Connectedness of Efficient Solutions in Multiple Objective Combinatorial Optimization*

Jochen Gorski† Kathrin Klamroth†⋆ Stefan Ruzika‡

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Abstract

Connectedness of efficient solutions is a powerful property in multiple objective combinatorial optimization since it allows the construction of the complete efficient set using neighborhood search techniques. However, we show that many classical multiple objective combinatorial optimization problems do not possess the connectedness property in general, including, among others, knapsack problems (and even several special cases) and linear assignment problems. We also extend known non-connectedness results for several optimization problems on graphs like shortest path, spanning tree and minimum cost flow problems. Different concepts of connectedness are discussed in a formal setting, and numerical tests are performed for two variants of the knapsack problem to analyze the likelihood with which non-connected adjacency graphs occur in randomly generated instances.

Keywords: Multiple objective combinatorial optimization; MOCO; connectedness; adjacency; neighborhood search

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†Bergische Universität Wuppertal, Arbeitsgruppe Optimierung und Approximation, Fachbereich C - Mathematik und Naturwissenschaften, Gaußstr. 20, 42119 Wuppertal

⋆ Corresponding author: kathrin.klamroth@math.uni-wuppertal.de

‡Department of Mathematics, University of Kaiserslautern, Postfach 3049, 67653 Kaiserslautern, Germany

1 Introduction

Typical examples of multiple criteria combinatorial optimization problems are multiple criteria knapsack problems with applications, among others, in capital budgeting, and optimization problems on networks like multiple criteria shortest path and minimum spanning tree problems, used, for example, within navigation systems and in supply chain management applications. Most of these problems are NP hard and intractable in the sense that the number of efficient solutions may grow exponentially with the size of the problem data; see, for example, [1] for a recent survey.

Structural properties of the efficient set of multiple criteria combinatorial optimization problems are of utmost importance for the development of efficient solution methods. In particular, the existence of a neighborhood structure between efficient solutions that would allow the generation of the complete efficient set by a simple neighborhood search would provide a theoretical justification for the application of fast neighborhood search methods. This paper provides answers to many open questions regarding the connectedness of the efficient set with respect to reasonable concepts of adjacency and for most of the classical problems in multiple criteria combinatorial optimization.

The literature on the connectedness of the set of efficient solutions in multiple objective optimization is scarce. The first publications appeared in the seventies together with the development of the multiple objective simplex method [2], see also [3] and [4] for general convex and locally convex problems. Later, research on the connectedness of efficient solutions of MOCO problems was coined by assertions and falsifications. [5] claimed that there always exists a sequence of adjacent efficient paths connecting two arbitrary efficient paths for MOSP. However, [6] demonstrated the incorrectness of the connectedness conjecture for MOSP and MOST problems by a counterexample. In [7], the example of [6] was used to show the incorrectness of the algorithm of [8] for biobjective network flow problems. Some comments on the connectedness of efficient solutions for biobjective multimodal assignment problems are also contained, but not further persued, in [9].

Positive connectedness results were so far only proven for some highly structured special cases. [10] show connectedness for biobjective $\{0,1\}$-knapsack problems with equal sums of coefficients, and [11] consider biobjective optimization problems on matroids where one objective is based on $\{0,1\}$-coefficients. Nevertheless, neighborhood search algorithms were applied and tested numerically by several authors also for other problems, for example, for different variants of bicriteria knapsack problems [12, 13] and for the bicriteria and multicriteria TSP [14, 15].

2 Problem Formulation

Multiple objective combinatorial optimization (MOCO) has become a quickly growing research topic, and has recently attracted the attention of researchers both from the fields of multiple objective and from single objective combinatorial optimization [1].

Formally, a general MOCO problem can be stated as

$$\begin{aligned} &\min \; f(x) = (f_1(x), \ldots, f_p(x)) \\ &\text{s.t.} \;\; x \in X, \end{aligned}$$

where the *decision space* X is a given feasible set with some additional combinatorial structure. The vector-valued objective function $f : X \longrightarrow \mathbb{Z}^p$ maps the set of *feasible solutions* into the *image space*. $Y := f(X)$ denotes the image of the feasible set in the image space.

The Pareto concept of optimality for MOCO problems is based on the componentwise ordering of $\mathbb{Z}^p$ defined for $y^1, y^2 \in \mathbb{Z}^p$ by

$$\begin{aligned} y^1 \leq y^2 \quad &:\Leftrightarrow \quad y^1_k \leq y^2_k, \quad k = 1, \ldots, p \quad \text{and} \quad y^1 \neq y^2, \\ y^1 < y^2 \quad &:\Leftrightarrow \quad y^1_k < y^2_k, \quad k = 1, \ldots, p. \end{aligned}$$

A point $y^2 \in \mathbb{Z}^p$ is called *dominated* by $y^1 \in \mathbb{Z}^p$ iff $y^1 \leq y^2$, and it is called *strongly dominated* by y^1 iff $y^1 < y^2$. The *efficient set* X_E and the *weakly efficient set* X_{wE} are defined by

$$\begin{aligned} X_E \;\; &:= \;\; \{x \in X : \text{there exists no } \bar{x} \in X \text{ with } f(\bar{x}) \leq f(x)\} \\ X_{wE} \;\; &:= \;\; \{x \in X : \text{there exists no } \bar{x} \in X \text{ with } f(\bar{x}) < f(x)\}. \end{aligned}$$

The images $Y_N := f(X_E)$ and $Y_{wN} := f(X_{wE})$ of these sets under the vector-valued mapping f are called the *nondominated set* and the *weakly nondominated set*, respectively. The task in MOCO is to find Y_N and for every $y \in Y_N$ at least one $x \in X_E$ with $f(x) = y$.

Structural properties of the efficient set of MOCO problems play a crucial role for the development of efficient solution methods. A central question relates to the connectedness of the efficient set with respect to combinatorially or topologically motivated neighborhood structures. A positive answer to this question would provide a theoretical justification for the application of fast neighborhood search techniques, not only for multiple objective but also for appropriate formulations of single objective problems.

Following the literature (see [6], [15]), we next introduce a graph theoretical definition of adjacency of efficient solutions MOCO problems.

Definition 2.1 *For a given MOCO problem the* adjacency graph of efficient solutions $\mathcal{G} = (\mathcal{V}, \mathcal{A})$ *of the MOCO problem is defined as follows:* $\mathcal{V}$ *consists of all efficient solutions of the given MOCO problem. An (undirected) edge is introduced between all pairs of nodes which are adjacent with respect to the considered definition of adjacency for the given MOCO problem. These edges form the set* $\mathcal{A}$*.*

The connectedness of X_E is now defined via the connectedness of an undirected graph. An undirected graph $\mathcal{G}$ is said to be connected if every pair of nodes is connected by a path.

Definition 2.2 *The set* X_E *of all efficient solutions of a given MOCO problem is said to be* connected *iff its corresponding adjacency graph* $\mathcal{G}$ *is connected.*

The remainder of this article is organized as follows. In Section 3, we discuss different definitions of adjacency of feasible solutions of a MOCO problem. On one hand, adjacency may be defined based on appropriate Integer Programming (IP-) formulations of a given problem and using the natural neighborhood of basic feasible solutions of linear programming. For many problems, however, it appears to be more convenient to consider a combinatorial neighborhood. In Section 4 we discuss and extend existing results for the multiple objective shortest path and spanning tree problem and present new results for other major classes of MOCO problems like the knapsack and the assignment problem with multiple objectives in Section 5. We report numerical tests on adjacency of efficient solutions for the binary multiple objective knapsack problem with bounded cardinalities and the binary multiple choice multiple objective knapsack problem in Section 6. Finally, we conclude the paper in Section 7 with a summary table of the state-of-the-art and with current and future research ideas.

