Planar Weber Location Problems with Line Barriers

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Abstract

The Weber problem for a given finite set of existing facilities in the plane is to find the location of a new facility such that the weighted sum of distances to the existing facilities is minimized.
A variation of this problem is obtained if the existing facilities are situated on two sides of a linear barrier. Such barriers like rivers, highways, borders or mountain ranges are frequently encountered in practice.
Structural results as well as algorithms for this non-convex optimization problem depending on the distance function and on the number and location of passages through the barrier are presented.

1 Introduction

Modern life encounters an ever growing concentration in many respects. Growing population, higher integration of electronic circuits or the economical need to choose an optimized site for new facilities have led to planar location problems with an ever growing number of obstacles (see e.g. [14]).

The classical *Weber problem* (median problem, minisum problem) which is the basis for many developments is stated as follows: Let $\mathcal{E}x = \{Ex_1, Ex_2, \ldots, Ex_M\}$ be a finite set of existing facilities represented by points in $\mathbb{R}^2$. A positive weight $w_m = w(Ex_m)$ is associated with each existing facility Ex_m ($m \in \mathcal{M} := \{1, \ldots, M\}$) which can be interpreted as the demand of facility Ex_m. The objective is to find a new facility $X^* \in \mathbb{R}^2$ such that the weighted sum of distances between X^* and the existing facilities

$$f(X) = \sum_{m=1}^{M} w_m \, d(X, Ex_m)$$

is minimized for some distance function d. With $\mathcal{X}^*$ we denote the set of optimal solutions of the Weber problem.

This problem, which has the classification $1/P/\bullet/d/\sum$ with respect to the classification scheme for location problems proposed in [6, 9] has already been thoroughly treated by

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many authors. For an overview see e.g. [5, 6, 13, 16]. In practice the modeling of the investigated region as the complete $\mathbb{R}^2$ is not realistic. There may be for example areas where the positioning of a new facility is not allowed (see e.g. [4, 7, 8]) and of course there may be regions where trespassing is prohibited. Such barriers may be for example buildings, lakes or mountain ranges. The idealized case that the barriers are linear and have only a finite set of passages is a special case which is frequently encountered in practice. Line barriers with passages may be used to model rivers, border lines, highways, mountain ranges or, on a smaller scale, conveyer belts in an industrial plant. Here trespassing is only allowed through a finite set of passages. Disregarding these types of barriers may lead to bad locational decisions since especially long and almost linear barriers have a big impact on travel distances and travel times.

The introduction of barriers significantly entails different treatment because the objective function is not convex as in the classical Weber problem. Literature has so far only treated some particular types of metrics and barrier shapes, like one circle as a barrier and the Euclidean distance [11] or closed polyhedra as barriers and the l_p-metric [1, 3]. Especially line barriers with passages have so far only been treated in the case of the Manhattan metric l_1 [12, 2] for which arbitrarily shaped barriers can be handled.

In this paper, general results as well as algorithms for Weber problems with line barriers, i.e. for problems of the type $1/P/\mathcal{B}_L/d_{\mathcal{B}_L}/\sum$, and for a large class of metrics including the class of l_p metrics are presented.

2 General Results

The Weber problem with line barriers $1/P/\mathcal{B}_L/d_{\mathcal{B}_L}/\sum$ can be modeled as follows: Let $L := \{(x,y) \in \mathbb{R}^2 \mid y = ax + b\}$ be a line and let $\{P_n \in L \mid n \in \mathcal{N} := \{1,\dots,N\}\}$ be a set of points on L. Then $\mathcal{B}_L := L \setminus \{P_1,\dots,P_N\}$ is called a *line barrier with passages* or shortly *line barrier*. The case that the barrier is a vertical line which is not included in this description can easily be transformed to this definition.

