Dominating Sets for Rectilinear Center Problems with Polyhedral Barriers

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Abstract

In planar location problems with barriers one considers regions which are forbidden for the siting of new facilities as well as for trespassing. These problems are important since they model various actual applications. The resulting mathematical models have a nonconvex objective function and are therefore difficult to tackle using standard methods of location theory even in the case of simple barrier shapes and distance functions.

For the case of center objectives with barrier distances obtained from the rectilinear or Manhattan metric it is shown that the problem can be solved in polynomial time by identifying a dominating set. The resulting genuinely polynomial algorithm can be combined with bound computations which are derived from solving closely connected restricted location and network location problems.

1 Introduction

In applications of location problems one often encounters situations in which regions are neither allowed for siting new facilities nor for trespassing. In accordance with most of the literature quoted below we call such regions

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barriers. Examples of barriers include lakes or nature parks when the location of industrial facilities is considered, high risk areas in the transportation and storing of chemicals, or obstacles in a production environment. An actual application of the latter was reported, for example, in Love and Yerex [18] who considered the problem of locating two new production facilities in the yard area of a precast-prestressed concrete products plant where traveling was restricted by other buildings.

In spite of this practical importance, there is only a relatively small amount of literature on location problems with barriers. Katz and Cooper [13] considered median (total cost) location problems using Euclidean distance and a forbidden region consisting of one circle. Klamroth [14] considered the median problem where distance is induced by a norm and with a barrier consisting of a line with passages. Aneja and Parlar [1] and Butt and Cavalier [4] developed heuristics for the median problem with l_p distance and barriers that are closed polyhedra. Larson and Sadiq [16], and Batta, Ghose and Palekar [2] obtained discretization results for median problems with l_1-distance and arbitrarily shaped barriers by transforming these problems into equivalent network location problems. Their results were generalized by Hamacher and Klamroth [9] for arbitrary block norms although it is not possible to transform these problems to the analogous network location problems. Location problems in which regions are excluded from siting new facilities, but trespassing is allowed are called *restricted location problems.* They have lately drawn some attention and have been successfully tackled for median and center problems, for instance, in Hamacher [8], Nickel [20], Hamacher and Nickel [10], and Hamacher and Schöbel [12].

This paper considers the weighted center problem with polyhedral barriers for which upto very recently no results where available in the literature. For the special case that N pairwise disjoint axis-aligned rectangles are given as barriers, Choi, Shin and Kim [5] presented an $O(N^2 M \log^2 M)$-time algorithm based on parametric search to compute the solution set of the weighted center problem with a fixed number M of existing facilities. Ben-Moshe, Katz and Mitchell [3] improved this result for the unweighted case by giving an $O(NM \log(N+M))$-time algorithm. More general problems with polyhedral (or general) barrier sets and general location objectives of which the weighted center problem with barriers can be viewed as a special case were discussed in Klamroth [15] and Segars [23].

In the next section we will formally introduce the problem and derive lower and upper bounds on the objective value by investigating the interrelation between center barrier problems on one hand and network location and restricted location problems on the other hand. A discretization result is developed in Section 3 for the special case that distances are measured by the Manhattan metric (l_1 metric) and that the barriers are pairwise disjoint convex polyhedra. Using the piecewise linearity of distances that distinguishes the Manhattan metric from round norms like the Euclidean norm it is shown that it is sufficient to consider a finite number of candidate sets, a dominating set (DS), to find an optimal location. The resulting polynomial time algorithm using this dominating set is given in Section 4. The paper is concluded by a final section in which the results of the paper are summarized and directions for future research are outlined.

2 Formal definition and bounds for center problems with barriers

In this section we first give a formal definition of center problems with barriers. Then we show that by considering the restricted location problem as a relaxation we get lower and upper bounds. Additional upper bounds are obtained by investigating a network location problem closely related to the input of the center problem with barriers.

Let $\{B_1, \ldots, B_N\}$ be a set of closed, convex sets in the plane, $\mathbb{R}^2$, with pairwise disjoint interior. Each set B_i, $i = 1, \ldots, N$ is called a barrier. Let $\mathcal{B} = \bigcup_{i=1}^{N} B_i$. The location of new facilities in the interior of $\mathcal{B}$ and travel through $\operatorname{int}(\mathcal{B})$ is forbidden. Thus the feasible region $F \subseteq \mathbb{R}^2$ for new facilities is given by

$$F = \mathbb{R}^2 \setminus \operatorname{int}(\mathcal{B}).$$

The distance $d_{\mathcal{B}}(X, Y)$ between two points $X, Y \in F$ is defined as the length of a shortest path from X to Y that does not meet the interior of a barrier. A finite set

$$\mathcal{E}x = \{Ex_m \in F : m \in \mathcal{M} = \{1, \ldots, M\}\}$$

of existing facilities is given in a connected subset of the feasible region F. A positive weight $w_m = w(Ex_m)$, $m \in \mathcal{M}$ is associated with each existing facility that represents the relative importance of facility Ex_m.

Define the function

$$f_{\mathcal{B}}(X) = \max_{m \in \mathcal{M}} \; w_m d_{\mathcal{B}}(X, Ex_m).$$

Then the weighted center problem with barriers is to minimize $f_{\mathcal{B}}(X)$ over all $X \in F$. To simplify further notation we will use the classification $Pos1/Pos2/Pos3/Pos4/Pos5$ of location problems according to Hamacher and Nickel [11]. Following their notation, the weighted center problem with barriers is classified as $1/P/\mathcal{B}/d_{\mathcal{B}}/\max$, where $Pos1$ gives the number of new facilities sought (1 for a single-facility problem), $Pos2$ denotes the type of location problem (P for planar location problems), $Pos3$ contains special assumptions ($\mathcal{B}$ for barrier regions), $Pos4$ contains the information about the distance function ($d_{\mathcal{B}}$ in case of barrier distances) and $Pos5$ indicates the objective function (max for the center objective).

While center location problems in the plane without barriers are extensively discussed in the literature (see, e.g. the books of Francis et. al. [6], Hamacher [8], and Love et. al. [17]) no references can be found on the corresponding barrier problems. The decisive distinction between the former and the latter problem is that the distance measure $d_{\mathcal{B}}$ for the problem considered in this paper reflects the fact that trespassing of the barriers is not allowed.