3 Defining Adjacency

We distinguish two different classes of adjacency definitions: definitions based on combinatorial structures and linear programming based definitions.

Combinatorial definitions of adjacency are problem-dependent and usually based on simple operations which transform one feasible solution into another, say "adjacent" feasible solution. We call such operations *elementary moves*. An elementary move is called *efficient*, if it leads from one efficient solution of the problem to another efficient solution. Two efficient solutions are called *adjacent*, if one can be obtained from the other by one efficient move. Examples for elementary moves for specific problem classes are the insertion and deletion of edges in a spanning tree, the modification of a matching along an alternating cycle, or simply the swap of two bits in a binary solution vector. In single objective optimization such elementary moves are frequently used in exact algorithms (e.g., the negative dicycle algorithm for the minimum cost flow problem) as well as in heuristic algorithms (e.g., the two-exchange heuristic for the traveling salesman problem).

We call an elementary move for a given problem class *canonical* iff the set of optimal solutions of the corresponding single objective problem is connected with respect to this elementary move for all problem instances. Although non-canonical moves immediately imply non-connectedness results also in the multiple objective case, they are used in heuristic methods based on neighborhood search [15].

For some classes of combinatorial problems, an elementary move corresponds to a move from one extreme point to another adjacent extreme point along an edge of the polytope which is obtained by the Linear Programming (LP) relaxation of a Mixed Integer Linear Programming (MILP) formulation of the given combinatorial problem. If the given MILP formulation represents the MOCO problem sufficiently well, a property which we will call *appropriate representation* in Definition 3.1 below, the corresponding elementary move is always canonical.

This observation motivates a more universal and less problem dependent adjacency concept which utilizes MILP formulations of MOCO problems and which is based on the topologically motivated adjacency of basic feasible solutions according to [2]. In order to define an *LP-based definition of adjacency* in a more general setting, the underlying MILP formulations of a given MOCO problem have to be selected carefully. In particular, a one-to-one correspondence between feasible solutions of the MOCO problem and basic feasible solutions of the LP relaxation of the MILP formulation used for the adjacency definition is required. Note that otherwise, the neighborhood structure induced by pivot operations on basic feasible solutions of the LP relaxation of the MILP cannot be transferred to the feasible solutions of the MOCO problem.

Definition 3.1 *An MILP formulation of a MOCO problem is called* appropriate *iff its LP relaxation, after transformation into standard form, has the following two properties:*

(M1) Every basic feasible solution corresponds to a feasible solution of the MOCO problem.

(M2) For every feasible solution of the MOCO problem there exists at least one basis such that the solution of the MOCO problem is equal to the corresponding basic feasible solution of the above LP relaxation of the MILP problem in standard form.

Properties (M1) and (M2) characterize MILP formulations of MOCO problems that are suitable for the definition of an LP-based concept of adjacency for these problems. In this context, two bases of an LP are called adjacent iff they can be obtained from each other by one pivot operation.

Definition 3.2 *Let an appropriate MILP formulation of a MOCO problem be given. Two feasible solutions x^1 and x^2 of the MOCO problem are called* adjacent with respect to the given MILP formulation *iff there exist two adjacent bases of the LP relaxation of the MILP problem (after transformation into standard form) corresponding to x^1 and x^2, respectively.*

In [16] it is shown that the *efficient* basic feasible solutions of the LP relaxation of the MILP problem in standard form are connected. Any of them can be obtained by the solution of some weighted sum problem and thus they are *supported* efficient solutions (see [17]). Thus, we can conclude that the adjacency graph of efficient solutions of a MOCO problem (with respect to an LP-based definition of adjacency) always contains a connected subgraph, the subgraph of supported efficient solutions.

Moreover, the set of optimal solutions of a corresponding single objective combinatorial optimization problem is always connected (or even unique) under this definition. Therefore, the question whether the corresponding multiple objective optimization problem has a connected adjacency graph is non-trivial.

The above definition of adjacency (and hence the resulting adjacency graph) depends on the chosen appropriate MILP formulation of the given MOCO problem, which is in general not unique. If different appropriate MILP formulations are used to model the same MOCO problem, we can expect different results concerning the connectedness of efficient solutions of the problem. In this context, Definitions 2.1 and 2.2 must always be understood with respect to the chosen appropriate MILP formulation of a MOCO problem.

Polyhedral theory implies that the transformation into standard form can be omitted in the case of bounded polyhedra. For more details we refer to [18].

4 Extensions of Known Results

In this section, we employ the classification of adjacency concepts developed in Section 3 to categorize definitions of adjacency used in the literature. Existing non-connectedness results are extended to the set of *weakly* efficient solutions and new results are derived.

In the following we refer to a graph with node set V and edge set A by $G = (V, A)$. Let $n := |V|$ and $m := |A|$ and let $s \in V$ and $t \in V$.

Let G be directed. For $c^1, \ldots, c^p : A \to \mathbb{R}^+$ the *multiple objective shortest path problem* can be formulated as

$$\begin{aligned} &\min \ (c^1 x, \ldots, c^p x)^T \\ &\text{s.t.} \ \sum_{j=1}^{n} x_{ij} - \sum_{j=1}^{n} x_{ji} = \begin{cases} 1, & \text{if } i = s, \\ 0, & \text{if } i \in \{1, \ldots, n\} \setminus \{s, t\}, \\ -1, & \text{if } i = t, \end{cases} \qquad (1) \\ &\quad x_{ij} \in \{0, 1\} \quad \forall (i, j) \in A. \end{aligned}$$

According to [6] two efficient paths are called adjacent iff they correspond to two adjacent basic feasible solutions of the linear programming relaxation of (1). Obviously this definition of adjacency corresponds to an MILP-based definition in the sense of Section 3. In [18] it is shown that this MILP formulation is appropriate in the sense of Definition 3.1.

For this problem, a combinatorial definition of adjacency can be derived which is equivalent to the MILP-based definition. Paths are associated with flows and the residual flow of two paths is used to decide whether they are adjacent. A shortest path P_1 is adjacent to a shortest path P_2 iff the symmetric difference of their edge set in the residual graph corresponds to a single cycle. Note that these definitions are canonical extensions of the single objective case in the sense of Section 3.

Ehrgott and Klamroth [6] showed that the adjacency graphs of efficient shortest paths are non-connected in general. However, the weakly efficient set in their counter-example turns out to be connected. A modification of the cost vectors in the counter-example of Ehrgott and Klamroth [6], depicted in Figure 1, proves that this set is in general also not connected.

Theorem 4.1 *The adjacency graphs of weakly efficient shortest paths are non-connected in general.*

Proof: The graph depicted in Figure 1 has twelve weakly efficient paths listed in Table 1.

It is easy to verify that the Path P_8 is not adjacent to any other weakly efficient shortest path since at least two of its intermediate nodes do not coincide with intermediate nodes of the other weakly efficient paths. Hence, the corresponding adjacency graph is non-connected.