The feasible region F for new locations is defined as the union of the two closed half-planes F^1 and F^2 on both sides of $\mathcal{B}_L$. Here $F^1 \cup F^2 = \mathbb{R}^2$ since the line $y = ax + b$ belongs to both half-planes F^1 and F^2. As all results can easily be transferred to the case that the line barrier has a finite width, for simplification this model will be used in the following although a new location placed directly on the barrier is not allowed in reality.

Furthermore a finite number of existing facilities $Ex^i_m \in F^i$, $m \in \mathcal{M}^i := \{1,\dots,M^i\}$ is given in each half-plane F^i, $i = 1,2$, represented by points in $\mathbb{R}^2$. A positive weight $w^i_m := w(Ex^i_m) \in \mathbb{R}_+$ is associated with each existing facility Ex^i_m representing the demand of Ex^i_m.

The major difference between this model and planar location problems without barriers is the modified distance function. If a distance function d, derived from a norm, is given for the unconstrained problem, the distance function $d_{\mathcal{B}_L}$ for a problem of the type $1/P/\mathcal{B}_L/d_{\mathcal{B}_L}/\sum$ is defined as the length of a shortest path (wrt. d) that does not cross the barrier. Therefore, $d_{\mathcal{B}_L}$ is given by

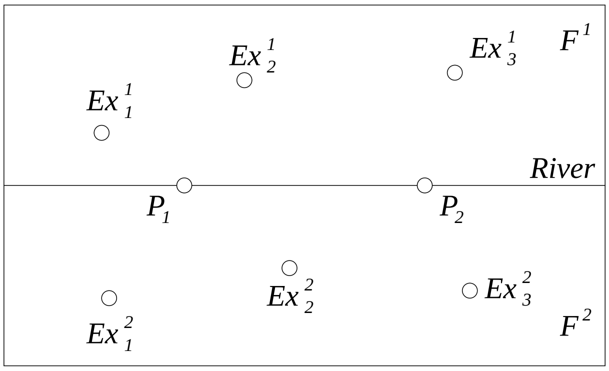

Figure 1: An example problem modeling a river with two bridges.

$$d_{\mathcal{B}_L}(X,Y) = \begin{cases} d(X,Y) & \text{if} \quad X,Y \in F^i \\ d(X,P_{n_{X,Y}}) + d(P_{n_{X,Y}},Y) \text{ for some } n_{X,Y} \in \mathcal{N} & \phantom{\text{if}} \quad X \in F^i,\ Y \in F^j, \end{cases} \tag{1}$$

where $P_{n_{X,Y}}$ is a passage point located on a shortest path connecting two points X and Y in opposite half-planes. Note that for $d_{\mathcal{B}_L}$ the triangle inequality is still satisfied but that $d_{\mathcal{B}_L}$ is not positively homogeneous in general. As a consequence the objective function of location problems with barriers is usually non-convex.

The general idea for solving problem $1/P/\mathcal{B}_L/d_{\mathcal{B}_L}/\sum$ can be summarized as follows: Assume that an optimal solution of the problem is located in half-plane F^1. Then the passages channel the flow from the existing facilities in F^2 to the new location. Interpreting these passages as artificial facilities carrying the weights of the assigned existing facilities in F^2, the new location can be retrieved as the solution of an unconstrained Weber problem in half-plane F^1. Finding the relevant subset of passage points and their respective weights is a combinatorial problem which will be discussed in Section 3. In the following we will derive the theoretical basis for this approach for a large class of Weber problems with distance functions including the class of all l_p metrics, $p \in [1,\infty]$.

We can use (1) to rewrite the objective function for a point $X \in F^i$. A similar formulation was given in [3], where the corner points of a set of polyhedral obstacles are used instead of the passage points.