Let d be a given distance function induced by a norm $\| \bullet \|_d$. Then the distance $d_{\mathcal{B}}(X, Y)$ between two points $X, Y \in F$ is defined as the length of a shortest path (with respect to the given distance function d) from X to Y that does not intersect the interior of a barrier. Formally, let p be a piecewise, continuously differentiable parameterization, $p : [a, b] \to \mathbb{R}^2$, $a, b \in \mathbb{R}$, $a < b$, of a *permitted path connecting X and Y*, i.e. a curve not intersecting the interior of a barrier, $p([a, b]) \cap \mathrm{int}(\mathcal{B}) = \emptyset$, with $p(a) = X$ and $p(b) = Y$. Then $d_{\mathcal{B}}$ is given by

$$d_{\mathcal{B}}(X, Y) := \inf \left\{ \int_a^b \| p'(t) \|_d \, \mathrm{d}t \; : p \text{ permitted path connecting } X \text{ and } Y \right\}.$$

Any path connecting X and Y with length $d_{\mathcal{B}}(X, Y)$ not intersecting the interior of $\mathcal{B}$ is called a *d-shortest permitted path connecting X and Y*. Any two points X and Y in F that satisfy $d_{\mathcal{B}}(X, Y) = d(X, Y)$ are called *d-visible*. If d is the Manhattan metric, $d_{\mathcal{B}}(X, Y)$ is denoted by $l_{1,\mathcal{B}}(X, Y)$.

Note that $d_{\mathcal{B}}$ is symmetric and satisfies the triangle inequality, but is in general not positively homogeneous. Moreover, the objective function $f_{\mathcal{B}}$ is non-convex. However, instead of tackling the problem with methods of non-convex optimization we will choose a different approach by investigating the structure of the problem in more detail.

Next, upper and lower bounds for the optimal objective value of the center problem with barriers, $1/P/\mathcal{B}/d_{\mathcal{B}}/\max$, will be discussed. These bounds are analogous to bounds given for the median objective function in Hamacher and Klamroth [9]. Since these results can be easily transferred to other and more general objective functions we refer to their work for a more detailed discussion.

Two different approaches are suggested. The first approach is based on a relaxation of the barrier problem to a restricted location problem of the type $1/P/\mathcal{R} = \mathcal{B}/d/\max$. Here trespassing through the barrier regions is allowed whereas the placement of a new facility within the region $\mathcal{R} = \mathcal{B}$ is prohibited. The second approach makes use of the visibility graph G of the problem to relate the barrier problem to a network location problem $1/G/\bullet/d_G/\max$ on G. In both cases the non-convex optimization problem $1/P/\mathcal{B}/d_{\mathcal{B}}/\max$ is relaxed to a location problem that is easier to solve.

Lemma 1 *Let $z^*_{\mathcal{B}}$ be the optimal objective value of the barrier problem $1/P/\mathcal{B}/d_{\mathcal{B}}/\max$ and let $X^*_{\mathcal{R}}$ be an optimal solution of the corresponding restricted problem $1/P/\mathcal{R} = \mathcal{B}/d/\max$. Then*

$$f(X^*_{\mathcal{R}}) = \max_{m \in \mathcal{M}} \{w_m \, d(Ex_m, X^*_{\mathcal{R}})\} \le z^*_{\mathcal{B}} \le \max_{m \in \mathcal{M}} \{w_m \, d_{\mathcal{B}}(Ex_m, X^*_{\mathcal{R}})\} = f_{\mathcal{B}}(X^*_{\mathcal{R}}).$$

Proof: The second inequality is trivial. For the first inequality, let $X^*_{\mathcal{B}}$ be an optimal solution of the barrier problem $1/P/\mathcal{B}/d_{\mathcal{B}}/\max$. Since $X^*_{\mathcal{R}}$ is an optimal solution of the restricted problem $1/P/\mathcal{R} = \mathcal{B}/d/\max$, we have that $f(X^*_{\mathcal{R}}) \le f(X^*_{\mathcal{B}})$. Furthermore, $f(X) \le f_{\mathcal{B}}(X)$ for all $X \in F$, and thus

$$f(X^*_{\mathcal{R}}) \le f(X^*_{\mathcal{B}}) \le f_{\mathcal{B}}(X^*_{\mathcal{B}}) = z^*_{\mathcal{B}}.$$

□

An immediate consequence of the preceding lemma is the following result.

Corollary 1 *Let $X^*_{\mathcal{R}}$ be an optimal solution of the restricted problem $1/P/\mathcal{R} = \mathcal{B}/d/\max$. If $z^*_{\mathcal{B}} \ge w_m d_{\mathcal{B}}(Ex_m, X^*_{\mathcal{R}})$ for all $m \in \mathcal{M}$, then $X^*_{\mathcal{R}} = X^*_{\mathcal{B}}$ is an optimal solution of $1/P/\mathcal{B}/d_{\mathcal{B}}/\max$.*

For the case that distances are measured with respect to the Manhattan metric $d = l_1$ or the Chebychev metric $d = l_\infty$, the restricted problem $1/P/\mathcal{R} = \mathcal{B}/d/\max$ can be solved by an algorithm developed in Hamacher and Nickel [10]. If distances are measured with respect to polyhedral gauges, the optimal solution of the restricted problem can be obtained using an algorithm proposed in Nickel [20].

