14}$$

The basic idea for an approximation procedure is - similar to the convex case - to minimize the maximum γ-distance between a nondominated point in Z

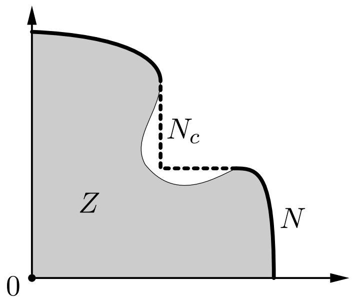


Figure 9: Nonconvex example with a nonconnected nondominated set N given by the union of the two bold curve segments. Its connected extension, the set N_c, includes the dashed line segments.

and the boundary of B. However, while the concept of using hyperplanes and their corresponding normal vectors for the generation of nondominated solutions works well in the convex case, it has to be replaced by a suitable alternative in the nonconvex case. Since the approximation itself will be constructed from dominating cones, it is natural to use variants of the Tchebycheff method for this purpose which theoretically allows the generation of the complete nondominated set (Steuer and Choo, 1983; Kaliszewski, 1987).

4.1 Inner Approximation

Let $d^1, \dots, d^s \in \mathbb{R}^n_{\geqq}$ be a nonempty and finite set of vectors generating the nonnegative orthant, i.e., $\{v \in \mathbb{R}^n : v = \sum_{i=1}^s \lambda_i d^i, \lambda \geqq 0\} = \mathbb{R}^n_{\geqq}$. We additionally assume that the set B defined by

$$B = \operatorname{cl}\left(\mathbb{R}^n_{\geqq} \setminus \bigcup_{i=1,\dots,s} (d^i + \mathbb{R}^n_{\geqq})\right)$$

is bounded, has nonempty interior and that $B \subseteq (Z_{\leqq} \cap \mathbb{R}^n_{\geqq})$. Even though the assumption of boundedness is quite restrictive in general, it will automatically be satisfied during all stages of the approximation algorithm that will be described at the end of this section.

Note that B could be symmetrically extended to all orthants of the coordinate system, yielding a compact set $\hat{B}$ that contains the origin in its interior. However, since we only consider points in the nonnegative orthant of the coordinate system, this extension has no impact on the following discussion and we omit it for the sake of simplicity.

If we interpret the vectors $d^1, \dots, d^s$ as local nadir points, they define a corresponding set of local utopia points $v^1, \dots, v^s$. The components v_i^j, $i = 1, \dots, n$ of these local utopia points v^j, $j = 1, \dots, s$ can be found as

$$\begin{aligned} v_i^j &= \max\left\{v_i \,:\, v_k = d_k^j \,\forall k \neq i,\, k \in \{1, \dots, n\};\, v \leqq z,\, z \in Z\right\} \\ &= \max\left\{z_i \,:\, z_k = d_k^j \,\forall k \neq i,\, k \in \{1, \dots, n\};\, z \in Z_{\leqq}\right\}. \end{aligned}$$

Each pair (d^j, v^j), $j = 1, \dots, s$ defines an n-dimensional axis-parallel rectangular box which can be used to define the weights for a local application of the Tchebycheff method. Consequently, a point $v \in N_c$, that is currently worst approximated with respect to the distance measure γ and that is generated by a variation of a "local Tchebycheff method", can be determined using the following disjunctive programming problem:

$$\begin{aligned} \max \quad & \gamma(v) \\ \text{s.t.} \quad & \bigvee_{i=1}^{s} \left(d^i + \lambda_i (v^i - d^i) = v \;\wedge\; \lambda_i \geq 0 \;\wedge\; v \leqq z^i \;\wedge\; z^i \in Z\right). \end{aligned} \tag{15}$$

Within a cone $d^j + \mathbb{R}^n_{\geqq}$, $j \in \{1, \dots, s\}$, solving (15) is equivalent to the application of the lexicographic weighted Tchebycheff method with the utopia point v^j and with the weights $w_i^j := \frac{1}{v_i^j - d_i^j}$, $i = 1, \dots, n$, i.e., to solving

$$\begin{aligned} \operatorname{lex\,min} \quad & \left(\left\|v^j - z\right\|_\infty^{w^j}, \left\|v^j - z\right\|_1\right) \\ \text{s.t.} \quad & z \in Z. \end{aligned} \tag{16}$$

The two problems can indeed be viewed as being equivalent since there exist optimal solutions $\bar{z}$ of (16) and $\bar{v}, \bar{\lambda}, \bar{z}^j$ of $\max\{\gamma(v) \,:\, d^j + \lambda(v^j - d^j) = v,\, \lambda \geq 0,\, v \leqq z^j,\, z^j \in Z\}$ such that $\bar{z} = \bar{z}^j$.

