Algebraic Properties of Location Problems with One Circular Barrier

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Dedicated to Prof. Dr. Rainer Burkard (Graz) on the occasion of his 60^{th} birthday.

Abstract

The consideration of barriers to travel plays an increasingly important role in the transportation and location literature. In one of the classical papers on location problems with barriers, Katz and Cooper (1981) considered the Weber problem (often also referred to as median problem) with one circular barrier region. Considering the same problem we develop new structural results showing that the set of feasible solutions can be subdivided into a polynomial number of cells of algebraic invariance, on every convex subset of which the - generally non-convex - objective function is convex. These results imply improved exact and heuristic solution procedures based on convex optimization methods.

Keywords: *location, barriers to travel*

1 Introduction

Growing transportation costs and tight delivery schedules let good locational decisions be more crucial than ever for the success or failure of industrial as well as public projects. The location of a new warehouse with respect to a given set of customers or the location of an emergency facility in an expanding neighborhood are only two examples for a wide range of applications.

The development of realistic location models is a crucial phase in every locational decision process. Especially in the case of continuous (planar) location models, a geometric representation of the problem is used and the geographical reality has to be incorporated into this representation (see, for example, Love *et al.*, 1988; Plastria, 1995). Restrictions of different types occur in almost every real-world application since there are in general regions to be excluded from the placement of new facilities.

Often, some of these regions can also not be used for transportation purposes. In this case we deal with barrier regions. To give only some examples of possible barrier regions, consider military areas, mountain ranges, lakes, big rivers, inter state highways, or, on smaller scales, machinery and conveyor-belts in an industrial plant. The introduction of barrier regions yields in general non-convex optimization problems so that most of the techniques used in classical location models cannot be applied and new solution strategies are needed.

The increasing interest in location models incorporating restrictions and barrier regions is reflected in the recent literature. A comprehensive overview about the state of the art in continuous location theory incorporating barriers is provided in Klamroth (2002).

Barrier regions were first introduced to location modeling by Katz and Cooper (1981) who considered the Weber problem with one circular barrier and developed a heuristic solution procedure for the case that distances are measured by the Euclidean distance function. Most of the subsequent work concentrated on special barrier shapes and/or special distance functions. In the case that all barrier sets are polyhedra a visibility graph of the demand points and the extreme points of the barrier polyhedra can be constructed. This graph was, for example, used by Aneja and Parlar (1994), Butt (1994) and Butt and Cavalier (1996) to efficiently evaluate the objective function value of the Weber problem at solution points in the context of iterative and/or heuristic algorithms. Also for the case of polyhedral barrier sets, Klamroth (2001a) and Klamroth (2001b) showed that an optimal solution of the non-convex barrier problem can be found by solving a finite and, in the case of line barriers, polynomial number of related unconstrained subproblems; a result that was generalized to the multicriteria case in Klamroth and Wiecek (2002).

From the point of view of special distance functions, rectilinear and, more general, block norm distances played a central role for the development of discretization based solution procedures. Larson and Sadiq (1983) identified an easily determined finite dominating set for rectilinear distances. This result was generalized by Batta *et al.* (1989) who also included forbidden regions into the model, and by Savaş *et al.* (2001) and Wang *et al.* (2002) who located finite size facilities acting as barriers themselves. Similar discretization results were developed by Hamacher and Klamroth (2000) for the Weber problem with general block norm distances and by Dearing *et al.* (2002) for the center problem with rectilinear distances. The computational efficiency of these methods was significantly improved by Segars Jr. (2000) who showed that the consideration of a much smaller dominating set is sufficient to solve the problem. Fekete *et al.* (2000) introduced Weber problems with continuous demand over some given polyhedral set, possibly with holes acting as barriers to travel, and recitilinear distances. Kusakari and Nishizeki (1997), Choi *et al.* (1998) and Ben-Moshe *et al.* (2001) focused on computationally efficient, polynomial solution approaches for specially structured problems based on rectilinear barrier sets and distance functions.

A different approach to handle the non-convexity of the objective function can be seen in the application of general global optimization methods, see, for example, Hansen *et al.* (1995). Krau (1996) generalized the big square small square method (a geometrical Branch and Bound algorithm, see Hansen *et al.* (1985) or Plastria (1992) for details) to handle the Weber problem with polyhedral barrier sets as well as forbidden regions, and Fliege (1997) suggested to model the physical barriers by suitable barrier functions (in the sense of nonlinear optimization).

Based on the work of Katz and Cooper (1981), in this paper we prove algebraic properties for continuous location problems with one circular barrier and Euclidean distances. Even though we mainly restrict ourselves to the case that only one barrier is given in the plane $\mathbb{R}^2$, most of the results of this paper are more general and can be transferred to the case that more than one barrier is given.

Let one circular barrier

$$B_C := \{X \in \mathbb{R}^2 \,:\, \|X\|_{l_2} \leq r\}$$

be given in the plane $\mathbb{R}^2$ that is, without loss of generality, centered at the origin and that has a finite positive radius $r \in \mathbb{R}_+$. B_C represents that region in the plane where neither trespassing nor the location of new facilities is allowed. The feasible region $\mathcal{F}$ for new locations is given by $\mathcal{F} = \mathbb{R}^2 \setminus \mathrm{int}(B_C)$. Furthermore a finite set of existing facilities $\mathcal{E}x = \{Ex_m \in \mathcal{F} \,:\, m \in \mathcal{M}\}$ with the index set $\mathcal{M} = \{1, \ldots, M\}$ is given in the feasible region $\mathcal{F}$. We want to determine the optimal location of one new facility with respect to some location objective that can be modelled by a convex function of the distances between the new and the existing facilities.

If distances are measured by the Euclidean metric l_2 the corresponding barrier distance $l_{2,B_C}(X,Y)$ for two points $X, Y \in \mathcal{F}$ is given by the length of a shortest path between X and Y not intersecting with the interior of B_C. More formally, let P be a *permitted X-Y-path* in $\mathcal{F}$, i.e., a curve connecting X and Y not intersecting with the interior of B_C. Furthermore, let p be a piecewise continuously differentiable parameterization of P, $p : [a,b] \to \mathbb{R}^2$ with $a, b \in \mathbb{R}$, $a < b$, $p(a) = X$, $p(b) = Y$ and $p([a,b]) \cap \mathrm{int}(B_C) = \emptyset$. Then $l_{2,B_C}(X,Y)$ can be defined as

$$l_{2,B_C}(X,Y) := \inf \left\{ \int_a^b \|p'(t)\|_{l_2} \, \mathrm{d}t \,:\, P \text{ permitted } X\text{-}Y\text{-path} \right\}. \tag{1}$$

Note that for l_{2,B_C} the triangle inequality is satisfied but that l_{2,B_C} is in general not positively homogeneous. A permitted X-Y-path with length $l_{2,B_C}(X,Y)$ will be called a *shortest permitted X-Y-path.*

The *location problem with a circular barrier* considered in this paper can now be formulated as

$$\begin{aligned} \min \quad & f_{\mathcal{B}}(X) = f(l_{2,B_C}(X, Ex_1), \ldots, l_{2,B_C}(X, Ex_M)) \\ \text{s.t.} \quad & X \in \mathcal{F}. \end{aligned} \tag{2}$$

We assume that the objective function $f(X) = f(l_2(X, Ex_1), \ldots, l_2(X, Ex_M))$ of the corresponding unconstrained problem is a convex and nondecreasing function of the distances $l_2(X, Ex_1), \ldots, l_2(X, Ex_M)$. This definition includes, for example, the well known Weber- or median objective as well as the Weber-Rawls- or center objective. Despite this convexity assumption for f, the objective function $f_{\mathcal{B}}$ of (2) is in general non-convex due to the non-convexity of the barrier distance l_{2,B_C}.