The second approach to derive bounds for the problem $1/P/\mathcal{B}/d_{\mathcal{B}}/\max$ makes use of the visibility graph of the problem in order to relax the non-convex barrier problem to a network location problem. In this case an additional assumption is needed, namely that the set of barriers is given by a set of polyhedra with extreme points $\mathcal{P}(\mathcal{B}) := \{p_i : \ i = 1, \ldots, g\}$. The *embedded visibility graph* of $\mathcal{E}x \cup \mathcal{P}(\mathcal{B})$ is defined as $G = (V, E)$ with node set $V(G) = \mathcal{E}x \cup \mathcal{P}(\mathcal{B})$ and weights $w(v) = 0$ if $v = p \in \mathcal{P}(\mathcal{B})$ and $w(v) = w(Ex_m)$ if $v = Ex_m \in \mathcal{E}x$. Two nodes $u, v \in V(G)$ are connected by an edge if the corresponding points are d-visible in the feasible region F, i.e. $d_{\mathcal{B}}(u, v) = d(u, v)$, and in this case the length of the edge is $d(u, v)$. The embedding of this edge is represented by a d-shortest permitted path between the points u and v. The length of a shortest network path between two vertices u and v is denoted by $d_G(u, v)$. Analogously the length of a shortest network path between a vertex v and a point X on an edge $e \in E(G)$ is denoted by $d_G(X, v)$. Then the network location problem $1/G/\bullet/d_G/\max$ on G is defined by

$$\begin{aligned} & \min_{X \in G} f_G(X) \\ \text{where} \quad & f_G(X) = \max_{v \in V(G)} w(v)\, d_G(X, v). \end{aligned}$$

Lemma 2 *If X_G^* is an optimal solution of the network location problem $1/G/\bullet/d_G/\max$ on G, then a point $X_{\mathcal{B}}$ in the feasible region F that corresponds to the point X_G^* on the embedded graph, is feasible for $1/P/\mathcal{B}/d_{\mathcal{B}}/\max$ and*

$$f_{\mathcal{B}}(X_{\mathcal{B}}) \le f_G(X_G^*).$$

Proof: The feasibility of X_G^* is trivial because X_G^* is either a vertex of G, i.e. $X_G^* \in (V(G) \cup \mathcal{P}(\mathcal{B})) \subseteq F$, or it is a point on an edge of G which is represented by a shortest permitted path in the feasible region F.

The upper bound on the optimal objective value of the barrier problem follows from

$$f_{\mathcal{B}}(X_{\mathcal{B}}) \quad = \quad \min_{X \in F} \left\{ \max_{m \in \mathcal{M}} \{ w_m\, d_{\mathcal{B}}(X, Ex_m) \} \right\}$$

$$\leq \min_{X \in G} \left\{ \max_{v \in V(G)} \{w(v)\, d_G(X, v)\} \right\} = f_G(X_G^*).$$

□

Observe that the upper bounds given in Lemmas 1 and 2 are in general independent of each other. An example where the optimal solution of problem $1/P/\mathcal{B}/l_{1,\mathcal{B}}/\max$ differs both from the solution of the corresponding restricted problem and of the corresponding network location problem is given in Figure 1. In this example, $X_{\mathcal{R}}^*$ is the optimal solution of problem $1/P/\mathcal{R} = \mathcal{B}/l_1/\max$ with $f(X_{\mathcal{R}}^*) = 4.5$ and $f_{\mathcal{B}}(X_{\mathcal{R}}^*) = 6.5$. The network location problem on the network the embedding of which is represented by all the lines in Figure 1 has two optimal solutions $X_{G,i}^*$, $i = 1, 2$ which both have an objective value of $f_G(X_{G,i}^*) = f_{\mathcal{B}}(X_{G,i}^*) = 7$. However, the optimal solution of the original problem $1/P/\mathcal{B}/l_{1,\mathcal{B}}/\max$ is given by $X_{\mathcal{B}}^*$ with objective value $f_{\mathcal{B}}(X_{\mathcal{B}}^*) = 5.5$.

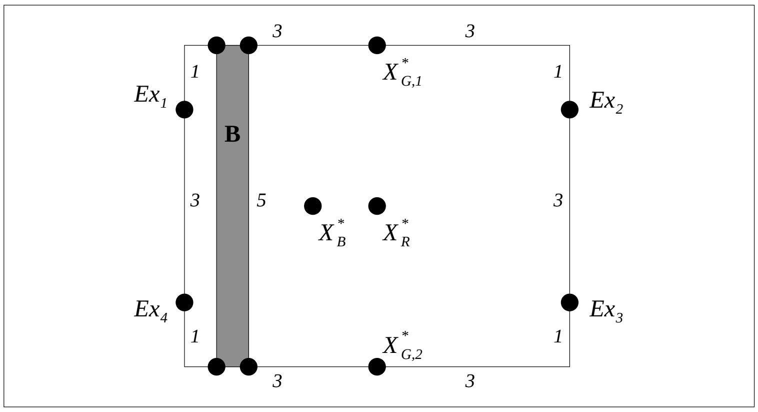

Figure 1: An example problem with four existing facilities and weights equal to 1.

3 The special case of the Manhattan metric and convex polyhedral barriers

In this section a different network than the one used in the previous section is constructed for the special case that distances are measured by the Manhattan metric $d = l_1$ and that all barriers are closed, convex polyhedra with pairwise disjoint interior. Using this network we will develop a polynomial time algorithm that determines at least one optimal solution of the problem $1/P/\mathcal{B}/l_{1,\mathcal{B}}/\max$.

3.1 Shortest l_1-paths in the presence of barriers

Let $l_{1,\mathcal{B}}(X,Y)$ denote the length of an l_1-*shortest permitted path* connecting X and Y in F, i.e. a shortest permitted path with respect to length $l_{1,\mathcal{B}}$. As a special case of the d-visibility definition above, any two points X and Y in F that satisfy

$$l_{1,\mathcal{B}}(X,Y) = l_1(X,Y)$$

are called l_1-*visible*. The set of points $Y \in F$ that are non l_1-visible from a point $X \in F$ is called the *shadow of X with respect to l_1*, i.e.

$$shadow_{l_1}(X) := \{Y \in F \ : \ l_{1,\mathcal{B}}(X,Y) > l_1(X,Y)\}.$$

Note that if a point $X \in F$ is l_1-visible from another point $Y \in F$, then Y is also l_1-visible from X, i.e. the concept of visibility is symmetric. For all $X \in F$ the set $shadow_{l_1}(X)$ is bounded by parts of the boundaries of barriers or by horizontal or vertical line segments or half-lines in F. Furthermore some l_1-visible points are obviously not l_2-visible, i.e. not visible in the usual sense of straight line visibility. On the other hand, the following result holds; see Figure 2 for an example.