Lemma 1 *Let d be a metric derived from a norm, $X \in F^i$ and $i,j \in \{1,2\}$, $i \neq j$. Then there exist passages $P_{n_1},\ldots,P_{n_{M^j}}$ such that*

$$f(X) = f^i_X(X) + g^j_X, \tag{2}$$

where

$$\begin{aligned} f^i_X(Y) &= \sum_{m=1}^{M^i} w^i_m d(Y,Ex^i_m) + \sum_{m=1}^{M^j} w^j_m d(Y,P_{n_m}), \quad Y \in F^i, \\ g^j_X &= \sum_{m=1}^{M^j} w^j_m d(Ex^j_m,P_{n_m}). \end{aligned}$$

Note that $f^i_X(Y)$ is the objective function of the corresponding unconstrained Weber problem in the half-plane F^i with existing facilities $Ex^i_1, \ldots, Ex^i_{M^i}, P_1, \ldots, P_N$. Observe also that the right hand side of (2) takes on different values depending on what passage points have been used to evaluate the distance from a point X to the existing facilities located in the opposite half-plane while passing through those passage points. Due to the definition of P_{n_m}, we have that

$$f(Y) = f^i_Y(Y) + g^j_Y \leq f^i_X(Y) + g^j_X \qquad \forall X, Y \in F^i. \tag{3}$$

Lemma 1 implies the following result specifying the possible locations of optimal solutions of problem $1/P/\mathcal{B}_L/d_{\mathcal{B}_L}/\sum$.

Lemma 2 *Consider a problem of type* $1/P/\mathcal{B}_L/d_{\mathcal{B}_L}/\sum$ *and let* d *be a metric derived from a norm such that*

(a) $\mathcal{X}^* \subseteq \mathrm{conv}\{Ex_m \mid m \in \mathcal{M}\}$ *holds for* $1/P/\bullet/d/\sum$. *Then*

$$\mathcal{X}^*_{\mathcal{B}} \subseteq \mathrm{conv}\{Ex^1_m, P_n \mid m \in \mathcal{M}^1,\ n \in \mathcal{N}\} \cup \mathrm{conv}\{Ex^2_m, P_n \mid m \in \mathcal{M}^2,\ n \in \mathcal{N}\};$$

(b) $\mathcal{X}^* \cap \mathrm{conv}\{Ex_m \mid m \in \mathcal{M}\} \neq \emptyset$ *holds for* $1/P/\bullet/d/\sum$. *Then*

$$\mathcal{X}^*_{\mathcal{B}} \cap \Big(\mathrm{conv}\{Ex^1_m, P_n \mid m \in \mathcal{M}^1,\ n \in \mathcal{N}\} \cup \mathrm{conv}\{Ex^2_m, P_n \mid m \in \mathcal{M}^2,\ n \in \mathcal{N}\}\Big) \neq \emptyset.$$

The assumptions of Lemma 2 are satisfied for a large class of distance functions, including the class of l_p-metrics, $p \in [1, \infty]$ (see [10, 15]).

Unfortunately it is not possible to restrict $\mathcal{X}^*_{\mathcal{B}}$ for example to that half-plane with the higher total weight as one may conjecture intuitively. This can be easily seen since, starting from an appropriate unconstrained Weber problem, a line barrier can be added so that the optimal solution of the unconstrained problem does not lie on the side with the higher total weight. If we now place passage points on the line so that the shortest paths from the existing facilities to the former optimum remain feasible, we obtain the same solution also for the problem including the barrier.

In the following we will restrict our discussion to such metrics d derived from a norm (and location problems $1/P/\bullet/d/\sum$) for which conditions (a) or (b) of Lemma 2 are satisfied. In the first case, the complete set of optimal locations $\mathcal{X}^*_{\mathcal{B}}$ of $1/P/\mathcal{B}_L/d_{\mathcal{B}_L}/\sum$ can be determined by solving a finite series of subproblems in each half-plane, whereas in the latter case at least one optimal solution $X^*_{\mathcal{B}} \in \mathcal{X}^*_{\mathcal{B}}$ is found. Note that Lemma 2 is crucial for the relaxation of the restriction to the respective half-planes in the subproblems.