Moreover, problem (15) can be simplified to

$$\begin{array}{ll} \max & \lambda \\ \text{s.t.} & \bigvee_{i=1}^{s} \left(d^i + \lambda \cdot \frac{v^i - d^i}{\gamma(v^i)-1} \leqq z^i \ \wedge \ z^i \in Z \right) \\ & \lambda \geq 0. \end{array} \tag{17}$$

In this formulation, the search directions $v^j - d^j$ are normalized by the expression $\gamma(v^j) - 1 = \gamma(v^j) - \gamma(d^j) = \min\{\frac{v_i^j - d_i^j}{d_i^j} \ : \ i \in \{1, \ldots, n\}\}$. Thus the distance information between the current approximation (given by B) and a point $d^j + \lambda \cdot \frac{v^j - d^j}{\gamma(v^j)-1}$ is captured in the value of λ. In particular, the optimal solutions $\bar{v}$ of (15) and $\bar{\lambda}$ of (17) satisfy $\gamma(\bar{v}) = 1 + \bar{\lambda}$. This is due to the fact that the optimal $\bar{v}$ has to be located in some cone $d^j + \mathbb{R}^n_{\geqq}$, $j \in \{1, \ldots, s\}$ (note that the same cone also contains the vector v^j) and satisfies

$$\begin{aligned} \gamma(\bar{v}) &= \gamma\left(d^j + \bar{\lambda} \frac{v^j - d^j}{\gamma(v^j) - 1} \right) \\ &= \gamma\left(\left[1 - \frac{\bar{\lambda}}{\gamma(v^j) - 1} \right] \cdot d^j \ + \ \left[\frac{\bar{\lambda}}{\gamma(v^j) - 1} \right] \cdot v^j \right) \\ &= \left[1 - \frac{\bar{\lambda}}{\gamma(v^j) - 1} \right] \cdot \underbrace{\gamma(d^j)}_{=1} \ + \ \frac{\bar{\lambda}}{\gamma(v^j) - 1} \cdot \gamma(v^j) \\ &= 1 + \bar{\lambda}. \end{aligned}$$

The third equality holds since d^j and v^j are located in the same fundamental cone of B and thus $\gamma(\alpha d^j + \beta v^j) = \alpha\gamma(d^j) + \beta\gamma(v^j)$ for all nonnegative scalars α and β. In fact, the nonnegativity constraint in problem (17) can be relaxed due to the defintion of the unit ball B. Figure 10 illustrates problem (17) and its optimal solution.

Note that the disjunctive programming problem (17) has a linear programming reformulation if the set Z has a linear programming representation, c.f. problems (4) and (5) in Section 3.1.

Theorem 4.1 Let $\bar{\lambda}$ be an optimal solution of (17), let J be the index set of all the constraints that are satisfied (and binding) at optimality, and let

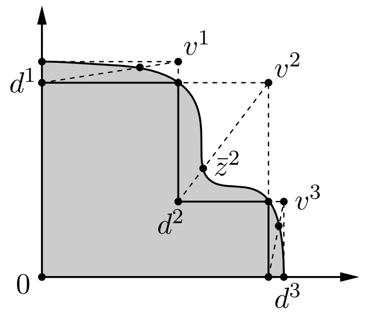


Figure 10: Inner approximation in the nonconvex case. The optimal solution of (17) is attained at $\bar{z}^2$ where $d^2 + \bar{\lambda} \cdot \frac{v^2 - d^2}{\gamma(v^2)-1} = \bar{z}^2$.

$\bar{v}^j := d^j + \bar{\lambda} \cdot \frac{v^j - d^j}{\gamma(v^j)-1}$, $j \in J$. Then B' defined as

$$B' := \left(B \cup \bigcup_{j \in J} (\bar{v}^j - \mathbb{R}^n_{\geqq}) \right) \cap \mathbb{R}^n_{\geqq} \tag{18}$$

satisfies $B \subseteq B' \subseteq (Z_{\leqq} \cap \mathbb{R}^n_{\geqq})$.