□

In all examples described in the literature, only two connected components of the adjacency graphs exist. One of them consists of a single element, while the second comprises all other (weakly)

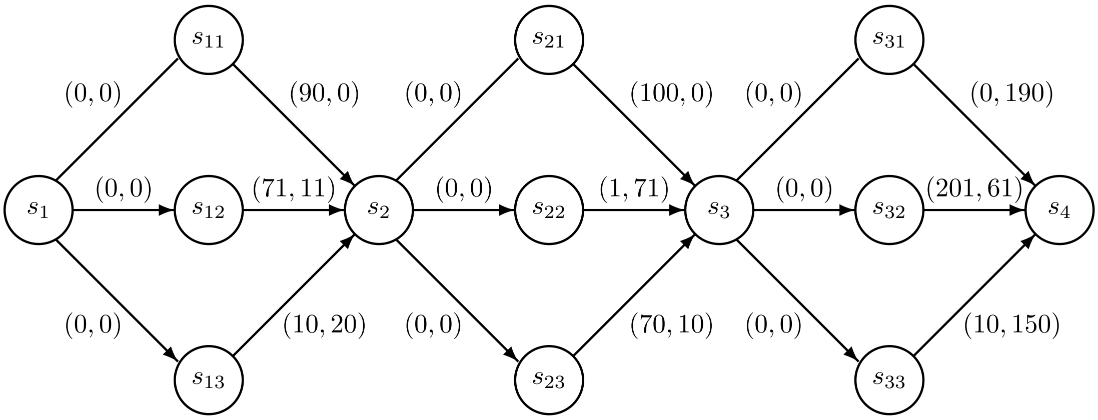

Figure 1: *Modified digraph from [6]*

efficient solutions. Yet, we can derive the following structural property.

Theorem 4.2 *In general, the number of connected components and the cardinality of the components are exponentially large in the size of the input data.*

Proof: Suppose we have k copies of the graph shown in Figure 1. The cost vectors of copy k are multiplied by the factor 1000^k. These k copies are connected sequentially by connecting node s_4 of copy i, $i = 1, \ldots, k-1$, with node s_1 of copy $i+1$ using an edge with costs $(0, 0)$. The resulting graph has $(19 \cdot k - 1)$ edges and the corresponding adjacency graph consists of 2^k different connected components. The largest component subsumes 11^k efficient solutions, the second largest 11^{k-1} efficient solutions, and so on. $\square$

Since the multiple objective shortest path problem is a special case of the multiple objective minimum cost flow problem, the results obtained above are also valid for the more general problem.

Let us now consider an undirected graph $G = (V, A)$. Let $A(S) := \{a = [i, j] \in A : i, j \in S\}$ denote the subset of edges in the subgraph of G induced by $S \subseteq V$. The *multiple objective spanning*

Efficient Path	**Interm. Nodes**			**Objective Vector**
P_1	s_{13}	$\boxed{s_{22}}$	s_{31}	$(11, 281)$
P_2	s_{13}	$\boxed{s_{22}}$	s_{33}	$(21, 241)$
P_3	s_{13}	s_{23}	s_{31}	$(80, 220)$
P_4	s_{13}	s_{23}	s_{33}	$(90, 180)$
P_5	s_{13}	s_{21}	s_{33}	$(120, 170)$
P_6	s_{11}	s_{23}	s_{33}	$(170, 160)$
P_7	s_{11}	s_{21}	s_{33}	$(200, 150)$
P_8	s_{12}	s_{22}	s_{32}	$(273, 143)$
P_9	s_{13}	s_{23}	$\boxed{s_{32}}$	$(281, 91)$
P_{10}	s_{13}	s_{21}	$\boxed{s_{32}}$	$(311, 81)$
P_{11}	s_{11}	s_{23}	$\boxed{s_{32}}$	$(361, 71)$
P_{12}	s_{11}	s_{21}	$\boxed{s_{32}}$	$(391, 61)$

Table 1: All weakly efficient paths of the graph depicted in Figure 1. The paths $\{P_1, \ldots, P_{12}\} \setminus \{P_8\}$ form a connected component in the adjacency graph $\mathcal{G}$ of the weakly efficient set and $\{P_8\}$ is an isolated node in $\mathcal{G}$.

tree problem can be formulated as

$$\begin{aligned} \min \;\; & (c^1 x, \ldots, c^p x)^T \\ \text{s.t.} \;\; & \sum_{a \in A} x_a \;=\; n-1, \\ & \sum_{a \in A(S)} x_a \;\leq\; |S| - 1 \quad \forall\, S \subseteq V, \\ & x_a \;\in\; \{0,1\} \quad \forall\, a \in A. \end{aligned} \tag{2}$$

[6] consider a combinatorial definition of adjacency: Two spanning trees are adjacent iff they have $n-2$ edges in common. They prove that there is a one-to-one correspondence between efficient shortest paths and efficient spanning trees for the graph in Figure 1. Hence, also the adjacency graph of weakly efficient spanning trees based on the combinatorial definition of adjacency is non-connected in general.

It can be shown that the MILP formulation (2) is appropriate in the sense of Definition 3.1 (see [18]). Moreover, it is easy to see that any graph containing a single cycle has a connected adjacency graph.

The multiple objective spanning tree problem is an optimization problem on matroids. A natural, combinatorial definition of adjacency for bases of matroids is to call two bases (of rank n) adjacent iff they have $n-1$ elements in common. Again, based on our findings for multiple objective spanning tree problem problems, we can conclude that the adjacency graph of the more general problem is in general non-connected. Our observations are summarized in the following corollary.

Corollary 4.1 *The adjacency graphs of (weakly) efficient spanning trees, cost flows and bases of matroids are non-connected in general. The number of connected components and the number of nodes in these components can grow exponentially wrt. the problem size.*

5 New Results for Special Classes of MOCO Problems

In this section we focus on (binary) knapsack, unconstrained binary optimization, linear assignment and traveling salesman problems. Using suitable combinatorial or MILP-based definitions of adjacency in the sense of Section 3, we show that all problems mentioned have non-connected adjacency graphs in general.

5.1 Binary Knapsack Problems

We examine two types of binary knapsack problems, the binary multiple choice knapsack problem with equal weights and the binary knapsack problem with bounded cardinality. While the investigation of the former problem is motivated by structural similarities to the counter-example in [6] and was thus expected to have a non-connected adjacency graph in general, the latter can be regarded as weakly structured MOCO since each combination of items is allowed as long as the cardinality constraint is met. Hence it was long conjectured that this problem has a connected adjacency graph.

The *multiple objective binary multiple choice knapsack problem with equal weights* is defined by

$$\begin{aligned} \max\ & \left(\sum_{i=1}^{n} \sum_{j=1}^{k_i} c_{ij}^1 x_{ij}, \ldots, \sum_{i=1}^{n} \sum_{j=1}^{k_i} c_{ij}^p x_{ij} \right)^T \\ \text{s.t. } & \sum_{j=1}^{k_i} x_{ij} \ = \ 1, \qquad i = 1, \ldots, n, \\ & \quad x_{ij} \ \in \ \{0,1\}, \ i = 1, \ldots, n, \ j = 1, \ldots, k_i, \end{aligned} \tag{3}$$

where $c_{ij}^1, \ldots, c_{ij}^p \geq 0$ for all $i = 1, \ldots, n$ and $j = 1, \ldots, k_i$.