To simplify further notation we will use the classification $Pos_1/Pos_2/Pos_3/Pos_4/$ Pos_5 of location problems as suggested in Hamacher and Nickel (1998). Following their notation, problem (2) is classified as $1/P/B_C/l_{2,B_C}/f$ *convex*. In this classification, Pos_1 gives the number of new facilities sought (1 for a single-facility problem), Pos_2 denotes the type of location problem (P for planar location problems), Pos_3 contains special assumptions (B_C for one circular barrier), Pos_4 contains the information about the distance function (l_{2,B_C} in case of barrier distances based on the Euclidean metric) and Pos_5 indicates the type of objective function (f *convex* in the given general setting).

Properties of shortest permitted paths in the presence of one circular barrier are discussed in the following section. In Section 3 the convexity of the objective function $f_{\mathcal{B}}$ on certain open convex subsets of $\mathcal{F}$, the *classes of algebraic invariance*, is proven. Algorithmic consequences of these results are presented in Section 4 and the paper is concluded with Section 5.

2 Properties of the Barrier Distance l_{2,B_C}

The aim of this section is to identify subsets of the feasible region $\mathcal{F}$ on which convexity of the barrier distance l_{2,B_C} can be proven. As a first step towards this goal we will discuss some general properties of the barrier distance $l_{2,B_C}(X, Ex)$ between an arbitrary but fixed existing facility $Ex \in \mathcal{E}x$ and a point $X \in \mathcal{F}$.

We call two points X and Y in $\mathcal{F}$ *visible* if they satisfy $l_{2,B_C}(X, Y) = l_2(X, Y)$, i.e., if the straight line segment connecting X and Y does not intersect with $\text{int}(B_C)$. The set of points that are visible from a point $X \in \mathcal{F}$ is denoted by

$$\text{visible}(X) := \{Y \in \mathcal{F} \ : \ l_{2,B_C}(X, Y) = l_2(X, Y)\}.$$

Similarly, we call the set of points $Y \in \mathcal{F}$ that are not visible from a point $X \in \mathcal{F}$ the *shadow* of X, i.e.,

$$\text{shadow}(X) := \{Y \in \mathcal{F} \,:\, l_{2,B_C}(X,Y) > l_2(X,Y)\}.$$

If $X \in \text{visible}(Ex)$ the straight line segment connecting X and Ex is a permitted X-Ex-path and therefore $l_{2,B_C}(X,Ex) = l_2(X,Ex)$ holds in this case, see Figure 1.

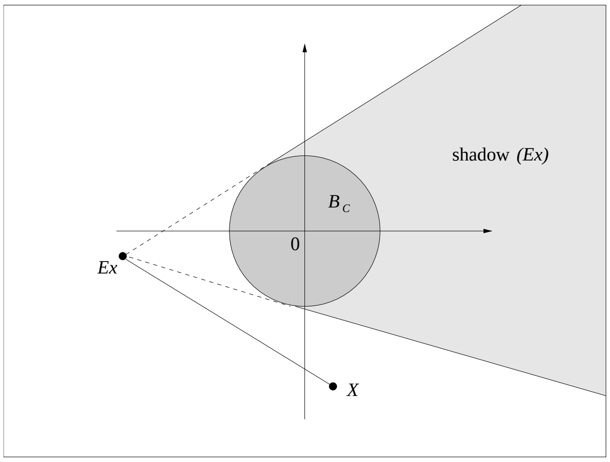

Figure 1: The shadow of an existing facility $Ex \in \mathcal{E}x$.

For $X \in \text{shadow}(Ex)$ it was shown in Smith (1974) that a shortest permitted X-Ex path consists of straight line segments and of circular sections on the boundary $\partial(B_C)$ of the barrier. Moreover it was proven in Elsgolc (1962) (see also Katz and Cooper, 1981) that the straight line segments of an optimal path must be tangent to the boundary $\partial(B_C)$ of the circular barrier at every point of intersection. Since only one circular barrier is given in our case we obtain two candidates for an optimal X-Ex-path as illustrated in Figure 2.

In the following we will refer to the *right point of tangency* with respect to a given point $X \in \mathcal{F}$ as $T_r(X)$ and to the *left point of tangency* as $T_l(X)$, respectively, where "right" and "left" are defined with respect to the half line starting at X and passing through the center of B_C (the origin in our case). Obviously the path through the points of tangency $T_l(Ex)$ and $T_r(X)$ is optimal in the example given in Figure 2. The length of the shortest permitted X-Ex-path in Figure 2 can be calculated as (see Katz and Cooper, 1981)

$$\begin{aligned} l_{2,B_C}(X,Ex) &= l_2(Ex,T_l(Ex)) + 2r \arcsin\left(\frac{l_2(T_l(Ex),T_r(X))}{2r}\right) + l_2(T_r(X),X) \\ &= l_2(Ex,T_l(Ex)) + r\theta + l_2(T_r(X),X) \end{aligned} \tag{3}$$

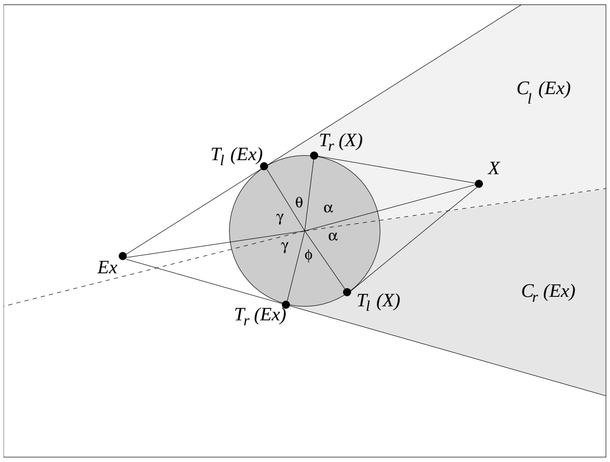

Figure 2: Two candidates for a shortest permitted X-Ex-path.

where θ is the angle (in radians) enclosed by the two line segments between the origin and the two points of tangency $T_l(Ex)$ and $T_r(X)$.