Theorem 1 *Under the assumptions of Lemma 2 (a), every optimal solution* $X^*_{\mathcal{B}} \in \mathcal{X}^*_{\mathcal{B}}$ *of* $1/P/\mathcal{B}_L/d_{\mathcal{B}_L}/\sum$ *is also an optimal solution of the corresponding unconstrained Weber problem* $1/P/\bullet/d/\sum$ *with existing facilities* $Ex^i_1, \ldots, Ex^i_{M^i}, P_1, \ldots, P_N$ $(i \in \{1, 2\})$ *and objective function* $f^i_{X^*_{\mathcal{B}}}$.

Proof: Let $X^*_{\mathcal{B}} \in F^i$ be an optimal solution of $1/P/\mathcal{B}_L/d_{\mathcal{B}_L}/\sum$. From Lemma 1 we have that $f(X^*_{\mathcal{B}}) = f^i_{X^*_{\mathcal{B}}}(X^*_{\mathcal{B}}) + g^j_{X^*_{\mathcal{B}}}$. Here $g^j_{X^*_{\mathcal{B}}}$ is constant and $f^i_{X^*_{\mathcal{B}}}(Y)$ is the objective function of an unconstrained Weber problem. Assume that $\exists Y \in F^i$ with $f^i_{X^*_{\mathcal{B}}}(Y) < f^i_{X^*_{\mathcal{B}}}(X^*_{\mathcal{B}})$. Then $f(Y) \leq f^i_{X^*_{\mathcal{B}}}(Y) + g^j_{X^*_{\mathcal{B}}} < f^i_{X^*_{\mathcal{B}}}(X^*_{\mathcal{B}}) + g^j_{X^*_{\mathcal{B}}} = f(X^*_{\mathcal{B}})$, contradicting the optimality of $X^*_{\mathcal{B}}$. $\square$

Theorem 1 can be easily transformed to the case that only the weaker condition (b) of Lemma 2 is true for the unconstrained Weber problem $1/P/\bullet/d/\sum$:

Corollary 1 *Under the assumptions of Lemma 2 (b), there exists at least one optimal solution $X^*_{\mathcal{B}} \in \mathcal{X}^*_{\mathcal{B}}$ of $1/P/\mathcal{B}_L/d_{\mathcal{B}_L}/\sum$ which is also an optimal solution of the corresponding unconstrained Weber problem $1/P/\bullet/d/\sum$ with existing facilities $Ex^i_1, \dots, Ex^i_{M^i}, P_1, \dots, P_N$ ($i \in \{1,2\}$) and objective function $f^i_{X^*_{\mathcal{B}}}$.*

3 Algorithms for Weber problems with line barriers

Theorem 1 implies a straight-forward solution approach for Weber problems with line barriers. The basic idea is to check for all existing facilities in one half-plane F^i all possible passages to the other half-plane F^j and to determine the sets of optimal solutions of the corresponding unconstrained location problems in F^j. This procedure must be carried out for each half-plane, yielding a total number of $O(N^{2M})$ subproblems (where $M = M^1 + M^2$ denotes the overall number of existing facilities and N denotes the number of passage points). In the following we will develop a polynomial time algorithm that disregards irrelevant subproblems from further investigation. Since the case that only 1 passage allows trespassing through $\mathcal{B}_L$ is trivial, we will concentrate on line barriers with 2 or more passages.

3.1 Line barriers with two passages

For both half-planes F^i, $i = 1, 2$ define the difference of distances $D^i(m)$ between an existing facility $Ex^i_m \in F^i$ and the two passages P_1 and P_2 as

$$D^i(m) := d(Ex^i_m, P_1) - d(Ex^i_m, P_2), \qquad m \in \mathcal{M}^i.$$

Wlog assume that the existing facilities are ordered such that $D^i(1) \leq \cdots \leq D^i(M^i)$. Furthermore let $j \in \{1,2\}$ with $j \neq i$ be the index of the opposite half-plane F^j. A shortest path SP from an existing facility $Ex^j_m \in F^j$ to a point $X \in F^i$ passes through one of the passages P_1 and P_2 depending on the following condition:

$$\begin{aligned} P_1 \in SP \;&\Leftrightarrow\; D^j(m) \;<\; d(P_2, X) - d(P_1, X) \\ P_2 \in SP \;&\Leftrightarrow\; D^j(m) \;>\; d(P_2, X) - d(P_1, X). \end{aligned} \tag{4}$$