This problem can be interpreted as follows: Given n disjoint baskets $B_1, \ldots, B_n$ each having exactly k_i items, $i = 1, \ldots, n$, the objective is to maximize the overall profit with the restriction that exactly one item is chosen from each basket. Problem (3) is well structured since items cannot be combined arbitrarily.

Definition 5.1 *Two (weakly) efficient solutions x and x' of the binary multiple choice knapsack problem with equal weights are called* adjacent *iff x' and x differ in one item in exactly one basket B_i for an $i \in \{1, \ldots, n\}$.*

This definition of adjacency is canonical: For single objective problems, any optimal solution must contain an item of maximum profit from each basket. Alternative optimal solutions may exist if at least one basket contains more than one item of maximum profit. All these optimal solutions are adjacent in the sense of Definition 5.1.

In the multiple objective case the situation is, however, different. The counter-example in [6] and its modification in Section 4 can be used to establish the following non-connectedness result.

Theorem 5.1 *The adjacency graph of a binary multiple choice knapsack problem with equal weights, where adjacency of two efficient solutions is defined according to Definition 5.1, is non-connected in general.*

Proof: In the counter-example for the multiple objective shortest path problem given in the proof of Theorem 4.1 we redefine the cost vectors c_{ij} of the three paths from node s_i to node s_{i+1}, $i = 1, 2, 3$, via s_{ij}, $j = 1, 2, 3$, by setting

$$\tilde{c}^q_{ij} \;=\; \max\{c^q_{ij} : i, j = 1, 2, 3;\, q = 1, 2\} - c^q_{ij}$$

for $i, j = 1, 2, 3$ and $q = 1, 2$ and interpret the resulting vectors of the three paths from the node s_i to node s_{i+1} as profit vectors for basket B_i, $i = 1, 2, 3$. Since we have transformed the minimization problem into a maximization problem by taking the negative value of each cost vector followed by a shift of these vectors by an amount of $\max\{c^q_{ij}\} = 201$, there is a one-to-one correspondence between the efficient solutions of the modified problem and the efficient solutions of the counter-example considered in Theorem 4.1. The profit vectors of the resulting solutions $K_1, \ldots, K_{12}$ are given by $(603, 603)^T - c(P_i)$ where $c(P_i)$ corresponds to the cost vector of P_i in Table 1 for $i = 1, \ldots, 12$. Items in at least two baskets have to be exchanged when transforming K_8 into K_j, $j \neq 8$ by elementary moves. Hence, K_8 is not adjacent to any other (weakly) efficient solution in the sense of Definition 5.1. □

Note that, since there is a one-to-one correspondence between the example used in the proof of Theorem 5.1 and the example given in Theorem 4.1, the above result can be generalized similar to Theorem 4.2 using the same extension of the original example:

Corollary 5.1 *In general, the number of connected components and the cardinality of the components in the adjacency graph of a binary multiple choice knapsack problem with equal weights, where adjacency of two efficient solutions is defined according to Definition 5.1, can grow exponentially in the size of the input data.*

In Section 6.2 we investigate empirically the frequency with which non-connected adjacency graphs for problem (3) occur in randomly generated instances.

In the following we consider binary knapsack problems where the number of items in each solution is fixed to a constant. The *multiple objective binary knapsack problem with bounded cardinality* is formally given by

$$\begin{aligned} \max \;\; & (c^1 x, \ldots, c^p x)^T \\ \text{s.t.} \;\; & \sum_{i=1}^{n} x_i \;=\; k, \\ & \qquad x_i \;\in\; \{0,1\},\;\; i = 1 \ldots, n, \end{aligned} \tag{4}$$

where $c_i^1, \ldots, c_i^p \geq 0$ for all $i = 1, \ldots, n$. Let $KP(n,k)$ denote an instance of Problem (4). According to [10], the problem formulation can also be relaxed to the case that at most k items have to be chosen. Since all item values are non-negative, every efficient solution will have maximum cardinality.

For this kind of problem few results concerning the connectedness of the set of efficient solutions can be found in the recent literature. In [10], three different models of binary knapsack problems were studied and some connectedness results using an MILP-based definition of adjacency were presented for very specific problem classes. In [12] two algorithms for solving Problem (4) in the biobjective case using a combinatorial definition of adjacency are proposed. These algorithms are only guaranteed to find the set of all efficient solutions under the assumption that this set is connected. Based on the categorization for the definition of adjacency of efficient solutions given in Section 3, we renew the ideas of the above mentioned papers and show that the set of efficient solutions is in general non-connected in both cases.

We start our analysis with the combinatorial definition of adjacency which is also used in [12].

Definition 5.2 *Two efficient solutions $x = (x_1, \ldots, x_n)^T$ and $x' = (x'_1, \ldots, x'_n)^T$ of $KP(n,k)$ are called* adjacent *iff x' can be obtained from x by replacing one item in x with one item of x' which is not contained in x.*

Note that this kind of elementary move is canonical. Two efficient solutions x and x' are adjacent if and only if $\sum_{i=1}^{n} |x_i - x'_i|=2$, i.e., if their Hamming distance is 2. For $n \in \{1, 2, 3, 4\}$ or $k \in \{0, 1, n-1, n\}$ it is easy to verify that $KP(n, k)$ has a connected adjacency graph.

Theorem 5.2 *The adjacency graph of $KP(n, k)$ is connected for $n \in \{1, 2, 3, 4\}$ or $k \in \{0, 1, n-1, n\}$.*

In [10] another sufficient condition yielding a connected adjacency graph is specified.

Theorem 5.3 ([10]) *Let an instance $KP(n, k)$ be given such that $c_i^1 + c_i^2 = \alpha$ for all $i = 1, \ldots, n$ and for some $\alpha \in \mathbb{N}$. Then all $\binom{n}{k}$ feasible solutions are efficient solutions of* (4) *and hence, the adjacency graph of the problem is connected.*

Unfortunately, this connectedness result is no longer valid for the general case.

Theorem 5.4 *The adjacency graph of a binary knapsack problem of the form* (4) *with adjacency defined as in Definition 5.2 is non-connected in general.*

Proof: Consider $KP(9, 3)$ with the objective function vectors

$$\binom{c^1}{c^2} = \begin{pmatrix} 44 & 36 & 27 & 10 & 8 & 5 & 3 & 1 & 0 \\ 0 & 8 & 9 & 21 & 23 & 29 & 31 & 32 & 34 \end{pmatrix}.$$

The problem has 84 feasible and 38 efficient solutions. All efficient solutions $S_1, \ldots, S_{38}$ and their corresponding objective function vectors are listed in Table 2. Using the plotted boxes it is easy to verify that the efficient solution S_{11} is not adjacent to any other solution in the sense of Definition 5.2. Consequently, the adjacency graph of the given problem is non-connected. $\Box$

Note that the given counter-example in Theorem 5.4 is minimal in the sense that deleting any combination of profit vectors from the problem always leads to a connected adjacency graph, assuming that $k = 3$.

As a direct consequence of Theorem 5.4, the algorithms proposed in [12] fail to compute the complete set of efficient solutions in general.

In Section 6, we report about numerical results indicating the likelihood that a non-connected adjacency graph of problem (4) appears in randomly generated instances. Note that for these investigations problems $KP(n, k)$ with $k > \frac{n}{2}$ are not of separate interest since they can be transformed into equivalent knapsack problems $KP(n, \tilde{k})$ with $\tilde{k} := n - k \leq \frac{n}{2}$ where the objective is to select $\tilde{k}$ items that are left out of the original knapsack.