Lemma 1. *Let $Ex \in \mathcal{E}x$ be an existing facility and let $X \in \mathcal{F}$. Then*

$$l_{2,B_C}(X, Ex) = \begin{cases} l_2(X, Ex) & \text{if } X \in \text{visible}(Ex) \\ l_2(Ex, T(Ex)) + r\theta + l_2(T(X), X) & \text{if } X \in \text{shadow}(Ex), \end{cases}$$

where $(T(Ex), T(X)) \in \{(T_r(Ex), T_l(X)), (T_l(Ex), T_r(X))\}$ such that the value of $l_{2,B_C}(X, Ex)$ is minimal.

In the example depicted in Figure 3 the points of tangency $T_r(X) = (t_1^r, t_2^r)^T$ and $T_l(X) = (t_1^l, t_2^l)^T$ for a given point $X \in \mathcal{F}$ can be obtained using the angles α and β in the following way: Let $X = (x_1, x_2)^T$ and let $\| X \| := \| X \|_{l_2} = \sqrt{x_1^2 + x_2^2}$ denote the Euclidean norm of X. Then the angles α and β are given by

$$\begin{aligned} \cos\alpha &= \frac{r}{\| X \|}, & \sin\alpha &= \frac{\sqrt{\| X \|^2 - r^2}}{\| X \|}, \\ \cos\beta &= \frac{x_1}{\| X \|}, & \sin\beta &= \frac{x_2}{\| X \|}. \end{aligned}$$

Depending on whether we want to find the right point of tangency $T_r(X)$ or the left point of tangency $T_l(X)$, the values of the coordinates t_1 and t_2 of the respective point of tangency are given by (see Katz and Cooper, 1981)

$$\begin{aligned} t_1^{r,l} &= r\,\cos(\beta \pm \alpha) = r\,(\cos\alpha\cos\beta \mp \sin\alpha\sin\beta)\,, \\ t_2^{r,l} &= r\,\sin(\beta \pm \alpha) = r\,(\cos\alpha\sin\beta \pm \sin\alpha\cos\beta)\,. \end{aligned}$$

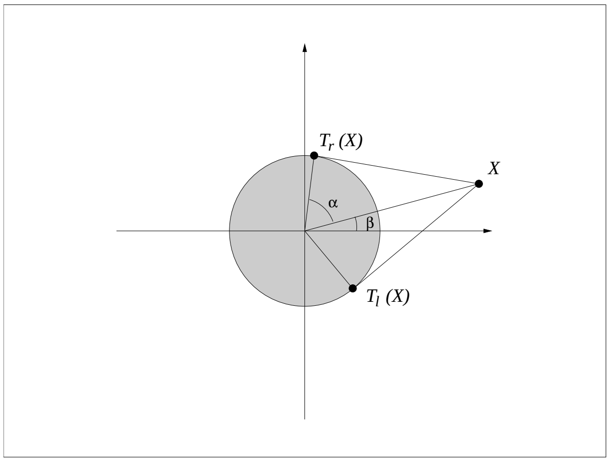

Figure 3: Determination of the points of tangency $T_r(X)$ and $T_l(X)$.

Summarizing the discussion above we obtain the following representation of the right and left point of tangency, respectively:

$$
\begin{aligned}
T_r(X) &= \left(\frac{r^2 x_1 - r x_2 \sqrt{x_1^2 + x_2^2 - r^2}}{x_1^2 + x_2^2} \, , \, \frac{r^2 x_2 + r x_1 \sqrt{x_1^2 + x_2^2 - r^2}}{x_1^2 + x_2^2} \right)^T \\
T_l(X) &= \left(\frac{r^2 x_1 + r x_2 \sqrt{x_1^2 + x_2^2 - r^2}}{x_1^2 + x_2^2} \, , \, \frac{r^2 x_2 - r x_1 \sqrt{x_1^2 + x_2^2 - r^2}}{x_1^2 + x_2^2} \right)^T .
\end{aligned}
\tag{4}
$$

Given a point $X \in \text{shadow}(Ex)$ as depicted in Figure 2, a shortest permitted X-Ex-path passes through the points of tangency $T_l(Ex), T_r(X)$ or through $T_r(Ex), T_l(X)$, depending on the relative magnitude of the angles θ and ϕ. For every existing facility $Ex \in \mathcal{E}x$, let a bisection line $(0 - Ex)_{B_C}$ be defined as

$$(0 - Ex)_{B_C} := \{-\lambda Ex \ : \ \lambda \in \mathbb{R}_+; \ \lambda \| Ex \| \geq r\}. \tag{5}$$

Then the bisection line $(0 - Ex)_{B_C}$ decomposes the set shadow(Ex) into two cells $C_r(Ex)$ and $C_l(Ex)$ (see Figure 2) so that for every point $X \in C_r(Ex)$ a shortest permitted X-Ex-path passes through $T_r(Ex), T_l(X)$ and for every point $X \in C_l(Ex)$, a shortest permitted X-Ex-path passes through $T_l(Ex), T_r(X)$. Observe that for all points on the bisection line $(0 - Ex)_{B_C}$ both paths have the same length, i.e., a shortest permitted X-Ex-path may pass along either side of the barrier.

Even though the points $T_r(Ex)$ and $T_l(Ex)$ only depend on the fixed location of the existing facility $Ex \in \mathcal{E}x$ and are therefore also fixed, the locations of the points of tangency $T_r(X)$ and $T_l(X)$ depend on the coordinates of X and change when X is moved around in the set shadow(Ex). This fact complicates the computation of the barrier distance in the case of round shaped barrier sets compared to polyhedral barrier sets where constant intermediate points and a visibility graph can be used to describe the barrier distance (see, for example, Klamroth, 2001b). However, the following result proves that the barrier distance $l_{2,B_C}(Ex, X)$ is nevertheless convex on every open convex subset of a cell $C_r(Ex)$ or $C_l(Ex)$, respectively.

Theorem 1. *Let $Ex \in \mathcal{E}x$ be an existing facility and let $C_r(Ex)$ and $C_l(Ex)$ be the two cells obtained from subdividing the set* shadow(Ex) *with the bisection line $(0 - Ex)_{B_C}$. Then $l_{2,B_C}(X, Ex)$ is a convex function of X on every open convex set O satisfying $O \subset C_r(Ex)$ or $O \subset C_l(Ex)$.*

Proof. To simplify further notation we will assume in the following that without loss of generality $r = 1$ and that the existing facility $Ex \in \mathcal{E}x$ is located in the fourth orthant of the Cartesian coordinate system such that $Ex = (a_1, a_2)^T$ with $a_1 = 1$ and $a_2 < 0$. Thus the right point of tangency $T_r(Ex)$ to the circle is given by $T_r(Ex) = (1, 0)^T$, see Figure 4. Due to the symmetry of the problem it is sufficient to prove the convexity of $l_{2,B_C}(X, Ex)$ on every open convex subset O of the cell $C_r(Ex)$.