Lemma 3 *Every point $X \in F$ that is l_2-visible from the origin is also l_1-visible from the origin. Furthermore, in this case the straight line segment connecting the origin and X is an l_1-shortest permitted path.*

Proof: Let $X \in F$ be a point that is l_2-visible from the origin. Then the straight-line segment connecting the origin and X is a permitted path P from the origin to $X = (x_1, x_2)^T$ with length $l_1(P) = |x_1| + |x_2| = l_1(0, X)$.

$\square$

Since we assume that all barriers are convex polyhedra, the relation between l_1-visibility and l_2-visibility can be used to obtain a simpler description of the barrier distance $l_{1,\mathcal{B}}$. The following lemma is a special case of a result of Viegas and Hansen [26] for l_p-distance functions ($1 \le p \le \infty$).

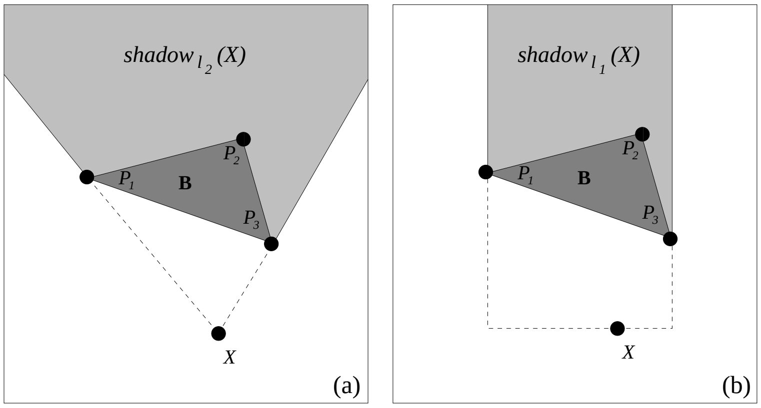

Figure 2: The l_2-shadow and the l_1-shadow of a point $X \in F$.

Lemma 4 *Let $X, Y \in F$. Then there exists an l_1-shortest permitted path, SP, connecting X and Y with the following property.*

$$\text{\textit{SP is a piecewise linear path with breaking points only in extreme points of barriers.}} \tag{1}$$

In the following we will refer to Property (1) as the *barrier touching property.*

3.2 Constructing a cell decomposition of the feasible region

In the following a network $\mathcal{N}$ will be constructed such that l_1-shortest permitted paths between all existing facilities and extreme points of barriers are represented by network paths in $\mathcal{N}$, similar to the visibility graph given in Section 2. Additional edges are added resulting in a decomposition of the feasible region into cells. The four *fundamental vectors* $e^1 = (0,1)^T$, $e^2 = (1,0)^T$, $e^3 = (0,-1)^T$ and $e^4 = (-1,0)^T$ defining the unit ball of the Manhattan metric and the corresponding *fundamental directions* $d^i = \{\lambda e^i \ : \ \lambda \geq 0\}$, $i = 1, \ldots, 4$ play a central role in the construction of $\mathcal{N}$ (see Figure 3).

Let $\mathcal{P}(\mathcal{B})$ and $\mathcal{F}(\mathcal{B})$ denote the set of extreme points and facets, respectively, of the convex barrier polyhedra. For every $X \in (\mathcal{E}x \cup \mathcal{P}(\mathcal{B}))$ and for every fundamental direction d^i, $i = 1, \ldots, 4$, define a *construction line*

$$(X + d^i)_{\mathcal{B}} := \{X + \lambda e^i \ : \ \lambda \in I\!R_+; \ (X + \mu e^i) \cap \mathrm{int}(\mathcal{B}) = \emptyset \ \forall 0 \leq \mu \leq \lambda\}$$

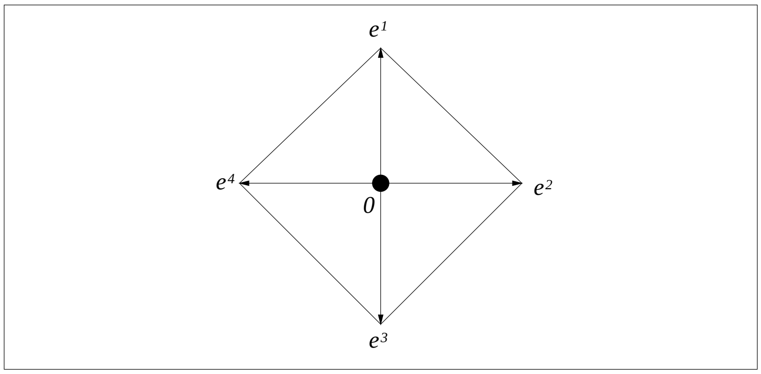

Figure 3: The unit ball of the l_1-norm and its four fundamental vectors.

as the set of points in the plane which are l_2-visible from X in the fundamental direction d^i. Then

$$\mathcal{G} := \left(\bigcup_{X \in \mathcal{E}x \cup \mathcal{P}(\mathcal{B})} \bigcup_{i=1}^{4} (X + d^i)_{\mathcal{B}} \right) \cup \mathcal{F}(\mathcal{B})$$

defines a grid which is a subset of F. Moreover, a network $\mathcal{N}$ corresponding to the grid $\mathcal{G}$ is defined as follows: All possible intersection points of construction lines, or the intersection points of construction lines and facets of a barrier in $\mathcal{G}$ define the set $V(\mathcal{N}) := V(\mathcal{G})$ of vertices of the corresponding network $\mathcal{N}$. Two vertices $u, v \in V(\mathcal{N})$ are connected by an edge in $E(\mathcal{N})$ if they are adjacent on some construction line or facet in $\mathcal{G}$. The length of this edge is then given by the l_1-length of the corresponding line-segment in $\mathcal{G}$.

The grid defined by $\mathcal{G}$ decomposes the feasible region F into a finite set of cells denoted by $C(\mathcal{G})$, i.e. the set of smallest 2-dimensional convex polyhedra with nonempty interior and with extreme points in $V(\mathcal{G})$ (see Figure 4 and compare Figure 11 for an example with two barrier sets). The extreme points of a cell $C \in C(\mathcal{G})$ are called *corner points* of the cell C. Note that the boundary of each cell consists of construction lines or facets of the barriers.