Cx		x									S
4	97	0	0	0	$\boxed{0}$	$\boxed{0}$	0	1	$\boxed{1}$	1	s_1
6	95	0	0	0	$\boxed{0}$	$\boxed{0}$	1	0	$\boxed{1}$	1	s_2
8	94	0	0	0	0	0	1	1	0	1	s_3
9	92	0	0	0	$\boxed{0}$	$\boxed{0}$	1	1	$\boxed{1}$	0	s_4
11	88	0	0	0	$\boxed{0}$	$\boxed{1}$	0	1	$\boxed{0}$	1	s_5
13	86	0	0	0	$\boxed{1}$	$\boxed{0}$	0	1	$\boxed{0}$	1	s_6
13	86	0	0	0	$\boxed{0}$	$\boxed{1}$	1	0	$\boxed{0}$	1	s_7
15	84	0	0	0	$\boxed{1}$	$\boxed{0}$	1	0	$\boxed{0}$	1	s_8
16	83	0	0	0	$\boxed{0}$	$\boxed{1}$	1	1	$\boxed{0}$	0	s_9
18	81	0	0	0	$\boxed{1}$	$\boxed{0}$	1	1	$\boxed{0}$	0	s_{10}
19	76	0	0	0	1	1	0	0	1	0	s_{11}
28	75	0	0	1	$\boxed{0}$	$\boxed{0}$	0	0	$\boxed{1}$	1	s_{12}
37	74	0	1	0	$\boxed{0}$	$\boxed{0}$	0	0	$\boxed{1}$	1	s_{13}
39	73	0	1	0	0	0	0	1	0	1	s_{14}
41	71	0	1	0	0	0	1	0	0	1	s_{15}
42	69	0	1	0	$\boxed{0}$	$\boxed{0}$	1	0	$\boxed{1}$	0	s_{16}
44	68	0	1	0	0	0	1	1	0	0	s_{17}
45	66	1	0	0	$\boxed{0}$	$\boxed{0}$	0	0	$\boxed{1}$	1	s_{18}
47	65	1	0	0	0	0	0	1	0	1	s_{19}
49	63	1	0	0	0	0	1	0	0	1	s_{20}
50	61	1	0	0	$\boxed{0}$	$\boxed{0}$	1	0	$\boxed{1}$	0	s_{21}
52	60	1	0	0	0	0	1	1	0	0	s_{22}
54	55	1	0	0	$\boxed{1}$	$\boxed{0}$	0	0	$\boxed{0}$	1	s_{23}
55	54	1	0	0	$\boxed{0}$	$\boxed{1}$	0	1	$\boxed{0}$	0	s_{24}
57	52	1	0	0	$\boxed{1}$	$\boxed{0}$	0	1	$\boxed{0}$	0	s_{25}
57	52	1	0	0	$\boxed{0}$	$\boxed{1}$	1	0	$\boxed{0}$	0	s_{26}
63	51	0	1	1	0	0	0	0	0	1	s_{27}
64	49	0	1	1	$\boxed{0}$	$\boxed{0}$	0	0	$\boxed{1}$	0	s_{28}
66	48	0	1	1	0	0	0	1	0	0	s_{29}
68	46	0	1	1	0	0	1	0	0	0	s_{30}
71	43	1	0	1	0	0	0	0	0	1	s_{31}
80	42	1	1	0	0	0	0	0	0	1	s_{32}
81	40	1	1	0	$\boxed{0}$	$\boxed{0}$	0	0	$\boxed{1}$	0	s_{33}
83	39	1	1	0	0	0	0	1	0	0	s_{34}
85	37	1	1	0	0	0	1	0	0	0	s_{35}
88	31	1	1	0	$\boxed{0}$	$\boxed{1}$	0	0	$\boxed{0}$	0	s_{36}
90	29	1	1	0	$\boxed{1}$	$\boxed{0}$	0	0	$\boxed{0}$	0	s_{37}
107	17	1	1	1	0	0	0	0	0	0	s_{38}

Table 2: All efficient solutions of the example used in the proof of Theorem 5.4.

Next, we concentrate on the MILP-based definition of adjacency which is considered in [10]. Let $P := \{x \in [0,1]^n : \sum_{i=1}^n x_i = k\}$ denote the feasible set of the LP relaxation of (4). Since P is a subset of the unit cube $[0,1]^n$, the MILP formulation (4) is appropriate in the sense of Definition 3.1. To decide whether two basic feasible solutions of the LP relaxation of (4) are adjacent, we state a necessary and sufficient condition for two extreme points of P being connected by an edge for $n \geq 5$. According to [19] two extreme points u and v of P are connected by an edge if and only if there do not exist two other extreme points w^1 and w^2 of P, i.e., other feasible solutions of $KP(n,k)$, such that

$$\frac{1}{2}(w^1 + w^2) = \frac{1}{2}(u + v). \tag{5}$$

Using (5) it is easy to verify the following result:

Lemma 5.1 *Let $n \geq 5$. Two extreme points u and v of the binary knapsack polytope $P = \{x \in [0,1]^n : \sum_{i=1}^n x_i = k\}$ are connected by an edge if and only if u and v are adjacent in the sense of Definition 5.2.*

According to Lemma 5.1, the adjacency structure of the efficient extreme points of P coincides with the adjacency structure induced by Definition 5.2. Hence, the adjacency graph with respect to the appropriate MILP formulation based on (4) and the adjacency graph resulting from Definition 5.2 are the same. Thus, Theorem 5.4 immediately implies the following result.

Corollary 5.2 *In general, the set of efficient solutions of $KP(n,k)$ is non-connected with respect to the appropriate MILP formulation based on* (4).

Since the general knapsack problem subsumes the binary case with bounded cardinality, we have also shown the non-connectedness for the general knapsack problem, if adjacency of efficient solutions for general knapsack problems is defined, for example, based on elementary moves similar to Definition 5.2. Note that the MILP-based definition of adjacency cannot be applied (directly) to general knapsack problems since a one-to-one correspondence between feasible solutions of the MOCO problem and basic feasible solutions of the LP relaxation of the MILP formulation (in standard form) cannot be guaranteed. Since the same reasoning also applies to integer programming problems with fixed (or bounded) cardinalities, the non-connectedness results are also valid for these types of problems.

5.2 Unconstrained Binary Optimization Problems

The unconstrained binary problem which we are going to study in this subsection possesses even less structure than all problems considered before. Formally, a *multiple objective unconstrained binary problem* is defined by

$$\begin{aligned} \max \quad & \left(c^1 x, \ldots, c^p x\right)^T \\ \text{s.t.} \quad & x_i \in \{0,1\}, \quad i = 1, \ldots, n. \end{aligned} \tag{6}$$

We concentrate on the case that $p = 2$ and we assume without loss of generality that $c_i^1 \cdot c_i^2 < 0$ (but not necessarily $c_i^1 < 0$ and $c_i^2 > 0$) for all $i = 1, \ldots, n$. Otherwise, either $x_i = 0$ or $x_i = 1$ in every efficient solution.