In the case that $D^j(m) = d(P_2, X) - d(P_1, X)$ a shortest path may pass through either passage P_1 or P_2. Note that the set of points for which both passages are equally good is in general not linear but may define an only implicitly available curve, depending on the distance function d. However, the computation of those subsets of F^i where the same passage points are used from an existing facility can be avoided by defining $k := \max\{m \in \{0, \ldots, M^j\} \mid D^j(m) < d(P_2, X) - d(P_1, X)\}$. Then, similar to (2), the value of the objective function $f(X)$ for the point $X \in F^i$ can be evaluated as

$$f(X) = f_k^i(X) + g_k^j, \tag{5}$$

where

$$\begin{aligned} f_k^i(X) &= \sum_{m=1}^{M^i} w_m^i d(X, Ex_m^i) + \left(\sum_{m=1}^{k} w_m^j\right) d(X, P_1) + \left(\sum_{m=k+1}^{M^j} w_m^j\right) d(X, P_2) \\ g_k^j &= \sum_{m=1}^{k} w_m^j d(P_1, Ex_m^j) + \sum_{m=k+1}^{M^j} w_m^j d(P_2, Ex_m^j). \end{aligned}$$

The only unknown parameters in (5) are the values of i and k. Therefore all possible values $i = 1, 2$ and $k = 0, \ldots, M^j$ are tested in the following algorithm to obtain a candidate set for the set of globally optimal solutions.

Algorithm for solving $1/P/\mathcal{B}_L,\ 2\,passages/d_{\mathcal{B}_L}/\sum$**:**

For $i = 1, 2$ do

1. Let $j \in \{1, 2\}$ with $j \neq i$, determine $D^j(m) := d(Ex_m^j, P_1) - d(Ex_m^j, P_2)$; $m \in \mathcal{M}^j$, and sort the existing facilities in F^j such that $D^j(1) \leq \cdots \leq D^j(M^j)$.
2. For $k = 0$ to M^j do

   Let $w(P_1) := \sum_{m=1}^{k} w(Ex_m^j)$, $w(P_2) := \sum_{m=k+1}^{M^j} w(Ex_m^j)$, and determine the set of optimal solutions $\mathcal{X}_k^i$ of $1/P/\bullet/d/\sum$ with existing facilities $\mathcal{E}x := \{P_1, P_2, Ex_1^i, \ldots, Ex_{M^i}^i\}$ and the corresponding objective values $f_k^i(\mathcal{X}_k^i) + g_k^j$.

Output: $\mathcal{X}_{\mathcal{B}}^* = \underset{X_k^i \in \mathcal{X}_k^i\,;\, k \in \mathcal{M}^i\,;\, i \in \{1,2\}}{\arg\min} f_k^i(X_k^i) + g_k^j.$

The time complexity of this algorithm can be calculated as $O(M \log M + MT)$, where $M = M^1 + M^2$ is the number of existing facilities and $O(T)$ is the time complexity of the corresponding unconstrained Weber problems.

Note that a solution X_k^i may be found during the algorithm for which the current value of k and therefore the current assignment of passages is not optimal. Anyhow the optimal assignment yielding a globally optimal solution $X_{\mathcal{B}}^*$ is used sometimes during the solution process so that equality, i.e. $f(X_{\mathcal{B}}^*) = f_k^i(X_{\mathcal{B}}^*) + g_k^j$, holds in this case.