Larson and Sadiq [16] defined a similar network omitting some of the construction lines introduced above by only considering horizontal and vertical tangents to the barrier polyhedra. Even though the properties of l_1-shortest permitted paths with respect to their smaller network need some further discussion, Larson and Sadiq showed that in case of the median objective function the problem can be transformed into a network location problem.

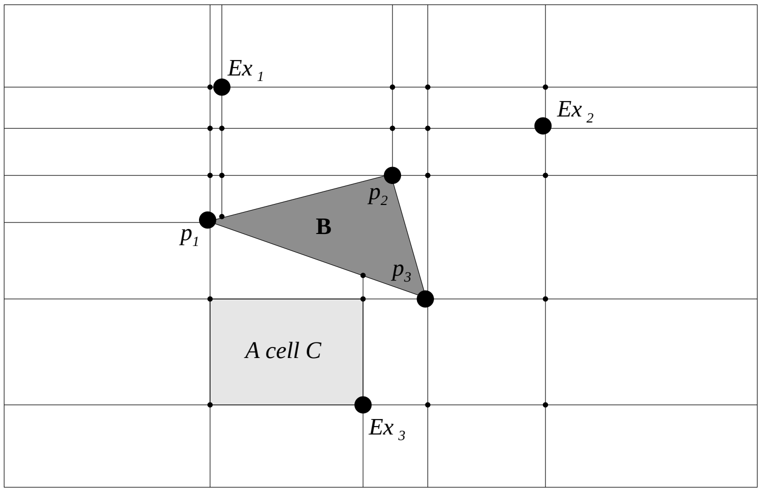

Figure 4: The grid $\mathcal{G}$ for an example problem with three existing facilities and one triangular barrier. The vertices of the corresponding network $\mathcal{N}$ are represented by small dots.

Observe that an analogous result is not true for the center objective function even in the unconstrained case as can be seen in Figure 5.

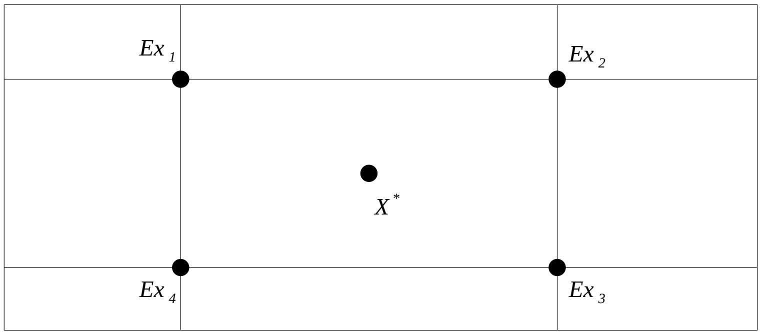

Figure 5: An example with four existing facilities with equal weights ($w_i = 1$, $i = 1, \dots, 4$). The unique optimal solution X^* of $1/P/\bullet/l_1/\max$ lies in the interior of a cell of the corresponding grid $\mathcal{G}$.

Nevertheless it can be shown that the set of optimal solutions $\mathcal{X}^*_{\mathcal{B}}$ of $1/P/\mathcal{B}/\,l_{1,\mathcal{B}}/\max$ is contained in the *rectangular hull* R of the existing facilities and the barrier regions, i.e. in the smallest rectangle R with sides parallel to the coordinate axes, containing all existing facilities, and such that the boundary $\partial(R)$ of R does not intersect with the interior of a barrier.

Theorem 1 *Let R be the smallest axes-parallel rectangle such that $\mathcal{E}x \subseteq R$ and $\partial(R) \cap \operatorname{int}(\mathcal{B}) = \emptyset$. Then the set of optimal solutions of $1/P/\mathcal{B}/l_{1,\mathcal{B}}/\max$ is contained in R, i.e.*

$$\mathcal{X}_{\mathcal{B}}^* \subseteq R.$$

Proof: Suppose that $X_{\mathcal{B}}^* \in \mathcal{X}_{\mathcal{B}}^*$ is an optimal solution not located in R, i.e. $X_{\mathcal{B}}^* \in F \setminus R$. Then the assumption that there exists no barrier in $\mathbb{R}^2 \setminus R$ does not increase the objective value of $X_{\mathcal{B}}^*$ and it has no influence on the objective values of points in R.

Let $R = [x_1, x_2] \times [y_1, y_2]$ and let $X_{\mathcal{B}}^* = (a, b)^T$. Since $X_{\mathcal{B}}^* \notin R$ we can conclude that a is not contained in the closed interval $[x_1, x_2]$ or that b is not contained in the closed interval $[y_1, y_2]$; without loss of generality let $a < x_1$.

Let P_m be an l_1-shortest permitted path from $X_{\mathcal{B}}^*$ to Ex_m with the barrier touching property of Lemma 4, and let $I_m \in \mathcal{P}(\mathcal{B}) \cup \{Ex_m\}$ be an intermediate point on P_m that is l_2-visible from $X_{\mathcal{B}}^*$, $m \in \mathcal{M}$. The straight-line segment connecting $X_{\mathcal{B}}^*$ and I_m, $m \in \mathcal{M}$ intersects $\partial(R)$ in a point $(a_m, b_m)^T$ with $a < a_m$. Thus moving $X_{\mathcal{B}}^*$ towards the boundary of R by increasing a to $a + \epsilon$ with a small $\epsilon > 0$ decreases the l_1-distance between $X_{\mathcal{B}}^*$ and $(a_m, b_m)^T$, and thus also between $X_{\mathcal{B}}^*$ and each of the intermediate points I_m and between $X_{\mathcal{B}}^*$ and Ex_m, $m \in \mathcal{M}$, contradicting the optimality of $X_{\mathcal{B}}^*$.

□

Note that since the barrier polyhedra are compact sets, R is a compact and convex set, the boundary of which is part of the network $\mathcal{N}$. Since the boundary $\partial(R)$ of the rectangular hull R is also feasible, i.e. $\partial(R) \subseteq F$, the following result for l_1-shortest permitted paths between two points in R can be proven:

Lemma 5 *Let X and Y be two points in $F \cap R$. Then every l_1-shortest permitted path connecting X and Y lies completely in $F \cap R$.*

Proof: The result follows from the fact that all barriers $B \subseteq \mathbb{R}^2 \setminus R$ can be discarded. Every permitted path connecting X and Y that leaves the set $F \cap R$ at some point $Z_1 \in \partial(R)$ has to reenter at some point $Z_2 \in \partial(R)$ and will be dominated by a path using the shortest path from Z_1 to Z_2 along the boundary of R instead.