In problem (6) the number of variables set equal to one is not fixed. Consequently, an appropriate notion of adjacency is not evident. Nevertheless, Definition 5.1 can be transferred to this problem. For this purpose, consider the following modification of (6):

$$\begin{array}{lll} \max & \left(c^1 x,\, c^2 x\right)^T & \\ \text{s.t.} & x_i + y_i = 1, & i = 1, \ldots, n \\ & x_i, y_i \in \{0,1\}, & i = 1, \ldots, n. \end{array} \tag{7}$$

By introducing additional zero cost vectors for each y_i, $i = 1, \ldots, n$, (7) can be interpreted as a binary multiple choice knapsack problem with equal weights consisting of n baskets where either x_i or y_i has to be included in the knapsack for $i \in \{1, \ldots, n\}$. Hence, Definition 5.1 can be directly applied to (7) and induces a single '1-to-0' or a single '0-to-1' swap in exactly one x_i for adjacent solutions of (6). Hence we define:

Definition 5.3 *Two efficient solutions x and x' of the unconstrained binary problem are called* adjacent *iff they differ in exactly one component, i.e., if $\sum_{i=1}^{n} |x_i - x'_i| = 1$.*

If we extend the last definition to all 2^n feasible solutions of the problem which can be identified with the set of all extreme points of the n-dimensional unit cube $W := [0,1]^n$, two feasible (efficient) solutions are adjacent if and only if they are connected by an edge in W. Since W in combination with (6) can be easily modeled by an appropriate MILP formulation, the adjacency graph which results from Definition 5.3 coincides with the adjacency graph of this appropriate MILP formulation.

Theorem 5.5 *The adjacency graph of an unconstrained binary problem* (6)*, where adjacency is defined according to Definition 5.3, is non-connected in general.*

Proof: Consider the following unconstrained binary problem with objective matrix

$$C = \begin{pmatrix} -126 & -121 & -120 & -103 & -100 & -97 & -17 & -13 \\ 100 & 94 & 90 & 74 & 73 & 68 & 23 & 7 \end{pmatrix}.$$

The set of all efficient solutions of this problem consists of 110 vectors. It can be shown that the efficient solution $x = (0,1,0,1,1,1,0,1)^T$ with objective value $Cx = (-434, 316)^T$ is not adjacent to any other efficient solution in the sense of Definition 5.3. $\square$

The counter-example of Theorem 5.5 is minimal in the sense that deleting any combination of profit vectors from the problem always leads to a connected adjacency graph.

5.3 Linear Assignment Problems

The *multiple objective linear assignment problem* can be formulated as

$$
\begin{array}{rrcll}
\min & \multicolumn{4}{l}{\left(\sum_{i,j=1}^{n} c_{ij}^{1} x_{ij}, \ldots, \sum_{i,j=1}^{n} c_{ij}^{p} x_{ij}\right)^{T}} \\
\text{s.t.} & \sum_{i=1}^{n} x_{ij} & = & 1, & j = 1, \ldots, n, \\
& \sum_{j=1}^{n} x_{ij} & = & 1, & i = 1, \ldots, n \\
& x_{ij} & \in & \{0,1\}, & i,j = 1, \ldots, n,
\end{array}
\tag{8}
$$

with objective coefficients $c_{ij}^1, \ldots, c_{ij}^p \geq 0$ for all $i, j = 1, \ldots, n$. We discuss a combinatorial definition based on swapping rows in the assignment matrix, and an MILP-based definition of adjacency.

We first investigate adjacency based on a simple swap of two rows of the assignment matrix. Unfortunately, this definition of adjacency is not canonical, i.e., it does not yield a connected graph of optimal solutions for the single objective version of the problem.

Theorem 5.6 *Swapping two rows of the assignment matrix of a single objective linear assignment problem does in general not permit to construct the complete set of optimal solutions starting from an arbitrary optimal solution.*

Proof: Consider a single objective linear assignment problem with $n = 4$ and cost matrix

$$
C = (c_{ij}^1)_{i,j=1,\ldots,n} = \begin{pmatrix} 1 & \infty & 1 & \infty \\ 1 & 1 & \infty & \infty \\ \infty & \infty & 1 & 1 \\ \infty & 1 & \infty & 1 \end{pmatrix}.
$$

This problem has two optimal assignments with value 4:
Assignment 1: $x_{11} = x_{22} = x_{33} = x_{44} = 1$ and $x_{ij} = 0$ otherwise.
Assignment 2: $x_{13} = x_{21} = x_{34} = x_{42} = 1$ and $x_{ij} = 0$ otherwise.
Clearly, these assignments cannot be obtained from each other by a single row swap. □

Looking at the MILP-based definition of adjacency based on formulation (8), we observe that the biobjective linear assignment problem is a special case of the minimum cost flow problem (cf. Section 4). Since the matrix describing the assignment polytope P_{as} is totally unimodular, formulation (8) is appropriate in the sense of Definition 3.1. Due to a result in [20], the MILP-based definition of adjacency induced by P_{as} corresponds to the following combinatorial definition of adjacency:

Definition 5.4 *Let $G = (V_1 \cup V_2, A)$ with $|V_1| = |V_2| = n$ be a bipartite graph with edge costs $c^1, c^2 : A \to \mathbb{R}$, representing an instance of the biobjective linear assignment problem. Let A_1 and A_2 be the edges corresponding to two different assignments. Two solutions A_1 and A_2 are* adjacent *iff the graph induced by $A_1 \cup A_2$ in G contains exactly one cycle.*

Equivalently, two assignments A_1 and A_2 are adjacent if and only if their symmetric difference $A_1 \triangle A_2 := (A_1 \cup A_2) \setminus (A_1 \cap A_2)$ consists of exactly one cycle the edges of which alternately belong to A_1 and A_2, respectively [21].

Theorem 5.7 *The adjacency graph $\mathcal{G}$ of the biobjective linear assignment problem using Definition 5.4 for characterizing adjacent assignments is not connected in general.*

Proof: We restructure the counter-example for the multiple objective shortest path problem given in the proof of Theorem 4.1. Consider the six cost-submatrices of a (9×9) biobjective linear assignment problem given by

$$C^1_{(1:3,1:3)} = \begin{pmatrix} 0 & 0 & \infty \\ 10 & 71 & 0 \\ \infty & 90 & 0 \end{pmatrix}, \quad C^1_{(4:6,4:6)} = \begin{pmatrix} 0 & 0 & \infty \\ 70 & 1 & 0 \\ \infty & 100 & 0 \end{pmatrix}, \quad C^1_{(7:9,7:9)} = \begin{pmatrix} 0 & 0 & \infty \\ 10 & 201 & 0 \\ \infty & 0 & 0 \end{pmatrix},$$

$$C^2_{(1:3,1:3)} = \begin{pmatrix} 0 & 0 & \infty \\ 20 & 11 & 0 \\ \infty & 0 & 0 \end{pmatrix}, \quad C^2_{(4:6,4:6)} = \begin{pmatrix} 0 & 0 & \infty \\ 10 & 71 & 0 \\ \infty & 0 & 0 \end{pmatrix}, \quad C^2_{(7:9,7:9)} = \begin{pmatrix} 0 & 0 & \infty \\ 150 & 61 & 0 \\ \infty & 190 & 0 \end{pmatrix}.$$

Let all remaining cost coefficients be set to infinity. This problem decomposes into three (3×3)-subproblems denoted by S_1, S_2, and S_3, where each subproblem S_i has three solutions G_1, G_2, and G_3 that have finite costs in both objectives. These three solutions have the same structure for all three subproblems and are depicted in Figure 2. Note that the cost vector of each solution G_j of subproblem S_i is chosen such that it corresponds to the cost vector of the path connecting node s_i with node s_{i+1} via node s_{ij} in Figure 1. Consequently, there is a one-to-one correspondence between the (weakly) efficient solutions of this instance of the biobjective linear assignment problem and the (weakly) efficient solutions listed in Table 1.