Proof. The first inclusion is trivial. To verify the second inclusion, suppose that a j^{th} constraint, $j \in \{1, \dots, s\}$ is satisfied at optimality. Hence there exists $\bar{z}^j \in Z$ such that $\bar{v}^j \in (\bar{z}^j - \mathbb{R}^n_{\geqq})$, implying the result. $\square$

If the j^{th} constraint of (17) is satisfied (and thus binding) at optimality, we can also conclude that $(\bar{v}^j + \text{int}\mathbb{R}^n_{\geqq}) \cap Z = \emptyset$, which immediately yields the following corollary:

Corollary 4.2 Under the assumptions of Theorem 4.1, if $j \in J$ and if $\bar{v}^j \leqq \bar{z}^j \in Z$, then $\bar{z}^j$ is weakly nondominated. Moreover, if Z is strictly $\text{int}\mathbb{R}^n_{\leqq}$ - convex, the solution $\bar{z}^j$ is properly nondominated.

From an algorithmic point of view we would like to iteratively solve problem (17) while increasing the size of the set B according to Theorem 4.1. Similar to the convex case, a repeated solution of (17) with a growing number of disjunctions is inefficient and will therefore be avoided by decomposing the problem into subproblems (P^j_{inner}), $j = 1, \dots, s$, that can be formulated as

$$\begin{array}{rcll} \delta_j & = & \max & \lambda \\ & & \text{s.t.} & d^j + \lambda \cdot \frac{v^j - d^j}{\gamma(v^j)-1} \leqq z \\ & & & \lambda \in \mathbb{R},\ z \in Z, \end{array} \tag{19}$$

and whose optimal $\bar{\lambda}$ determines the vector

$$\bar{v} = d^j + \bar{\lambda}\frac{v^j - d^j}{\gamma(v^j) - 1}. \tag{20}$$

Having the solution of all the subproblems (P^j_{inner}), $j = 1, \ldots, s$ available, the optimal solution value of (17) equals the maximum value of δ_j which also yields the related vector $\bar{v}$ as given above.

Figure 11 illustrates the proposed approximation algorithm that consists of the preprocessing phase and the main phase. As the result of the preprocessing phase, an initial feasible approximating unit ball is constructed. The preprocessing phase is initiated with only one vector d^1 which makes the interior of the corresponding set B empty (see Figure 11(a)). Nevertheless, problem (17) is well defined yielding the optimal solution $d^1 + \bar{\lambda} \cdot \frac{v^1 - d^1}{\gamma(v^1)-1} = \bar{z}^1$ as shown in Figure 11(b). Including the point $\bar{v} := d^1 + \bar{\lambda} \cdot \frac{v^1-d^1}{\gamma(v^1)-1}$ into the set B by updating it according to (18) defines the initial feasible approximation with which the main phase of the algorithm starts, see Figure 11(c). In the main phase, problem (17) is solved for the currently approximating ball B yielding the optimal solution $\bar{z}^1$ as shown in Figuree 11(d). The point $\bar{v} := d^1 + \bar{\lambda} \cdot \frac{v^1-d^1}{\gamma(v^1)-1}$ is included into the current approximation B by updating the set B according to (18). The new approximation is then used in the next iteration of the main phase of the algorithm (see Figure 11(e)). In comparison with the corresponding step in the convex case where the Beneath-Beyond Algorithm was used to update the current approximation, the update operation is significantly easier in this case. In particular, the point $\bar{v}$ generates a set $\bar{d}^1, \ldots, \bar{d}^n$ of new local nadir points replacing the point d^j of a box in which the maximum of (17) has been attained. The components $\bar{d}^j_i$, $i = 1, \ldots, n$, $j = 1, \ldots, n$ of these points can be computed as

$$\begin{aligned}\bar{d}^j_j &= \max\{0, \{d_j : d_i = \bar{v}_i \, \forall i \neq j, \, i \in \{1, \dots, n\}; \, d \leqq b, \, b \in B\}\}, \\ \bar{d}^j_i &= \bar{v}_i, \qquad i \neq j.\end{aligned}$$

In the subsequent iterations, problem (19) has to be solved only in the newly generated rectangular boxes (defined by the added local nadir and utopia points) while the remaining values of δ_j remain unchanged, see Figures 11(d)-11(f).