□

Note that Theorem 1 implies not only that it is sufficient to consider only cells within the rectangle R, but that, using Lemma 5, we can also reduce the network $\mathcal{N}$ to a subnetwork $\mathcal{N}' \subseteq \mathcal{N}$ that results from the intersection of the embedding of $\mathcal{N}$ in F with the rectangle R.

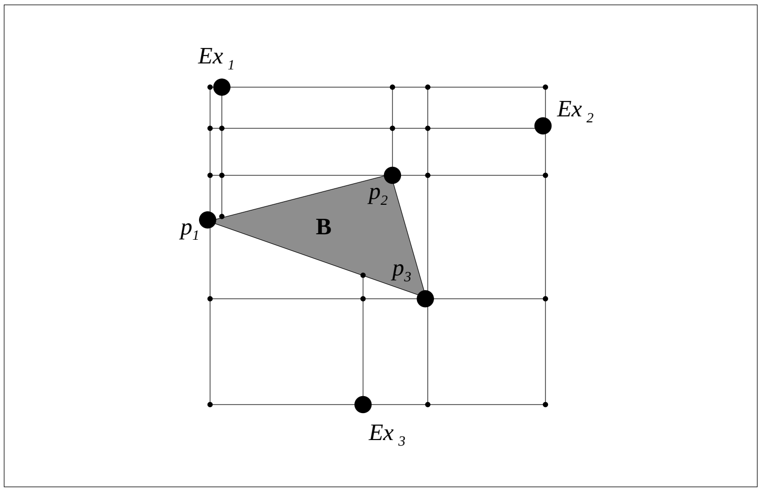

Figure 6: The rectangle R and the network $\mathcal{N}'$ for the example problem introduced in Figure 4.

3.3 Grid vertices on l_1-shortest permitted paths

The decomposition of F into cells $C(\mathcal{G})$ will be used in this section to derive further properties of l_1-shortest permitted paths from arbitrary feasible points to the existing facilities. Since an extended network is used compared to that defined in Larson and Sadiq [16], the following two results which can also be found in this reference can be proven in a more straight forward way.

Lemma 6 *Let $Y \in (\mathcal{E}x \cup \mathcal{P}(\mathcal{B}))$ be an existing facility or an extreme point of a barrier, and let C be a cell in $C(\mathcal{G})$. If Y is l_1-visible from some point in* $\mathrm{int}(C)$*, then Y is l_1-visible from all points in C.*

Proof: Let $Y \in (\mathcal{E}x \cup \mathcal{P}(\mathcal{B}))$. Then the l_1-shadow of Y is bounded by facets of the barriers and by construction lines rooted at extreme points of the barriers. Thus the result follows directly from the construction of the grid $\mathcal{G}$. □

Lemma 7 *Let $Ex_m \in \mathcal{E}x$ be an existing facility and let C be a cell in $C(\mathcal{G})$ with $X \in \text{int}(C)$. Then there exists an l_1-shortest permitted path connecting Ex_m and X that passes through a corner point of C.*

Proof: Let $Ex_m \in \mathcal{E}x$ and let $X = (a,b)^T \in \text{int}(C)$. Furthermore let $P(X, Ex_m)$ be an l_1-shortest permitted path connecting Ex_m and X and satisfying the barrier touching property of Lemma 4. Then there exists an intermediate point $I_m \in (\mathcal{E}x \cup \mathcal{P}(\mathcal{B}))$ on $P(Ex_m, X)$ that is l_1-visible from X. From Lemma 6 it follows that I_m is l_1-visible from all points in C.

Let $x_1 := \min\{x : (x,y)^T \in C\}$, $x_2 := \max\{x : (x,y)^T \in C\}$, $y_1 := \min\{y : (x,y)^T \in C\}$, and $y_2 := \max\{y : (x,y)^T \in C\}$ and let $I_m = (a_m, b_m)^T$. Then a_m can not be contained in the open interval $]x_1, x_2[$ and b_m can not be contained in the open interval $]y_1, y_2[$ since otherwise there would exist a construction line intersecting $\text{int}(C)$.

Since C is a convex polyhedron the boundary of which consists of vertical and horizontal line-segments and of boundary segments of barriers, there exists a corner point $C_m = (c_1, c_2)^T$ of C such that $c_1 \in [a_m, a]$ and $c_2 \in [b_m, b]$. Using additionally the fact that I_m is l_1-visible from every corner point of C and that the corner points of C are l_1-visible from every point in C, we can conclude that there exists a permitted X-C_m-path of length $l_1(X, C_m)$ and a permitted C_m-I_m-path of length $l_1(C_m, I_m)$, the combination of which is an l_1-shortest permitted path connecting X and I_m with the desired property (see Figure 7 for an example).

□

Lemma 7 implies that we can always find l_1-shortest permitted paths to the existing facilities with the following property: Whenever the path enters the interior of a cell, it leaves the cell through a corner point.

However, there might exist cells $C \subseteq (F \cap R)$ having two corner points C_1 and C_2 that are not connected by a network path of length $l_1(C_1, C_2)$ in $\mathcal{N}'$ even though they satisfy $l_{1,\mathcal{B}}(C_1, C_2) = l_1(C_1, C_2)$; see Figure 8 for an example.