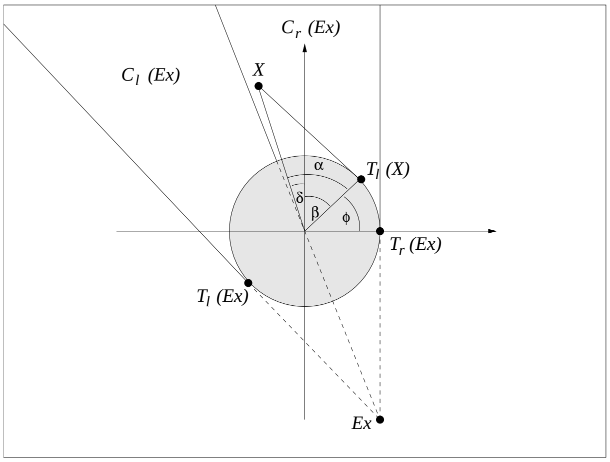

Figure 4: The special situation considered in the proof of Theorem 1.

Under these assumptions the distance $l_{2,B_C}(X, Ex)$ between an arbitrary point $X = (x_1, x_2)^T \in C_r(Ex)$ and the existing facility $Ex = (1, a_2)^T$ can be calculated as

$$\begin{aligned} l_{2,B_C}(X, Ex) &= l_2(X, T_l(X)) + r\phi + l_2(T_r(Ex), Ex) \\ &= g_1(X) + g_2(X) + |a_2| \\ &=: g(X), \end{aligned}$$

where $l_2(T_r(Ex), Ex) = |a_2| = -a_2$ is a constant not depending on X. The functions

$g_1(X)$ and $g_2(X)$ are given by

$$\begin{aligned} g_1(X) &:= l_2(X, T_l(X)) = \sqrt{\| X \|^2 - r^2} = \sqrt{x_1^2 + x_2^2 - 1} \\ g_2(X) &:= r\phi = \phi. \end{aligned}$$

The angle ϕ can be obtained as

$$\phi = \frac{\pi}{2} - \left(\arccos \frac{1}{\| X \|} + \arctan \frac{x_1}{x_2} \right),$$

(see Figure 4), since $\cos \alpha = \frac{1}{\|X\|}$, $\tan \delta = \frac{x_1}{x_2}$ and $\operatorname{sgn}(\arctan(\frac{x_1}{x_2})) = \operatorname{sgn}(x_1)$. Therefore

$$g_2(X) = \frac{\pi}{2} - \arccos \left(\frac{1}{\sqrt{x_1^2 + x_2^2}} \right) - \arctan \left(\frac{x_1}{x_2} \right).$$

Let $O \subset C_r(Ex)$ be an open convex set. Then $l_{2,B_C}(X, Ex) = g(X)$ is a convex function of X on the set O if and only if

$$(Y - X)^T \cdot (\nabla g(Y) - \nabla g(X)) \geq 0 \qquad \forall X, Y \in O$$

(see, for example, Rockafellar and Wets, 1998). The gradient of g satisfies $\nabla g(X) = \nabla g_1(X) + \nabla g_2(X)$, and the partial derivatives of g_1 and g_2 can be calculated as

$$\begin{aligned} \frac{\partial g_1}{\partial x_1} &= \frac{x_1}{\sqrt{x_1^2 + x_2^2 - 1}} \\ \frac{\partial g_1}{\partial x_2} &= \frac{x_2}{\sqrt{x_1^2 + x_2^2 - 1}} \end{aligned}$$

$$\begin{aligned} \frac{\partial g_2}{\partial x_1} &= \frac{1}{\sqrt{1 - \frac{1}{x_1^2 + x_2^2}}} \cdot \left(-\frac{1}{2} (x_1^2 + x_2^2)^{-\frac{3}{2}} \right) \cdot 2x_1 - \frac{1}{1 + \frac{x_1^2}{x_2^2}} \cdot \frac{1}{x_2} \\ &= \frac{-x_1}{\sqrt{\frac{x_1^2 + x_2^2 - 1}{x_1^2 + x_2^2}} \cdot (x_1^2 + x_2^2)^{\frac{3}{2}}} - \frac{x_2}{x_1^2 + x_2^2} \\ &= \frac{-x_1}{\sqrt{x_1^2 + x_2^2 - 1}\,(x_1^2 + x_2^2)} - \frac{x_2}{x_1^2 + x_2^2} \end{aligned}$$

$$\begin{aligned} \frac{\partial g_2}{\partial x_2} &= \frac{1}{\sqrt{1 - \frac{1}{x_1^2 + x_2^2}}} \cdot \left(-\frac{1}{2} (x_1^2 + x_2^2)^{-\frac{3}{2}} \right) \cdot 2x_2 - \frac{1}{1 + \frac{x_1^2}{x_2^2}} \cdot \left(-\frac{x_1}{x_2^2} \right) \\ &= \frac{-x_2}{\sqrt{x_1^2 + x_2^2 - 1}\,(x_1^2 + x_2^2)} + \frac{x_1}{x_1^2 + x_2^2}. \end{aligned}$$

Observe that $x_1^2 + x_2^2 \geq 1$ for all $X \in O$ since otherwise X would be located in the interior of the barrier $B_C = \{X \in \mathbb{R}^2 : \| X \| \leq 1\}$. For two points $X = (x_1, x_2)^T$ and $Y = (y_1, y_2)^T$ in O, the scalar product $(Y - X)^T \cdot (\nabla g(Y) - \nabla g(X))$ can now be evaluated as

$$
\begin{aligned}
&(Y - X)^T \cdot (\nabla g(Y) - \nabla g(X)) \\
&= (y_1 - x_1, y_2 - x_2) \cdot \begin{pmatrix} \frac{y_1(y_1^2+y_2^2)-y_1}{\sqrt{y_1^2+y_2^2-1}\,(y_1^2+y_2^2)} - \frac{y_2}{y_1^2+y_2^2} - \frac{x_1(x_1^2+x_2^2)-x_1}{\sqrt{x_1^2+x_2^2-1}\,(x_1^2+x_2^2)} + \frac{x_2}{x_1^2+x_2^2} \\ \frac{y_2(y_1^2+y_2^2)-y_2}{\sqrt{y_1^2+y_2^2-1}\,(y_1^2+y_2^2)} + \frac{y_1}{y_1^2+y_2^2} - \frac{x_2(x_1^2+x_2^2)-x_2}{\sqrt{x_1^2+x_2^2-1}\,(x_1^2+x_2^2)} - \frac{x_1}{x_1^2+x_2^2} \end{pmatrix} \\
&= \frac{1}{(y_1^2+y_2^2)\sqrt{y_1^2+y_2^2-1}} \Big(y_1^2(y_1^2+y_2^2) - x_1y_1(y_1^2+y_2^2) - y_1^2 + x_1y_1 \\
&\qquad + y_2^2(y_1^2+y_2^2) - x_2y_2(y_1^2+y_2^2) - y_2^2 + x_2y_2 \\
&\qquad + \sqrt{y_1^2+y_2^2-1}\,(-y_1y_2 + x_1y_2 + y_1y_2 - x_2y_1) \Big) \\
&\quad + \frac{1}{(x_1^2+x_2^2)\sqrt{x_1^2+x_2^2-1}} \Big(-x_1y_1(x_1^2+x_2^2) + x_1^2(x_1^2+x_2^2) + x_1y_1 - x_1^2 \\
&\qquad - x_2y_2(x_1^2+x_2^2) + x_2^2(x_1^2+x_2^2) + x_2y_2 - x_2^2 \\
&\qquad + \sqrt{x_1^2+x_2^2-1}\,(x_2y_1 - x_1x_2 - x_1y_2 + x_1x_2) \Big) \\
&= \frac{1}{(y_1^2+y_2^2)\sqrt{y_1^2+y_2^2-1}} \Big((y_1^2+y_2^2-1)(y_1^2+y_2^2-x_1y_1-x_2y_2) \\
&\qquad + \sqrt{y_1^2+y_2^2-1}\,(x_1y_2 - x_2y_1) \Big) \\
&\quad + \frac{1}{(x_1^2+x_2^2)\sqrt{x_1^2+x_2^2-1}} \Big((x_1^2+x_2^2-1)(x_1^2+x_2^2-x_1y_1-x_2y_2) \\
&\qquad + \sqrt{x_1^2+x_2^2-1}\,(-x_1y_2 + x_2y_1) \Big) \\
&= \frac{1}{y_1^2+y_2^2} \left(\sqrt{y_1^2+y_2^2-1}\,(y_1^2+y_2^2-x_1y_1-x_2y_2) + x_1y_2 - x_2y_1 \right) \\
&\quad + \frac{1}{x_1^2+x_2^2} \left(\sqrt{x_1^2+x_2^2-1}\,(x_1^2+x_2^2-x_1y_1-x_2y_2) - x_1y_2 + x_2y_1 \right).
\end{aligned}
$$