From Figure 2 it can be seen that the pairwise union of two subgraphs G_i and G_j, $i \neq j$, contains exactly one cycle. According to Definition 5.4, two (weakly) efficient assignments of the overall problem are thus adjacent if they differ in exactly one subproblem S_i, $i \in \{1, 2, 3\}$. Since the (weakly) efficient path P_8 in Table 1 differs from all other (weakly) efficient paths in at least

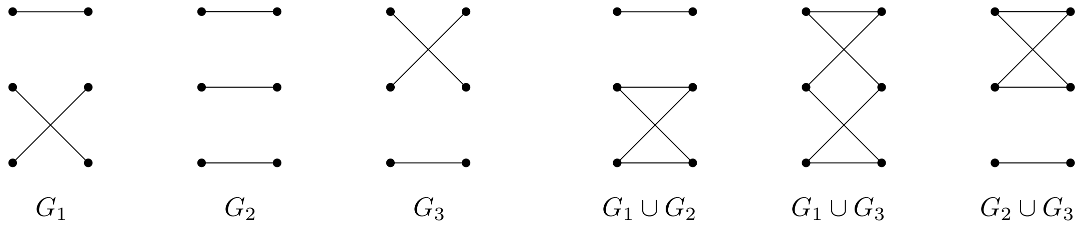

Figure 2: *All feasible assignments with finite costs for the subproblems S_i in the proof of Theorem 5.7 and their pairwise union.*

two connections, the corresponding assignment (consisting of G_2 in all three subproblems) differs from all other efficient assignments in at least two subproblems and is thus not adjacent to any other (weakly) efficient assignment. □

Note that according to [20] and [21] any pair of nodes of the assignment polytope is connected by a path of length at most 2. Hence any extension of the definition of adjacency for efficient solutions to a 2-neighborhood, based on Definition 5.4, trivially yields a connected adjacency graph.

Since the linear assignment problem is a special class of *transportation* and *transshipment problems* (see, for example, [1]), respectively, the non-connectedness results also apply to these two more general problems.

Note also that, similar to the case of the binary multiple choice knapsack problem with equal weights, the fact that there is a one-to-one correspondence between the example used in Theorem 5.7 and the example given in Theorem 4.1 can be used to generalize the above result similar to Theorem 4.2, using again the same extension of the original example.

Corollary 5.3 *In general, the number of connected components and the cardinality of the components in the adjacency graph of the biobjective linear assignment problem, where adjacency of two efficient solutions is defined according to Definition 5.4, grow exponentially in the size of the input data.*

We conclude this section with a comment regarding a large class of other problems. If the definition of adjacency considered for some problem is not canonical, then the corresponding adjacency graph of efficient solutions is non-connected. This is for example the case for the traveling salesperson problem and the 2-edge-exchange neighborhood (see [14] and [15]).

	10/5/10	20/10/10	30/15/10	40/20/10	60/30/10	80/40/20	100/50/20
Method 1	0 / 50000	0 / 20000	0 / 1000	2 / 1000	-	-	-
Method 2	0 / 50000	2 / 20000	1 / 1000	1 / 1000	-	-	-
Method 3	0 / 50000	2 / 20000	0 / 1000	0 / 1000	-	-	-
Method 4	0 / 50000	1 / 20000	0 / 1000	0 / 1000	-	-	-
Method 5	0 / 50000	1 / 20000	1 / 1000	0 / 1000	-	-	-
Method 6	0 / 50000	0 / 50000	0 / 50000	0 / 10000	0 / 10000	1 / 10000	0 / 1000

Table 3: Setup of computational experiments for the biobjective binary knapsack problem with bounded cardinality and frequencies of instances with a non-connected adjacency graph

6 Numerical Results

The results in the previous sections are obtained from a worst-case perspective. Therefore, it is an interesting question how frequently the phenomenon “non-connected adjacency graph” occurs in numerical examples. We exemplarily conduct numerical studies for the biobjective binary knapsack problem with bounded cardinality and the biobjective binary multiple choice knapsack problem introduced in Section 5.1. All in all, more than six million randomly generated problem instances have been analyzed.

6.1 Biobjective Binary Knapsack Problems with Bounded Cardinality

For the computation of the efficient set of the biobjective binary knapsack problem with bounded cardinality $KP(n,k)$ we used a dynamic programming approach for general multiple objective knapsack problems developed in [22]. Recent numerical tests in [23] on a slightly extended version of this approach proved its efficiency for solving even large scale bi- and multiple objective binary knapsack problems.

We generated seven problem setups and for each setup we used six different methods to generate the objective coefficients. In the first row of Table 3, we use a scheme of the form $Pos1/Pos2/Pos3$ to code these seven setups. $Pos1$ specifies the total number n of items. The upper bound k for the right hand side parameter of the knapsack constraint is specified under $Pos2$. For each problem instance, we determined the adjacency graph for all possible right hand sides $i \in \{1,\dots,Pos2\}$. Finally, the coefficients of the first objective c^1 were chosen in the interval $[0,r]$, where $r = Pos1 \cdot Pos3$. The coefficients of the second objective c^2 were chosen according to six different methods motivated by the study in [24]:

Method 1: c^1 was sorted in decreasing, c^2 in increasing order to obtain pairwise non-dominated profit vectors. Weakly-dominated vectors were omitted.

Method 2: The profit vectors $p_1 := (c_1^1, c_1^2)^T = (r, 0)^T$ and $p_n := (c_n^1, c_n^2)^T = (0, r)^T$ were fixed at the beginning. The remaining vectors were chosen within the triangle $(0, 0)^T$, p_1 and p_n. Note that a small number of the generated profit vectors were dominated (approx. 5 % on average over all examples).

Method 3: The profit vectors were generated as in Method 2, but now within the triangle $(r, r)^T$, p_1 and p_n.

Method 4: The profit vectors p_1 and p_n were fixed like in Method 2. The remaining vectors were generated spread around the concave part of the half circle with midpoint $(0, 0)^T$ connecting the points p_1 and p_n. Note that a small number of the generated profit vectors were dominated (approx. 10 % on average over all examples).

Method 5: The profit vectors were generated as in Method 4, but now spread around the convex half circle with midpoint $(r, r)^T$ connecting p_1 and p_n.

Method 6: The entries of the profit matrix were generated uniformly at random.

The two numbers in each entry of Table 3 correspond to the number of instances possessing a nonconnected adjacency graph and the total number of processed instances for each of the setups, respectively. A dash indicates that this setup was not tested due to its numerical difficulty.