Extending $\mathcal{N}'$ by edges of length $l_1(C_1, C_2)$ that connect two corner points C_1 and C_2 of the same cell C that are not yet connected by a network path of length $l_1(C_1, C_2)$ leads to a network $\mathcal{N}''$ with the following property:

Corollary 2 *The length of an l_1-shortest permitted path between a corner point of a cell and an existing facility in $\mathcal{E}x$ is equal to the length of a shortest network path connecting the corresponding vertices in $\mathcal{N}''$.*

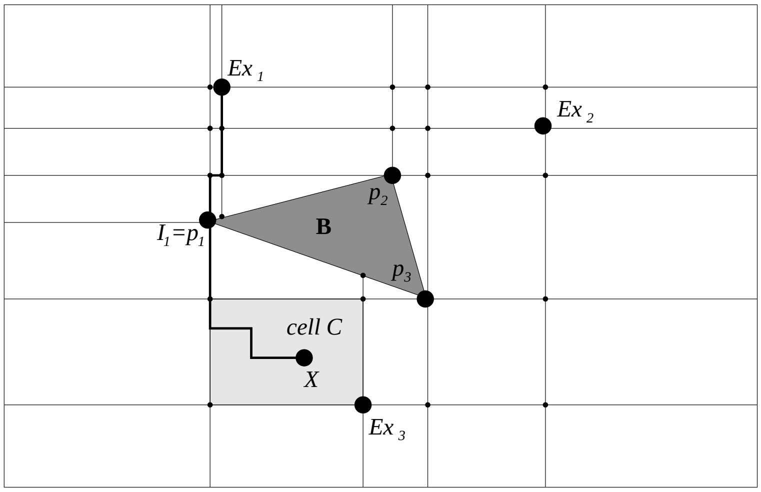

Figure 7: An example of an l_1-shortest permitted path connecting Ex_1 and X that passes through a corner point of the cell C.

Moreover, the length of an l_1-shortest permitted path between two existing facilities in $\mathcal{E}x$ is equal to the length of a shortest network path connecting the corresponding vertices in $\mathcal{N}''$.

Observe that Corollary 2 can in general not be extended to points on lines or line-segments of the grid $\mathcal{G}$ corresponding to points on edges of $\mathcal{N}''$. This can be seen for example in Figure 8 where an l_1-shortest permitted path from points on the facet of B_2 bounding the cell C to the corner point C_2 is not represented in the corresponding network $\mathcal{N}''$. However, the network $\mathcal{N}''$ can be used similar to the visibility graph (cf. Lemma 2) to derive an improved upper bound for the optimal objective value of $1/P/\mathcal{B}/l_{1,\mathcal{B}}/\max$:

Corollary 3 *Let $\mathcal{N}''$ be the extension of $\mathcal{N}'$ as defined above. If $X^*_{\mathcal{N}''}$ is an optimal solution of the network location problem $1/\mathcal{N}''/\bullet/d_{\mathcal{N}''}/\max$ on $\mathcal{N}''$, then the point in the plane corresponding to the point $X^*_{\mathcal{N}''}$ in the embedding of $\mathcal{N}''$ is feasible for $1/P/\mathcal{B}/l_{1,\mathcal{B}}/\max$ and*

$$f_{\mathcal{B}}(X^*_{\mathcal{B}}) \leq f_{\mathcal{B}}(X^*_{\mathcal{N}''}) \leq f_{\mathcal{N}''}(X^*_{\mathcal{N}''}).$$

In general, contrary to the median case (see [16]), this bound is not sharp (see Figure 5). Therefore additional arguments are needed in order to efficiently find an optimal solution to problems of type $1/P/\mathcal{B}/l_{1,\mathcal{B}}/\max$.

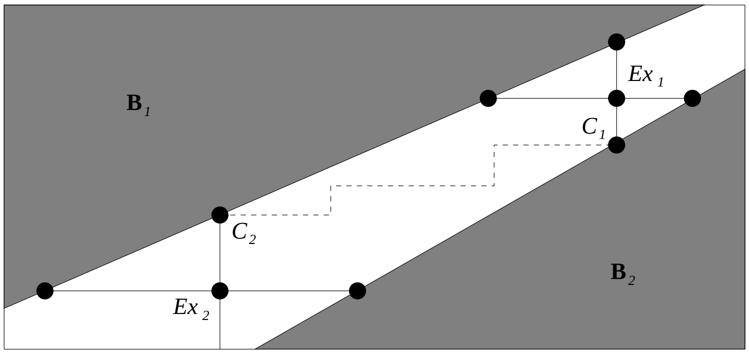

Figure 8: The two corner points C_1 and C_2 are not connected by a network path of length $l_1(C_1, C_2)$ in $\mathcal{N}'$.

3.4 A dominating set for an optimal solution

Corollary 2 enables us to calculate barrier distances between corner points of $\mathcal{G}$ and existing facilities in an efficient way by evaluating network distances in $\mathcal{N}''$. We can now draw our attention to the properties of an optimal solution of $1/P/\mathcal{B}/l_{1,\mathcal{B}}/\max$.

In the following we will use an extension of the concept of weighted bisectors (see, for example, [19] or [22]) to problems involving barriers. For two points $Y_1, Y_2 \in F$ with positive weights $w_1, w_2 \in I\!R_+$ let the *weighted bisector of* Y_1 *and* Y_2 be defined as

$$b(w_1Y_1, w_2Y_2) := \{X \in F \, : \, w_1 l_{1,\mathcal{B}}(X, Y_1) = w_2 l_{1,\mathcal{B}}(X, Y_2)\}.$$

To simplify a further discussion of intermediate points on l_1-shortest permitted paths we additionally define for the constants d_1 and d_2 the *weighted bisector of* Y_1, d_1 *and* Y_2, d_2 as

$$\begin{aligned} &b(w_1(Y_1, d_1), w_2(Y_2, d_2)) := \\ &\qquad \{X \in F \, : \, w_1(l_{1,\mathcal{B}}(X, Y_1) + d_1) = w_2(l_{1,\mathcal{B}}(X, Y_2) + d_2)\}. \end{aligned}$$

A well known result for center problems, that also applies to center problems with barriers, is that every optimal solution has to be located on the weighted bisector of two existing facilities. Otherwise the objective value could be improved by moving the new location towards the existing facility at maximum weighted distance. Note that therefore an optimal solution can always be found as a point on the farthest-point Voronoi diagram (see e.g. [21]) with respect to the existing facilities taking into account the barrier regions (see e.g. [24] for the unconstrained case).

Since the construction of the farthest-point Voronoi diagram is difficult in the presence of barriers, this result is strengthened in the following theorem yielding a solution strategy to solve $1/P/\mathcal{B}/l_{1,\mathcal{B}}/\max$ that avoids the construction of the weighted bisectors of all pairs of existing facilities or the corresponding Voronoi diagram.