This expression is nonnegative for all X and Y in O satisfying $\| X \| = \| Y \|$ since in this case

$$
\begin{aligned}
&(Y-X)^T \cdot (\nabla g(Y) - \nabla g(X)) \\
&= \frac{\sqrt{x_1^2+x_2^2-1}}{x_1^2+x_2^2}\left(y_1^2+y_2^2-x_1y_1-x_2y_2+x_1^2+x_2^2-x_1y_1-x_2y_2\right) \\
&= \frac{\sqrt{x_1^2+x_2^2-1}}{x_1^2+x_2^2}\left((x_1-y_1)^2+(x_2-y_2)^2\right) \;\geq\; 0.
\end{aligned}
$$

If $\| X \| \neq \| Y \|$, a more sophisticated analysis of the above system is needed. In the following, let

$$x := x_1^2+x_2^2 \qquad \text{and} \qquad y := y_1^2+y_2^2.$$

Recall that $x \geq 1$ and $y \geq 1$ since X and Y are not located in the interior of B_C. Moreover,

$$x_1y_1+x_2y_2 = (x_1,x_2)\cdot(y_1,y_2)^T = \sqrt{x}\sqrt{y}\cos\rho$$

and

$$x_1y_2-x_2y_1 = \sqrt{x}\sqrt{y}\sin\rho,$$

where ρ is the angle enclosed by the two vectors X and Y (see Bronstein and Semendjajew, 1985). Substituting this in the above formula we obtain

$$
\begin{aligned}
&(Y-X)^T \cdot (\nabla g(Y) - \nabla g(X)) \\
&= \frac{1}{y}\left(\sqrt{y-1}\;\left(y-\sqrt{x}\sqrt{y}\cos\rho\right)+\sqrt{x}\sqrt{y}\sin\rho\right) \\
&\quad +\frac{1}{x}\left(\sqrt{x-1}\;\left(x-\sqrt{x}\sqrt{y}\cos\rho\right)-\sqrt{x}\sqrt{y}\sin\rho\right).
\end{aligned}
$$

This implies that $(Y-X)^T \cdot (\nabla g(Y)-\nabla g(X)) \geq 0$ if and only if

$$\sqrt{x-1}+\sqrt{y-1} \geq \sqrt{x}\sqrt{y}\left(\left(\frac{\sqrt{y-1}}{y}+\frac{\sqrt{x-1}}{x}\right)\cos\rho+\left(\frac{1}{x}-\frac{1}{y}\right)\sin\rho\right)$$

which is equivalent to

$$\sqrt{x}\sqrt{y}\left(\sqrt{x-1}+\sqrt{y-1}\right) \geq \cos\rho\left(x\sqrt{y-1}+y\sqrt{x-1}\right)+\sin\rho\,(y-x). \tag{6}$$

Obviously the left-hand side of (6) is nonnegative since $x, y \geq 1$. If the right-hand side of (6) is negative, then (6) is trivially satisfied. Otherwise both sides of the inequality can be squared, implying the equivalent expression

$$xy\left(\sqrt{x-1}+\sqrt{y-1}\right)^2 \geq \left(\cos\rho\left(x\sqrt{y-1}+y\sqrt{x-1}\right)+\sin\rho\,(y-x)\right)^2. \tag{7}$$

At this point Schwarz's inequality given by $\left(\sum_{i=1}^n a_i b_i\right)^2 \leq \left(\sum_{i=1}^n a_i^2\right)\left(\sum_{i=1}^n b_i^2\right)$ for $a_i, b_i \in \mathbb{R}$, $i = 1, \ldots, n$ (see Bronstein and Semendjajew, 1985) can be applied to the right-hand side of (7), yielding that

$$\begin{aligned}
&\left(\cos^2\rho + \sin^2\rho\right) \cdot \left(\left(x\sqrt{y-1} + y\sqrt{x-1}\right)^2 + (y-x)^2\right) \\
&\geq \quad \left(\cos\rho\left(x\sqrt{y-1} + y\sqrt{x-1}\right) + \sin\rho(y-x)\right)^2.
\end{aligned}$$

Using this inequality to bound the right-hand side of (7) and using the fact that $\cos^2\rho + \sin^2\rho = 1$ we can conclude that (7) is satisfied if

$$xy\left(\sqrt{x-1} + \sqrt{y-1}\right)^2 \geq \left(x\sqrt{y-1} + y\sqrt{x-1}\right)^2 + (y-x)^2,$$

or, equivalently, if

$$\begin{aligned}
&xy\left(x + y - 2 + 2\sqrt{x-1}\sqrt{y-1}\right) \\
&\quad \geq \quad x^2(y-1) + y^2(x-1) + 2xy\sqrt{x-1}\sqrt{y-1} + x^2 - 2xy + y^2.
\end{aligned}$$

An easy calculation proves that both sides of this last expression are equal and thus the inequality is satisfied for all $x, y \geq 1$. □

3 Cells of Algebraic Invariance

In Theorem 1 we have shown that for a single existing facility $Ex \in \mathcal{E}x$ and one circular barrier the barrier distance $l_{2,B_C}(X, Ex)$ is a convex function of X on every open convex subset of $C_r(Ex)$ and of $C_l(Ex)$. Moreover, $l_{2,B_C}(X, Ex) = l_2(X, Ex)$ is a convex function of X on every open convex subset of $\mathcal{F} \setminus \text{shadow}(Ex)$. In the following several existing facilities will be combined and the objective function $f_{\mathcal{B}}(X)$ is analyzed over suitable subsets of $\mathcal{F}$.