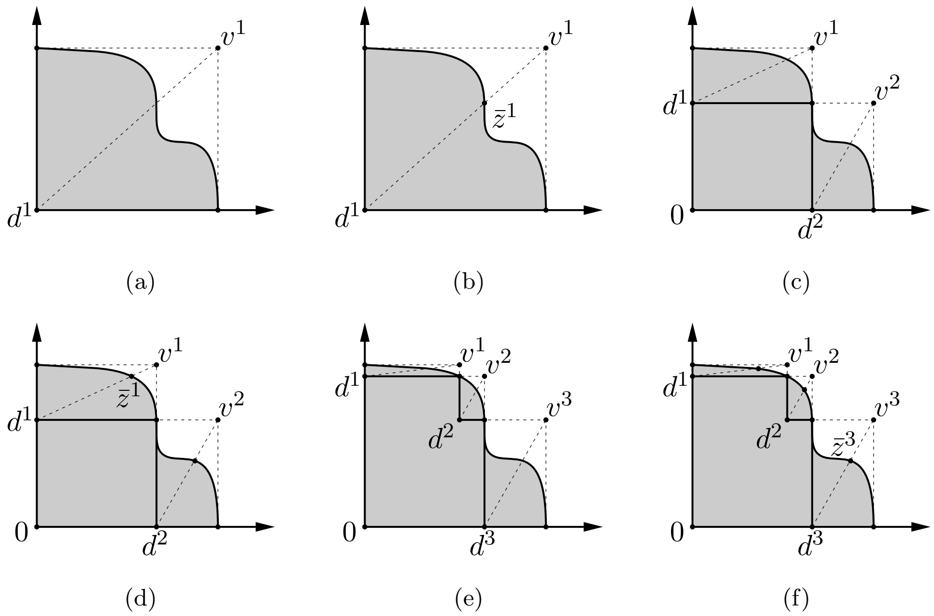


Figure 11: Inner approximation algorithm for general nonconvex problems: (a)-(b) preprocessing phase, (c)-(f) main phase.

As stopping criteria we can use, similar to the convex case, a bound ϵ on the required approximation accuracy which is again measured in a problem dependent way, or a bound `maxBoxNo` on the number of rectangular boxes that are generated during the algorithm. Figure 12 gives an outline of the main phase of the inner approximation algorithm for general nonconvex problems.

PROCEDURE: INNER APPROXIMATION; NONCONVEX CASE

Read/generate stopping criteria: $\epsilon > 0$, `maxBoxNo`;
Read/generate an initial inner approximation based on a set of local nadir points $d^1, \dots, d^s \in \mathbb{R}^n_{\geqq}$;
Construct s axis-parallel rectangular boxes by finding the local utopia points $v^1, \dots, v^s$ corresponding to $d^1, \dots, d^s$
for all boxes **do**
  Solve (19) to find $\bar{\lambda}$ and $\bar{v}$
end for
while #boxes < `maxBoxNo` and $|\gamma(\text{next point})-1| \geq \epsilon$ **do**
  Update the approximation B according to (18);
  Identify new and modified boxes
  for all new or modified boxes **do**
    Solve (19) to find $\bar{\lambda}$ and $\bar{v}$
  end for
end while
Output inner approximation

Figure 12: Pseudo code of the main phase of the inner approximation algorithm for general nonconvex problems

4.2 Outer Approximation

Let B be defined by a nonempty and finite set of fundamental vectors $v^1, \dots, v^t \in \mathbb{R}^n_{\geq}$ as

$$B = \mathbb{R}^n_{\geqq} \cap \bigcup_{i=1,\dots,t} (v^i - \mathbb{R}^n_{\geqq})$$

and let $(Z_{\leqq} \cap \mathbb{R}^n_{\geqq}) \subseteq B$. Note that the set B is always closed and bounded. Moreover, since B as defined above encloses the unit ball used in the inner approximation, independently of the choice of the vectors $d^1, \dots, d^s$ (inner approximation) and $v^1, \dots, v^t$ (outer approximation), the corresponding distance measure γ used in the outer approximation is always a lower bound on that used in the inner approximation.