For the generated instances, only very few adjacency graphs are non-connected. Nevertheless, we found for each of the six data generation methods at least one instance possessing a non-connected adjacency graph. Based on the small number of components, there do not seem to exist significant trends - neither with respect to increasing k or n nor with respect to some particular generation method for the data. For the case of randomly generated profit matrices (Method 6), non-connected adjacency graphs seem to occur extremely rarely. As mentioned in [10], an item x_i corresponding to a dominated profit vector p_i can only be contained in an efficient knapsack if at least one of the items x_j corresponding to profit vectors p_j dominating p_i is also contained in the knapsack. The set of all efficient solutions of such a problem consists of a few number of elements and is more structured than in the case when only pairwise non-dominated profit vectors are considered. For the problem size (40/20/10), the maximum number of elements of an efficient set for a problem instance generated by Method 6 is given by 290 while for the other methods the maximum number

Setup of test instances	20/5/10	20/10/10	20/15/10
Number of instances generated	10000	5000	1000
Number of instances having a non-connected adjacency graph	118	295	111

Table 4: Setup of computational experiments for the biobjective binary multiple choice knapsack problem with uniform weights and number of instances with a non-connected adjacency graph

does not fall below 1392 and has a maximum value of over 5300 elements for a problem instance generated by Method 2. Unfortunately, Method 6 seems to be the "standard" way to generate data when testing an algorithm numerically. Yet, for algorithms based on neighborhood search, this problem class seems to be quite uninteresting.

6.2 Biobjective Binary Multiple Choice Knapsack Problems

The biobjective binary multiple choice knapsack problem (see Section 5.1) is closely related to the biobjective binary knapsack problem with bounded cardinality. Yet, it behaves quite differently with respect to adjacency issues.

Consider a biobjective binary multiple choice knapsack problem with n baskets and k possible items per basket. To obtain the set of efficient solutions X_E, we use a simple dynamic programming scheme. In the i-th step, $i = 1, \ldots, n$, we combine every solution being efficient for the problem with baskets $B_1, \ldots, B_{i-1}$, with the items in basket i. Dominated solutions are deleted. The remaining solutions form the set of efficient solutions for the problem with baskets $B_1, \ldots, B_i$. Note that the items in each basket should be pairwise nondominated since dominated items are never included in an efficient solution. We study the frequency of problems with non-connected adjacency graphs when the number of baskets increases from 1 to n, and the (fixed) number of items per basket increases. As in Section 6.1, we use a scheme $Pos1/Pos2/Pos3$ coding the setup of the instances. $Pos1$ stands for the number of baskets while the number of items per basket is given in $Pos2$. The integer cost coefficients are taken from the interval $[1, Pos1 \cdot Pos3]$ according to Method 1 in Section 6.1. Table 4 reports the setups and the number of instances we have tested.

For each of the setups, Table 4 contains the number of instances possessing a non-connected adjacency graph cumulated over all $i = 1, \ldots, 20$ baskets. One obvious difference to the results obtained in Section 6.1 is that non-connected adjacency graphs occur far more often for multiple choice knapsack problems. Figure 3 shows the number of instances with non-connected adjacency graph per thousand instances tested. Detailed results for each problem with i baskets, $i = 1, \ldots, 20$,

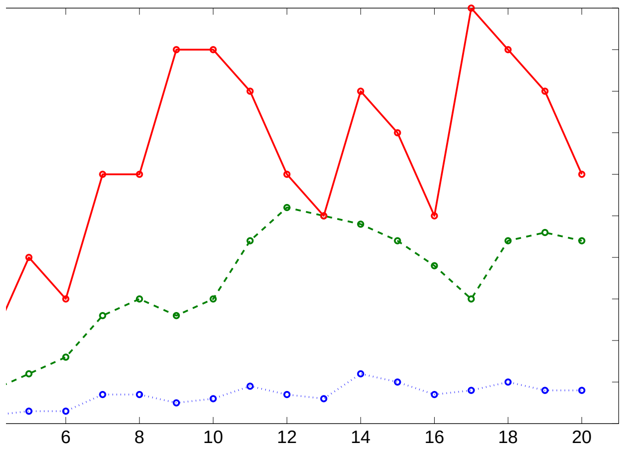

Figure 3: Number of instances per thousand (y-axis) with non-connected adjacency graph for $i = 1, \ldots, 20$ baskets (x-axis) for 5 (dotted line), 10 (dashed line), and 15 (solid line) items per basket

are reported. The dotted line, the dashed line, and the solid line correspond to the setups with 5, 10, and 15 items per basket, respectively. All curves are (slightly) increasing, i.e., “non-connectedness” happens more often with an increasing number of baskets. Furthermore, the more items per basket, the higher the likelihood for having a non-connected adjacency graph.

Table 5 provides more details about the character of the clusters. Among those instances with non-connected adjacency graphs, two clusters appear more often than three clusters. However, with increasing number of items per basket three clusters are getting more likely. Interestingly, we never generated an example where the maximum distance between clusters of instances with two clusters is greater than 2. The maximum (pairwise) distance between three clusters, however, happened to be as much as 8.

Setup of test instances	20/5/10	20/10/10	20/15/10
Instances with one cluster	199882	99705	19889
Instances with two clusters	115	282	100
Instances with three clusters	3	13	11
Maximal distance between two clusters	2	2	2
Maximal distance between three clusters	3	4	8

Table 5: Number of connected components (clusters) in the adjacency graph and maximum distance between two components for the biobjective binary multiple choice knapsack problem

Problem	Adjacency Definition	Connected
Shortest paths, Minimum cost flows	LP & CD	No
Spanning trees	LP & CD	No
Matroids	CD	No
Binary multiple choice knapsacks	CD	No
Binary knapsacks with bounded cardinality	LP & CD	No
Bin. knapsacks, bounded card., constant cost sum	LP & CD	Yes
Unconstrained binary optimization	LP & CD	No
Linear assignments	LP & CD	No

Table 6: Overview of connectedness results for MOCO problems. CD: Combinatorial definition of adjacency; LP: LP-based definition of adjacency (see Section 3).

7 Conclusions

As in the case of single objective combinatorial optimization, the question of adjacency of solutions is one of the core aspects in multiple objective combinatorial optimization (MOCO). The concept of adjacency of optimal solutions in multiple objective problems certainly exceeds its single criterion analogon in terms of complexity because of a more involved optimality concept. Maybe it is due to this increased complexity that research on this subject has widely been neglected. To the best of our knowledge there do not exist (correct) exact algorithms for computing the set of efficient solutions based on neighborhood structures, nor does the literature formalize different notions of adjacency.

The aim of our work is threefold. First, we formally introduce two different concepts of adjacency. One class of adjacency concepts relies on problem-dependent combinatorial structures, while the other one is based on appropriate models for the problem and, ultimately, goes back to the definition of adjacency for multiple objective linear programs. Second, we survey the current state of the art and supplement it with our own findings. As a result, we list eleven combinatorial optimization problems and discuss their adjacency properties, see Table 6 for an overview of these results. Third, we conduct numerical experiments to analyze the adjacency structure of two special types of bi-objective knapsack problems. Although being structurally related and possessing a non-connected adjacency graph in general, these knapsack problems differ significantly in the practical occurrence of adjacency.

Our work should be understood as a first step towards an in-depth investigation of adjacency in MOCO problems.

Although we prove the non-connectedness of many fundamental MOCO problems in general, special variants of these problems might possess a connected adjacency graph (cf. [10]). Further explorations of the structure of counter-examples as well as of the structure, the size and the geometry of connected components and especially their dependencies on the choice of the problem data should be in the main focus.

Another interesting stream of research is the development of new definitions of adjacency possibly yielding connected adjacency graphs for a wider class of MOCO problems. Especially, the ordered generation of the set of nondominated solutions for problems with two objective functions seems to be an appealing stream of research.

The structural results presented in Section 4 are based on a worst-case analysis. Studying theoretically the average case gives detailed information about the expected occurrence of adjacency in practical problems and might justify the application of adjacency-based algorithms even for problems having non-connected adjacency graphs in general.

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