Consider the grid $\mathcal{G}_{l_2}(B_C)$ defined by the boundaries of the shadows of all existing facilities, the bisection lines $(0 - Ex)_{B_C}$, plus the boundary of the barrier region B_C:

$$\mathcal{G}_{l_2}(B_C) := \bigcup_{Ex_m \in \mathcal{E}x} \left(\partial(\text{shadow}(Ex_m)) \cup (0 - Ex_m)_{B_C}\right) \cup \partial(B_C).$$

The set $\mathcal{C}(\mathcal{G}_{l_2}(B_C))$ of cells induced by the grid $\mathcal{G}_{l_2}(B_C)$ is defined as the set of those (largest) subsets of $\mathcal{F}$ that are bounded by curves or line segments of $\mathcal{G}_{l_2}(B_C)$ and the interior of which is not intersected by any curve or line segment of $\mathcal{C}(\mathcal{G}_{l_2}(B_C))$, see Figure 5. Each cell of $C \in \mathcal{C}(\mathcal{G}_d)$ can be interpreted as a *cell of algebraic invariance* (CAI) since for any point $X \in C$ the set $\mathcal{E}x \cap \text{visible}(X)$ of existing facilities visible

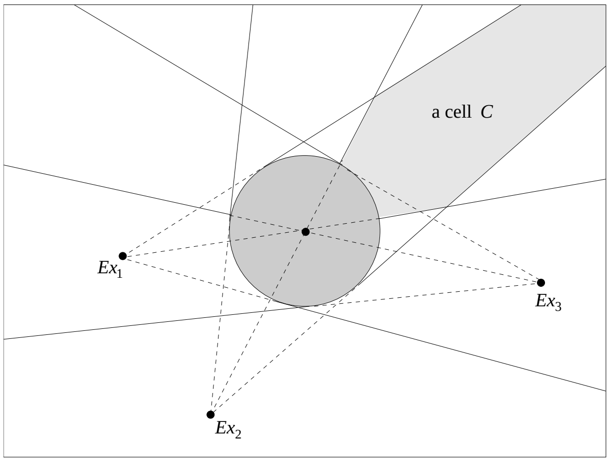

Figure 5: The grid $\mathcal{G}_{l_2}(B_C)$.

from X is identical, and for all nonvisible existing facilities in $\mathcal{E}x \cap \text{shadow}(X)$ either a right point of tangency $T_r(X)$ or a left point of tangency $T_l(X)$ is used throughout the cell on the shortest permitted path connecting X and the respective facility.

Lemma 2. *The overall number $|\mathcal{C}(\mathcal{G}_{l_2}(B_C))|$ of CAI's is $O(M^2)$, where $M = |\mathcal{E}x|$ denotes the total number of existing facilities.*

Proof. The value of $|\mathcal{C}(\mathcal{G}_{l_2}(B_C))|$ can be bounded in the following way: Every existing facility $Ex \in \mathcal{E}x$ generates three half lines that are part of the grid $\mathcal{G}_{l_2}(B_C)$, namely two half lines starting at the left and right point of tangency, respectively, bounding the shadow of Ex, plus the bisection line $(0 - Ex)_{B_C}$. The half lines induced by the existing facilities in $\mathcal{E}x$ can intersect in at most $O(M^2)$ intersection points, and the boundary of B_C is intersected by these half lines in not more than $O(M)$ different points. We construct a planar graph by approximating $\partial(B_C)$ by line segments between adjacent intersection points of $\partial(B_C)$ with the half lines building the grid $\mathcal{G}_{l_2}(B_C)$. If we define a vertex at each intersection point of line segments in $\mathcal{G}_{l_2}(B_C)$, the number of vertices is $O(M^2)$ and analogously the number of edges is $O(M^2)$. The number of CAI's in $\mathcal{C}(\mathcal{G}_{l_2}(B_C))$ can now be estimated using Euler's formula for planar graphs: The number of vertices minus the number of edges plus the number of cells of every planar graph equals 2 (see, for example, Harary, 1969). This implies that the maximal number of CAI's in $\mathcal{C}(\mathcal{G}_{l_2}(B_C))$ is $O(M^2) = O(|\mathcal{E}x|^2)$. $\square$

For the construction of the grid $\mathcal{G}_{l_2}(B_C)$ we first have to determine the points of tangency $T_r(Ex_m)$ and $T_l(Ex_m)$ for all existing facilities $Ex_m \in \mathcal{E}x$ according to equations (4). Then the half lines defining the boundary of the shadow of an existing facility $Ex_m \in \mathcal{E}x$ can be found as $\{X \in \mathbb{R}^2 : X = T_r(Ex_m) + \lambda(T_r(Ex_m) - Ex_m),\ \lambda \geq 0\}$ and $\{X \in \mathbb{R}^2 : X = T_l(Ex_m) + \lambda(T_l(Ex_m) - Ex_m),\ \lambda \geq 0\}$, respectively.

The bisection lines $(0 - Ex_m)_{B_C}$ are half lines that can be easily calculated according to equation (5) for all $m = 1, \ldots, M$. In order to obtain a planar graph representing $\mathcal{G}_{l_2}(B_C)$ and having a vertex at every intersection point of line segments in $\mathcal{G}_{l_2}(B_C)$ (the boundary of B_C is again approximated by a piecewise linear curve), the intersection points of the $O(M)$ line segments of $\mathcal{G}_{l_2}(B_C)$ can be determined by an algorithm of Chazelle and Edelsbrunner (1992) in optimal $O(M \log M + k)$ time, where k denotes the total number of intersection points. If, in the worst case, $k = O(M^2)$ this implies a running time of the algorithm of $O(M^2)$. Alternatively, either a randomized algorithm as suggested by Clarkson and Shor (1989) with the same expected running time but improved space requirements could be used to find the intersection points of all line segments in $\mathcal{G}_{l_2}(B_C)$, or the sweep-line algorithm of Bentley and Ottmann (1979) with a running time of $O((M + k) \log M)$ could be utilized.

In the following we will show that the grid $\mathcal{G}_{l_2}(B_C)$ decomposes the feasible region into a finite number of CAI's which have another desirable property: The objective function of any problem of type $1/P/B_C/l_{2,B_C}/f\ convex$ is convex on every open convex subset of a CAI in $\mathcal{C}(\mathcal{G}_{l_2}(B_C))$.

Theorem 2. *Let $C \in \mathcal{C}(\mathcal{G}_{l_2}(B_C))$ be a cell of the grid $\mathcal{G}_{l_2}(B_C)$ for a problem of the type $1/P/B_C/l_{2,B_C}/f\ convex$. Then the objective function $f_{\mathcal{B}}(X)$ of the problem is convex on every open convex subset of the cell C.*

Proof. Let $C \in \mathcal{C}(\mathcal{G}_{l_2}(B_C))$ be an arbitrary cell and let $O \subset C$ be an open convex subset of C.