Analogously to the inner approximation approach, we can interpret the vectors $v^1, \dots, v^t$ as local utopia points defining a corresponding set of local nadir points $d^1, \dots, d^t$ and thereby the desired Tchebycheff boxes. Even though the concept of nadir points is not unique for higher dimensional problems, we can use the symmetry to the inner approximation approach and compute the components d_i^j, $i = 1, \dots, n$ of local nadir points d^j, $j = 1, \dots, t$ as

$$\begin{aligned} d_i^j &= \max\left\{d_i \,:\, d_k = v_k^j \,\forall k \neq i,\, k\in\{1,\dots,n\};\, d \leqq z,\, z\in Z\right\} \\ &= \max\left\{z_i \,:\, z_k = v_k^j \,\forall k \neq i,\, k\in\{1,\dots,n\};\, z\in Z_{\leqq}\right\}. \end{aligned}$$

Using this definition, each pair (d^j, v^j), $j = 1, \dots, t$ again defines an n-dimensional axis-parallel rectangular box and thus the weights needed for the Tchebycheff method. Consequently, the disjunctive programming problems (15) and (17) can also be applied in the case of an outer approximation since these programs are solely based on pairs of local nadir and utopia points. However, since the current approximation (given by B) encloses the set $Z_{\leqq} \cap \mathbb{R}^n_{\geqq}$, the orientation of the search direction as well as its normalization as used in (17) have to be adapted to the new situation. This leads to the following variation of (17) in which the nonnegativity constraint for λ can also be relaxed:

$$\begin{array}{ll} \min & \lambda \\ \text{s. t.} & \bigvee_{i=1}^{s}\left(v^i - \lambda \cdot \frac{v^i - d^i}{1-\gamma(d^i)} \leqq z^i \ \wedge\ z^i \in Z\right) \\ & \lambda \in \mathbb{R}. \end{array} \tag{21}$$

Note that the search within each of the cones $d^j + \mathbb{R}^n_{\geqq}$ is now initiated at the point v^j, outside the set $Z_{\leqq} \cap \mathbb{R}^n_{\geqq}$, and thus the search is directed "inward". The normalization term can be evaluated as $1 - \gamma(d^j) = \gamma(v^j) - \gamma(d^j) = \min\{\frac{v_i^j - d_i^j}{v_i^j} \,:\, i \in \{1, \dots, n\}\}$. A similar analysis as in Section 4.1 shows that the optimal solutions $\bar{v}$ of (15) and $\bar{\lambda}$ of (21) satisfy $\gamma(\bar{v}) = 1 - \bar{\lambda}$. An example where problem (21) is applied in the outer approximation approach is given in Figure 13.

Problem (21) again allows a linear programming formulation in the case that Z has a linear programming representation, c.f. Section 4.1.

Due to the similarities between the inner and outer approximation ap-

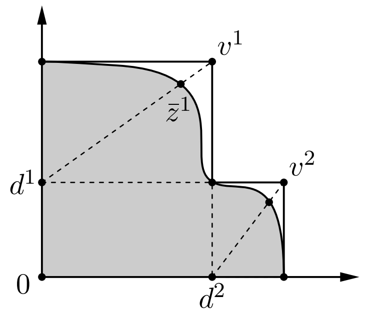


Figure 13: Outer approximation in the nonconvex case. The optimal solution of (21) is attained at $v^1 - \bar{\lambda} \cdot \frac{v^1 - d^1}{1-\gamma(d^1)} = \bar{z}^1$.

proaches, Theorem 4.1 and Corollary 4.2 can be easily transferred:

Theorem 4.3 Let $\bar{\lambda}$ be an optimal solution of (21), let J be the index set of all the constraints that are satisfied (and binding) at optimality and let $\bar{d^j} := v^j - \bar{\lambda} \cdot \frac{v^j - d^j}{1-\gamma(d^j)}$, $j \in J$. Then B' defined as

$$B' := B \setminus \bigcup_{j \in J} (\bar{d^j} + \mathrm{int}\mathbb{R}^n_{\geqq}) \tag{22}$$

satisfies $(Z_{\leqq} \cap \mathbb{R}^n_{\geqq}) \subseteq B' \subseteq B$.

Proof. The second inclusion is trivial. To prove the first inclusion, let a j^{th} constraint, $j \in \{1, \dots, t\}$ be binding at optimality. Thus $(\bar{d^j} + \mathrm{int}\mathbb{R}^n_{\geqq}) \cap Z = \emptyset$, and the result follows. $\square$

The proof of Theorem 4.3 immediately implies the following result:

Corollary 4.4 Under the assumptions of Theorem 4.3, if $j \in J$ and if $\bar{d^j} \leqq \bar{z}^j \in Z$, then $\bar{z}^j$ is weakly nondominated. If additionally Z is strictly $\mathrm{int}\mathbb{R}^n_{\leqq}$ - convex, the solution $\bar{z}^j$ is properly nondominated.