3.2 Line barriers with N passages, $N > 2$

In the case that more than 2 passages are available, some additional considerations have to be made. Wlog we assume that the passages are in consecutive order, i.e. there is no other passage between P_i and P_{i+1} for $1 \leq i \leq N-1$. Again the differences of distances between the existing facilities and every pair of two adjacent passages P_n and P_{n+1} are needed. For each half-plane F^i, $i \in \{1,2\}$ and $n = 1, \ldots, N-1$ we define

$$D_n^i(m) := d(Ex_m^i, P_n) - d(Ex_m^i, P_{n+1}), \qquad m \in \mathcal{M}^i.$$

For $n = 1, \ldots, N-1$ let $\pi_n^j : \mathcal{M}^j \to \mathcal{M}^j$ be a permutation of $\mathcal{M}^j$ such that $D_n^j(\pi_n^j(1)) \leq \cdots \leq D_n^j(\pi_n^j(M^j))$. Unfortunately two permutations π_n^j and $\pi_{\tilde{n}}^j$ need not be the same for $n \neq \tilde{n}$. Nevertheless, a shortest path SP from an existing facility $Ex_m^j \in F^j$, $j \in \{1,2\}$, $i \neq j$ in the opposite half-plane to a point $X \in F^i$ has to pass through one of the passages $P_1, \ldots, P_N$ depending on condition (6):

$$\begin{aligned}
P_1 \in SP &\Leftrightarrow & D_1^j(m) < d(P_2, X) - d(P_1, X)\\
P_n \in SP &\Leftrightarrow\ d(P_n, X) - d(P_{n-1}, X) < D_{n-1}^j(m)\ \wedge & D_n^j(m) < d(P_{n+1}, X) - d(P_n, X)\\
P_N \in SP &\Leftrightarrow\ d(P_N, X) - d(P_{N-1}, X) < D_{N-1}^j(m). &
\end{aligned} \tag{6}$$

To take into account the different orderings $\pi_1^j, \ldots, \pi_{N-1}^j$ we define

$$k_n := \operatorname*{argmax}_{m \in \mathcal{M}^j} \left\{0, \pi_n^j(m) \mid D_n^j(\pi_n^j(m)) < d(P_{n+1}, X) - d(P_n, X)\right\}, \qquad n = 1, \ldots, N-1,$$

and $k_N := M^j$. Furthermore, let $\mathcal{M}_1^j := \mathcal{M}^j$ and $\mathcal{M}_n^j := \mathcal{M}_{n-1}^j \setminus \left\{\pi_{n-1}^j(m) \mid \pi_{n-1}^j(m) \leq k_{n-1}\right\}$, $n = 2, \ldots, N$. Now we can rewrite the objective function similar to (5):

$$f(X) = f_{k_1,\ldots,k_N}^i(X) + g_{k_1,\ldots,k_N}^j,$$

where

$$\begin{aligned}
f_{k_1,\ldots,k_N}^i(X) &= \sum_{m=1}^{M^i} w_m^i d(X, Ex_m^i) + \sum_{n=1}^{N} \left(\sum_{\substack{\pi_n^j(m) \in \mathcal{M}_n^j \\ \pi_n^j(m) \leq k_n}} w_{\pi_n^j(m)}^j \right) d(X, P_n)\\
g_{k_1,\ldots,k_N}^j &= \sum_{n=1}^{N} \sum_{\substack{\pi_n^j(m) \in \mathcal{M}_n^j \\ \pi_n^j(m) \leq k_n}} w_{\pi_n^j(m)}^j d(P_n, Ex_{\pi_n^j(m)}^j).
\end{aligned}$$

The unknown parameters in this case are the values of i and of $k_1, \ldots, k_N$. Therefore all possible combinations $i = 1, 2$ and $k_1, \ldots, k_N \in \{0, \ldots, M^j\}$ satisfying $k_1 + \cdots + k_N = M^j$ have to be tested to obtain the globally optimal solution of $1/P/\mathcal{B}_L/d_{\mathcal{B}_L}/\sum$. As in the special case of $N = 2$ this leads to a polynomial time algorithm with a complexity of $O(N(M \log M) + \binom{M+N-1}{N-1}T)$, where $O(NM \log M)$ is the time needed to find the permutations π_n, $n = 1, \ldots, N-1$ and $O(\binom{M+N-1}{N-1})$ is an upper bound on the number of subproblems being solved. Note that with an increasing number of passages N the time complexity of this algorithm grows exponentially whereas it remains polynomial if the number of passages is fixed.