We first show that, as a consequence of Theorem 1, the distance $l_{2,B_C}(X, Ex_m)$ between a point $X \in O$ and every existing facility $Ex_m \in \mathcal{E}x$, $m = 1, \ldots, M$ is a convex function of X on O: Since C is a cell in $\mathcal{C}(\mathcal{G}_{l_2}(B_C))$, the interior of C is neither intersected by the boundary of the shadow of Ex nor by the bisection line $(0-Ex)_{B_C}$. Therefore Ex is either visible from all points in $O \subset C$ which immediately implies the convexity of $l_{2,B_C}(X, Ex)$ on O, or C is a subset of one of the sets $C_r(Ex)$ or $C_l(Ex)$ of the shadow of Ex (see Figure 4) and the convexity of $l_{2,B_C}(X, Ex)$ on O follows from Theorem 1.

Hence

$$\begin{aligned} f_{\mathcal{B}} &= f(l_{2,B_C}(X, Ex_1), \ldots, l_{2,B_C}(X, Ex_M)) \\ &= f(\varphi_1, \ldots, \varphi_M) \end{aligned}$$

where $\varphi_m : \mathbb{R}^2 \to \mathbb{R}$ with $\varphi_m(X) := l_{2,B_C}(X, Ex_m)$ are convex functions of X on O for every $m = 1, \ldots, M$. Therefore also the composition $f_{\mathcal{B}}$ of the convex and nondecreasing function f and the convex functions $\varphi_1, \ldots, \varphi_M$ is a convex function of X on O (see, for example, Rockafellar and Wets, 1998). $\square$

Theorem 2 is illustrated using a *Weber problem with a circular barrier* $1/P/B_C/l_{2,B_C}/\sum$ as an example. In this case a positive weight $w_m = w(Ex_m)$ is identified with every existing facility $Ex_m \in \mathcal{E}x$ representing the demand of the facility Ex_m, and problem $1/P/B_C/l_{2,B_C}/\sum$ is given by

$$\begin{array}{ll} \min & f_{\mathcal{B}}(X) = \sum_{m\in\mathcal{M}} w_m l_{2,B_C}(X, Ex_m) \\ \text{s.t.} & X \in \mathcal{F}. \end{array} \tag{8}$$

The level sets

$$L_{\leq}(z, f_{\mathcal{B}}) := \{X \in \mathcal{F} \ : \ f_{\mathcal{B}}(X) \leq z\}$$

of an example problem with the Weber objective function (8) and with four existing facilities having equal weights $w_m = 1$, $m = 1, \ldots, 4$ are depicted in Figure 6. Since $f_{\mathcal{B}}$ is a convex function on every open convex subset O of a cell $C \in \mathcal{C}(\mathcal{G}_{l_2}(B_C))$, the intersection of a level set $L_{\leq}(z, f_{\mathcal{B}})$ with an open convex set $O \subseteq C$ is a convex set for all $z \in \mathbb{R}$.

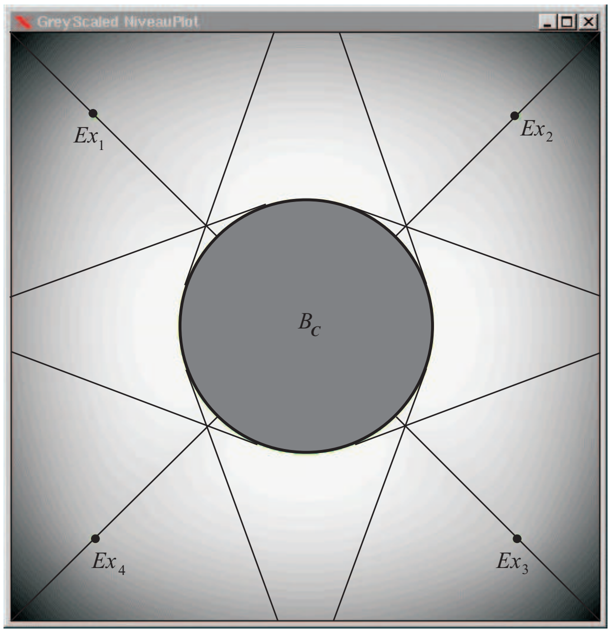

Figure 6: Level sets of a Weber problem (8) with four existing facilities having equal weights.

4 Algorithms and Heuristics

The convexity of the objective function on all open convex subsets of CAI's allows the development of efficient algorithms for the solution of problems of the type $1/P/B_C/l_{2,B_C}/f$ *convex*. The general idea for solving $1/P/B_C/l_{2,B_C}/f$ *convex* can be summarized as follows:

First the grid $\mathcal{G}_{l_2}(B_C)$ is constructed and all of the at most $O(M^2) = O(|\mathcal{E}x|^2)$ CAI's are identified. Then the objective function $f_{\mathcal{B}}$ is minimized over each CAI $C \in \mathcal{C}(\mathcal{G}_{l_2}(B_C))$ by adapting some available method for convex optimization problems, see, for example, the textbooks of Bazaraa *et al.* (1993) and Polak (1997). A global minimum of $1/P/B_C/l_{2,B_C}/f$ *convex* can be calculated as the minimum of all these subproblems solved on the individual cells.

Algorithm 1 (Algorithm for solving $\boldsymbol{1/P/B_C/l_{2,B_C}/f}$ *convex*).

Input: *Location problem* $1/P/B_C/l_{2,B_C}/f$ *convex.*

Step 1: *Construct the grid* $\mathcal{G}_{l_2}(B_C)$.

Step 2: *For all CAI's* $C_k \in \mathcal{C}(\mathcal{G}_{l_2}(B_C))$, $k = 1, \ldots, K$ *do:*

Find an optimal solution (or an approximation of the optimal solution) X_k^* *of the subproblem*

$$\min\{f_{\mathcal{B}}(X) \,:\, X \in C_k\} \tag{9}$$

by adapting an available method for convex optimization problems.

Step 3: *Determine an optimal solution*

$$X_{B_C}^* := \operatorname*{argmin}_{k=1,\ldots,K} f_{\mathcal{B}}(X_k^*).$$

Output: *Optimal solution (or its approximation)* $X_{B_C}^*$ *of* $1/P/B_C/l_{2,B_C}/f$ *convex.*

Since the number K of CAI's is $O(M^2)$ where $M = |\mathcal{E}x|$ denotes the total number of existing facilities (cf. Lemma 2), the time complexity of Algorithm 1 can be estimated as $O(M^2 \cdot T)$ where $O(T)$ is the time complexity for solving the subproblems (9). Note that the time complexity of constructing the grid $\mathcal{G}_{l_2}(B_C)$ in Step 1 of the algorithm is dominated by the time complexity of Step 2.