The outer approximation algorithm is again based on an iterative solution of problem (21) which leads to decreasing sizes of the approximating sets B, c.f. Theorem 4.3. For this purpose, problem (21) is decomposed into subproblems (P^j_{outer}), $j = 1, \dots, t$, given by

$$\begin{array}{rcll}\delta_j &=& \min & \lambda \\ && \text{s.t.} & v^j - \lambda \cdot \frac{v^j - d^j}{1-\gamma(d^j)} \leqq z \\ &&& \lambda \in \mathbb{R},\ z \in Z.\end{array} \tag{23}$$

The optimal solution of (21) equals to the minimum δ_j over all the subproblems (P^j_{outer}), $j = 1, \dots, t$.

The outer approximation algorithm does not need a preprocessing phase since the global utopia point of the problem can yield an initial feasible approximation, as it is shown in Figure 14(a) where the global utopia point is denoted by v^1. In this example, problem (21) generates the optimal solution $v^1 - \bar{\lambda} \cdot \frac{v^1 - d^1}{1-\gamma(d^1)} = \bar{z}^1$, see Figure 14(b). The corresponding point $\bar{d} = v^1 - \bar{\lambda} \cdot \frac{v^1 - d^1}{1-\gamma(d^1)}$ is then included into the current approximation by updating the set B according to (22), see Figure 14(c). In analogy to the inner approximation algorithm, the point $\bar{d}$ generates a set $\bar{v}^1, \dots, \bar{v}^n$ of new local utopia points replacing the point v^j of that box where the minimum over all subproblems (23) was attained. Their components $\bar{v}^j_i$, $i = 1, \dots, n$, $j = 1, \dots, n$ are given by

$$\begin{array}{rcl}\bar{v}^j_j &=& \max\left\{v_j \,:\, v_i = \bar{d}_i\ \forall i \neq j,\ i \in \{1, \dots, n\};\ v \leqq b,\ b \in B\right\}, \\ \bar{v}^j_i &=& \bar{d}_i, \qquad i \neq j.\end{array}$$

Note that the new local utopia points found according to the formula above are renamed in Figure 14 in order to simplify the notation. During the course of the algorithm (see Figures 14(d)-14(f)), problem (23) has to be solved in all the newly generated rectangular boxes in each iteration while reusing the unchanged values of δ_j.

Utilizing the same stopping criteria as in the inner approximation algorithm, the outer approximation algorithm is summarized in Figure 15.

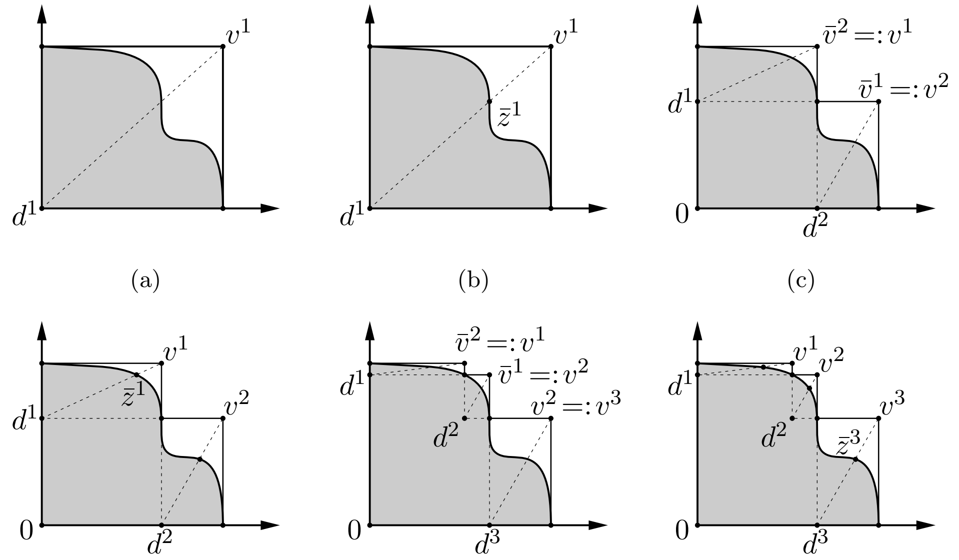


Figure 14: Outer approximation algorithm for general nonconvex problems.