4 Conclusions

The concepts developed in this paper allow the introduction of line barriers with a finite number of passages into the theory of planar location problems. For a broad class of location problems including the Weber problem with the l_p-metric, $p \in [1, \infty]$, the solution of the non-convex Weber problem with line barriers can be reduced to the solution of a polynomial number of unconstrained Weber problems and thus convex optimization problems.

The simultaneous introduction of forbidden regions as well as the consideration of line barriers with a finite positive width are examples of generalizations of the described model that can be easily incorporated. A generalization to higher dimensional problems seems to be more of theoretical than of practical interest. Anyhow a generalization to the case that the barriers are hyper-planes in $\mathbb{R}^n$ which allow trespassing only through a finite set of points is easily possible.

References

[1] Y. P. Aneja and M. Parlar. Algorithms for Weber facility location in the presence of forbidden regions and/or barriers to travel. *Transportation Science*, 28:70–76, 1994.

[2] R. Batta, A. Ghose, and U. S. Palekar. Locating facilities on the Manhatten metric with arbitrarily shaped barriers and convex forbidden regions. *Transportation Science*, 23:26–36, 1989.

[3] S. E. Butt and T. M. Cavalier. An efficient algorithm for facility location in the presence of forbidden regions. *European Journal of Operational Research*, 90:56–70, 1996.

[4] L. R. Foulds and H. W. Hamacher. Optimal bin location in printed circuit board assembly. *European Journal of Operational Research*, 66:279–290, 1993.

[5] R. L. Francis, L. F. McGinnis, and J. A. White. *Facility Layout and Location: An Analytical Approach.* Prentice-Hall, New York, 2nd edition, 1992.

[6] H. W. Hamacher. *Mathematische Lösungsverfahren für planare Standortprobleme.* Vieweg Verlag, Braunschweig, 1995.

[7] H. W. Hamacher and S. Nickel. Combinatorial algorithms for some 1-facility median problems in the plane. *European Journal of Operational Research*, 79:340–351, 1994.

[8] H. W. Hamacher and S. Nickel. Restricted planar location problems and applications. *Naval Research Logistics*, 42:967–992, 1995.

[9] H. W. Hamacher and S. Nickel. Multi-facility and restricted location problems. In *Encyclopedia of Optimization*, 1999. To appear.

[10] H. Juel and R. F. Love. Hull properties in location problems. *European Journal of Operational Research*, 12:262–265, 1983.

[11] I. N. Katz and L. Cooper. Facility location in the presence of forbidden regions, I: Formulation and the case of Euclidean distance with one forbidden circle. *European Journal of Operational Research*, 6:166–173, 1981.

[12] R. C. Larson and G. Sadiq. Facility locations with the Manhattan metric in the presence of barriers to travel. *Operations Research*, 31:652–669, 1983.

[13] R. F. Love, J. G. Morris, and G. O. Wesolowsky. *Facilities Location: Models & Methods*. North Holland, New York, 1988.

[14] F. Plastria. Continuous location problems. In Z. Drezner, editor, *Facility Location*, pages 225–262. Springer Series in Operations Research, 1995.

[15] R. E. Wendell and A. P. Hurter. Location theory, dominance and convexity. *Operations Research*, 21:314–320, 1973.

[16] G. O. Wesolowsky. The Weber problem: History and perspectives. *Location Science*, 1:5–23, 1993.