Algorithm 1 was implemented in the diploma thesis of Ochs (1998). In this implementation the subproblems are solved adapting the method of Hooke and Jeeves (see Hooke and Jeeves, 1961). To compare a special case of Algorithm 1 with the results of Katz and Cooper (1981) for Weber problems with a circular barrier, a reference problem introduced in Katz and Cooper (1981) was used. In this problem, five existing facilities with weights 1 are given at the coordinates $Ex_1 = (-8.0, -6.0)^T$, $Ex_2 = (-7.0, 13.0)^T$, $Ex_3 = (-1.0, -5.0)^T$, $Ex_4 = (6.6, -0.5)^T$, $Ex_5 = (4.4, 10.0)^T$, and one circular barrier with radius 2 centered at the origin $(0,0)^T$ is located within the considered region. The optimal solution was approximated by Algorithm 1 at the point $X_1 = (-1.18602, 2.06044)^T$ with an objective value of $z_1 = 48.2548$. This result slightly improves the solution $X_2 = (-1.2016, 2.0776)^T$ with $z_2 = 48.2560$ as found in Butt and Cavalier (1996) who approximated the circular barrier by a polyhedral set and used a heuristic algorithm that iteratively solves unconstrained location problems. The solution determined in the original work of Katz and Cooper (1981) at the point $X_3 = (-0.08130, 2.4833)^T$ with an objective value of $z_3 = 48.3524$ turned out to be a local minimum located relatively far from the approximated optimum at the point $X_1 = (-1.18602, 2.06044)^T$.

Besides the exact solution procedure given in Algorithm 1 the following heuristic approach was implemented in Ochs (1998): Like in Algorithm 1, the grid $\mathcal{G}_{l_2}(B_C)$ is constructed in a first step. The planar graph representing $\mathcal{G}_{l_2}(B_C)$ with vertices at all intersection points of line segments in $\mathcal{G}_{l_2}(B_C)$ is determined as discussed above. However, different from Algorithm 1 not all the cells are investigated during the following heuristic procedure. Instead, the objective value is first determined only at the at most $O(M^2)$ vertices of the graph representing $\mathcal{G}_{l_2}(B_C)$. These vertices are then stored in a list which is sorted according to their objective values. For a given percentage $p \in (0, 100)$, the best $p\%$ of vertices of this list are selected for further investigation: The selected points are used as starting points for the method of Hooke and Jeeves applied to the minimization problem

$$\begin{aligned} \min \quad & f_{\mathcal{B}}(X) \\ \text{s.t.} \quad & X \in \mathcal{F}, \end{aligned}$$

where the constraint to CAI's as imposed in Algorithm 1 is relaxed to the weaker feasibility constraint $X \in \mathcal{F}$. Therefore several CAI's may be visited during one application of the method of Hooke and Jeeves. Moreover, the objective function $f_{\mathcal{B}}$ is in general non-convex on the whole feasible region $\mathcal{F}$. Since the objective function is convex on every open convex subset of a CAI we may expect the method of Hooke and Jeeves to converge to a minimum in a CAI that contains a global minimum of $1/P/B_C/l_{2,B_C}/f\ convex$. Therefore we stop the minimization of a subproblem whenever a CAI is reached during the procedure which was already visited in an earlier iteration and thus a minimum was already sought starting from this CAI. In this case we proceed with the next selected point and use it as starting point for a new application of the method of Hooke and Jeeves.

Algorithm 2 (Heuristic algorithm to solve $1/P/B_C/l_{2,B_C}/f$ *convex*).

> ***Input:*** *Location problem* $1/P/B_C/l_{2,B_C}/f$ *convex; parameter* $p \in (0,100)$ *specifying the percentage of starting points considered.*
>
> *Step 1:* *Construct the grid* $\mathcal{G}_{l_2}(B_C)$ *and the corresponding planar graph representing* $\mathcal{G}_{l_2}(B_C)$.
>
> *Step 2:* *Determine the objective value for all intersection points in* $\mathcal{G}_{l_2}(B_C)$ *and store these points in a list sorted according to their objective values. Select the best* p *percent of points in this list for further investigation.*
>
> *Step 3:* *For all points* Y *in the list of starting points determined in Step 2, do:*
>
> *Apply the method of Hooke and Jeeves to the problem*
>
> $$\min\{f_{\mathcal{B}}(X) \,:\, X \in \mathcal{F}\} \tag{10}$$
>
> *with starting point* Y. *Add all CAI's visited during the procedure to a list until either a CAI is reached that was already visited by an earlier run of the method of Hooke and Jeeves, or until the method converges to a local minimum of (10).*
>
> *Step 4:* *Determine a minimizer of the solutions of the subproblems solved in Step 3.*
>
> ***Output:*** *A heuristic solution* X_{B_C} *of* $1/P/B_C/l_{2,B_C}/f$ *convex.*

The heuristic Algorithm 2 was tested against the exact Algorithm 1 for the case of Weber problems $1/P/B_C/l_{2,B_C}/\sum$ in Ochs (1998). In all of the 54 randomly generated examples with a set of 20 to 60 existing facilities an optimal solution was approximated by Algorithm 2. The percentage p of investigated starting points was set to $p = 10\%$ in these tests.

The average CPU time consumed by Algorithm 2 for a set of randomly generated problems with different numbers of existing facilities is given in Table 1. As in the above comparison the parameter p was set to $p = 10\%$ in all runs of the algorithm. The computations were performed on a machine of type i586 Linux 2.0.30 with 32MB RAM and 100MHz.

M	10	20	30	40	50	60
CPU time in seconds	0.880	4.275	10.798	22.702	42.202	71.234

Table 1: Average CPU time used by Algorithm 2 for a set of randomly generated test problems with $M = 10, 20, \ldots, 60$ existing facilities.

5 Conclusions

This paper considers a location problem with one circular barrier and a very general objective function of the Euclidean distances between a set of existing facilities and one new facility. Based on a decomposition of the feasible region into a polynomial number of cells of algebraic invariance (CAI's) it is shown that the generally non-convex problem can be decomposed into a polynomial number of subproblems which have a convex objective function on every open convex subset of their domain. Two algorithms, one exact and one heuristic method, are suggested and first computational results are presented that improve on earlier results by Katz and Cooper (1981) and Butt and Cavalier (1996).

A similar decomposition can be developed for the case that multiple circular barriers are given. If the main property of the CAI's is transferred, that is, if for any point X in a cell C the set $\mathcal{E}x \cap \text{visible}(X)$ of existing facilities visible from X is identical, and for all nonvisible existing facilities in $\mathcal{E}x \cap \text{shadow}(X)$ a specific point of tangency (possibly moving along the boundary of a barrier, but staying at the same side of it) is used throughout the cell on the shortest permitted path connecting X and the respective facility, then Theorems 1 and 2 can be generalized. This involves in particular the consideration of the inner and outer tangents on pairs of barriers (and of the respective points of tangency) since shortest permitted paths passing around several barriers may partially coincide with these tangents.

Future research should also focus on more general barrier shapes and on other distance functions. It is, for example, an interesting open question whether a similar decomposition of the feasible set is possible for other l_p distances with $1 \leq p \leq \infty$.

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