Procedure: Outer Approximation; Nonconvex Case

Read/generate stopping criteria: $\epsilon > 0$, `maxBoxNo`;
Read/generate an initial outer approximation based on a set of local utopia points $v^1, \ldots, v^t \in \mathbb{R}^n_{\geqq}$;
Construct t axis-parallel rectangular boxes by finding the local nadir points $d^1, \ldots, d^t$ corresponding to $v^1, \ldots, v^t$
for all boxes **do**
  Solve (23) to find $\bar{\lambda}$ and $\bar{d}$
end for
while #boxes < `maxBoxNo` and $|\gamma(\text{next point}) - 1| \geq \epsilon$ **do**
  Update the approximation B according to (22);
  Identify new and modified boxes
  for all new or modified boxes **do**
    Solve (23) to find $\bar{\lambda}$ and $\bar{d}$
  end for
end while
Output outer approximation

Figure 15: Pseudo code of the outer approximation algorithm for general nonconvex problems

4.3 Simultaneous Inner and Outer Approximation in the $\mathbb{R}^n_{\leqq}$ - Nonconvex Case

Even though Figures 11 and 14 suggest that the pairs of local nadir and utopia points generating the approximating boxes in the inner as well as in the outer approximation approach coincide if both algorithms are initialized accordingly, this is not true in general for two reasons: On one hand, the distance measure γ and the approximation B differ between the two algorithms and different boxes may contain the optimal solution of (17) and (21), respectively. On the other hand, the new local nadir and utopia points computed within one iteration of the procedure may not coincide in higher dimensional problems, a fact that immediately leads to different approximations. This indicates that a combination of the two procedures may be beneficial also for nonconvex problems.

5 Conclusions

In this paper we have developed inner as well as outer approximation algorithms that generate piecewise linear approximations of the nondominated set of convex and nonconvex multicriteria programs. In all cases, the approximation itself is used to define a problem dependent distance measure, leading to unbiased and scale-independent approximations. Moreover, the approximation is always improved where it is needed most, that is, where the current approximation error is maximal. This self-correcting property of the approximation was not present in the algorithms proposed for the nonconvex problems in Schandl *et al.* (2001c) and is a significant improvement.

The algorithms limit the involvement of the decision maker only to the initialization when the starting approximation has to be given. If such an interaction was desirable while approximation is being constructed, the algorithms could be easily modified. The authors however believe that decision makers may appreciate an interaction-free approximating technique releasing them from interrogation and queries.

A byproduct of the developed algorithms are the new scalarization techniques for generating (weakly) nondominated points. These techniques search the objective space by means of properly defined directions.

While quadratic convergence of the developed algorithms is proven for convex bicriteria problems, similar results can only be conjectured for the multicriteria case. Future research should focus, among others, on convergence results under more general assumptions as well as practical studies and comparisons of the proposed algorithms with other approximation approaches.

References

E. Balas. Disjunctive programming and a hierarchy of relaxations for discrete optimization problems. *SIAM J. Algebraic Discrete Methods* **6** 466–486 (1985).

Y. A. Belov and S. V. Shafranskii. Application of a differentiable family of convolutions in the solution of problems of multicriterial optimization. *Dokl. Akad. Nauk Ukrain. SSR* **163** 49–52 (1991). Russian.

H. P. Benson and S. Sayin. Towards finding global representations of the efficient set in multiple objective mathematical programming. *Naval Research Logistics* **44** 47–67 (1997).

P. Czyzak and A. Jaszkiewicz. Pareto Simulated Annealing—A metaheuristic technique for multiple-objective combinatorial optimization. *Journal of Multi-Criteria Decision Analysis* **7** 34–47 (1998).

I. Das. An improved technique for choosing parameters for Pareto surface generation using normal-boundary intersection. In *Short Paper Proceedings of the Third World Congress of Structural and Multidisciplinary Optimization*, volume 2, pp. 411–413 (1999).

K. Deb. *Multi-objective Optimization using Evolutionary Algorithms*. John Wiley & Sons Ltd., Chichester, England (2001).

H. Edelsbrunner. *Algorithms in Combinatorial Geometry.* Springer-Verlag, Berlin (1987).

E. A. Galperin and M. M. Wiecek. Retrieval and use of the balance set in multiobjective global optimization. *Computers and Mathematics with Applications* **37** 111–123 (1999).

X. Gandibleux, N. Mezdaoui, and A. Fréville. A tabu search procedure to solve multiobjective combinatorial optimization problems. In *Advances in Multiple Objective and Goal Programming*, (Edited by R. Caballero, F. Ruiz, and R. E. Steuer), pp. 291–300. Springer-Verlag, Berlin (1997).

A. M. Geoffrion. Proper efficiency and the theory of vector maximization. *Journal of Mathematical Analysis and Applications* **22** (3) 618–630 (1968).

P. M. Gruber. Approximation of convex bodies. In *Handbook of Convex Geometry*, (Edited by P. M. Gruber and J. M. Wills), pp. 321–345. North-Holland, Amsterdam (1992).

S. Helbig. Approximation of the efficient set by perturbation of the ordering cone. *ZOR – Methods and Models of Operations Research* **35** 197–220 (1991).

I. Kaliszewski. A modified weighted Tchebycheff metric for multiple objective programming. *Computers and Operations Research* **14** 315–323 (1987).

I. Kaliszewski. *Quantitative Pareto Analysis by Cone Separation Technique.* Kluwer Academic Publishers, Dordrecht (1994).

M. M. Kostreva, Q. Zheng, and D. Zhuang. A method for approximating solutions of multicriteria nonlinear optimization problems. *Optimization Methods and Software* **5** 209–226 (1995).

B. Lemaire. Approximation in multiobjective optimization. *Journal of Global Optimization* **2** 117–132 (1992).

H. Minkowski. *Gesammelte Abhandlungen, Band 2*. Editor: D. Hilbert. Teubner Verlag, Leipzig und Berlin (1911). Also in: Chelsea Publishing Company, New York, 1967.

V. N. Nefedov. On the approximation of a Pareto set. *U.S.S.R. Computational Mathematics and Mathematical Physics* **24** (4) 19–28 (1984).

N. M. Popov. Approximation of a Pareto set by the convolutions method. *Vestnik Moskov. Univ. Ser. XV Vychisl. Mat. Kibernet.* **81** 35–41 (1982). Russian.

R. T. Rockafellar. *Convex Analysis*. Princeton University Press, Princeton, NJ (1970).

G. Rote. The convergence rate of the sandwich algorithm for approximating convex functions. *Computing* **48** 337–361 (1992).

B. Schandl. *Norm-Based Evaluation and Approximation in Multicriteria Programming*. Ph.D. thesis, Clemson University, Clemson, SC (1999). Available at `http://www.math.clemson.edu/affordability/publications.html` (15.12.1999).

B. Schandl, K. Klamroth, and M. M. Wiecek. Introducing oblique norms into multiple criteria programming. *Journal of Global Optimization* (2001a). To appear.

B. Schandl, K. Klamroth, and M. M. Wiecek. Norm-based approximation in bicriteria programming. *Computational Optimization and Applications* (2001b). To appear.

B. Schandl, K. Klamroth, and M. M. Wiecek. Norm-based approximation in multicriteria programming. *Computers and Mathematics with Applications* (2001c). To appear.

M. M. Smirnov. On a logical convolution of a criterion vector in the Pareto set approximation problem. *Computational Mathematics and Mathematical Physics* **36** 605–614 (1996).

I. M. Sobol' and Y. L. Levitan. Error estimates for the crude approximation of the trade-off curve. In *Multiple Criteria Decision Making*, (Edited by G. Fandel and T. Gal), pp. 83–92. Springer-Verlag, Berlin (1997).

R. B. Statnikov and J. B. Matusov. Use of P_τ-nets for the approximation of the Edgeworth-Pareto set in multicriteria optimization. *Journal of Optimization Theory and Applications* **91** 543–560 (1996).

R. E. Steuer and E. U. Choo. An interactive weighted Tchebycheff procedure for multiple objective programming. *Mathematical Programming* **26** 326–344 (1983).

E. L. Ulungu, J. Teghem, P. H. Fortemps, and D. Tuyttens. MOSA method: A tool for solving multiobjective combinatorial optimization problems. *Journal of Multi-Criteria Decision Analysis* **8** 221–236 (1999).

E. Zitzler, K. Deb, L. Thiele, C. A. Coello, and D. Cotne (editors). *Evolutionary Multi-Criterion Optimization*, volume 1993 of *Lecture Notes in Computer Science*. Springer, Berlin (2001).