In the following we distinguish two different scenarios:

<u>Scenario 1</u>: There exists an optimal solution $X^*_{\mathcal{B}} \in \mathcal{X}^*_{\mathcal{B}}$ of $1/P/\mathcal{B}/l_{1,\mathcal{B}}/\max$ with objective value $z^*_{\mathcal{B}}$ so that $X^*_{\mathcal{B}} \in \mathcal{N}''$ and

$$w_p l_{1,\mathcal{B}}(X^*_{\mathcal{B}}, Ex_p) = z^*_{\mathcal{B}} = w_q l_{1,\mathcal{B}}(X^*_{\mathcal{B}}, Ex_q)$$

is satisfied for *exactly two* different existing facilities $Ex_p, Ex_q \in \mathcal{E}x$.

In this case $X^*_{\mathcal{B}}$ must lie on the intersection of the weighted bisector $b(w_p Ex_p, w_q Ex_q)$ of Ex_p and Ex_q with the network $\mathcal{N}''$.

<u>Scenario 2</u>: Otherwise, there does not exist an optimal solution $X^*_{\mathcal{B}}$ of $1/P/\mathcal{B}/l_{1,\mathcal{B}}/\max$ with optimal objective value $z^*_{\mathcal{B}}$ on $\mathcal{N}''$ that is of maximal weighted distance $z^*_{\mathcal{B}}$ from only two existing facilities in $\mathcal{E}x$. Then an optimal solution $X^*_{\mathcal{B}}$ may exist in the interior of a cell $C \subseteq (F \cap R)$ that lies on only one weighted bisector $b(w_i Ex_i, w_j Ex_j)$ of two different existing facilities $Ex_i, Ex_j \in \mathcal{E}x$. An example for this situation is given in Figure 9.

However, the following theorem proves that in this case an optimal solution of $1/P/\mathcal{B}/l_{1,\mathcal{B}}/\max$ can also be found in the intersection of two weighted bisectors $b(w_i Ex_i, w_j Ex_j)$ and $b(w_j Ex_j, w_k Ex_k)$, determined by three pairwise different existing facilities $Ex_i, Ex_j, Ex_k \in \mathcal{E}x$.

Theorem 2 *Let $\mathcal{X}^*_{\mathcal{B}}$ be the set of optimal solutions of $1/P/\mathcal{B}/l_{1,\mathcal{B}}/\max$ according to Scenario 2, i.e. there does not exist $X^*_{\mathcal{B}} \in \mathcal{X}^*_{\mathcal{B}}$ with $X^*_{\mathcal{B}} \in \mathcal{N}''$*

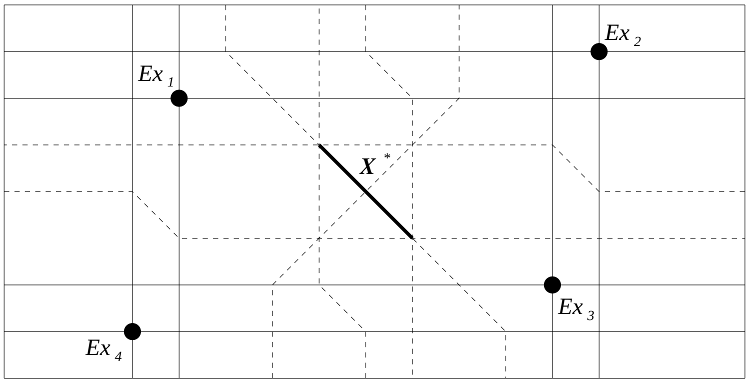

Figure 9: An example problem with four existing facilities having equal weights $w_m = 1$, $m = 1, \ldots, 4$. The weighted bisectors of all pairs of existing facilities are represented by dashed lines.

such that $X^*_{\mathcal{B}}$ is at maximum weighted distance from only two of the existing facilities in $\mathcal{E}x$. Let $z^*_{\mathcal{B}}$ be the optimal objective value of the problem.

Then there exists at least one optimal solution $X^*_{\mathcal{B}} \in \mathcal{X}^*_{\mathcal{B}}$ that has the weighted distance $z^*_{\mathcal{B}}$ from at least three different existing facilities in $\mathcal{E}x$.

Proof: First suppose that there exists $X^*_{\mathcal{B}} \in \mathcal{X}^*_{\mathcal{B}}$ with $X^*_{\mathcal{B}} \in \mathcal{N}''$. Then the fact that every optimal solution of $1/P/\mathcal{B}/l_{1,\mathcal{B}}/\max$ has to lie on at least one weighted bisector of two existing facilities and the assumption that there exists no optimal solution on $\mathcal{N}''$ being at maximum weighted distance from exactly two existing facilities implies the result.

Now suppose that $X^*_{\mathcal{B}}$ is an optimal solution of $1/P/\mathcal{B}/l_{1,\mathcal{B}}/\max$ and that C is a cell such that $X^*_{\mathcal{B}} \in \text{int}(C)$. Then Theorem 1 implies that $C \subseteq R \cap F$. Obviously there exist at least two existing facilities Ex_i and Ex_j in $\mathcal{E}x$ with $z^*_{\mathcal{B}} = w_i l_{1,\mathcal{B}}(X^*_{\mathcal{B}}, Ex_i) = w_j l_{1,\mathcal{B}}(X^*_{\mathcal{B}}, Ex_j)$. Furthermore, let C_i (and C_j, respectively) be a corner point of C such that there exists an l_1-shortest permitted path connecting Ex_i and $X^*_{\mathcal{B}}$ (Ex_j and $X^*_{\mathcal{B}}$, respectively) passing through C_i (C_j, respectively), see Lemma 7.

Now assume that there is no existing facility $Ex_m \in \mathcal{E}x$ other than Ex_i and Ex_j such that $w_m l_{1,\mathcal{B}}(X^*_{\mathcal{B}}, Ex_m) = z^*_{\mathcal{B}}$. Then $C_i \neq C_j$ since otherwise the objective value $z^*_{\mathcal{B}}$ could be improved by moving X^* towards C_i in C.

Defining $d_i := l_{1,\mathcal{B}}(C_i, Ex_i)$ and $d_j := l_{1,\mathcal{B}}(C_j, Ex_j)$ we get that

$$w_i(l_1(X^*_{\mathcal{B}}, C_i) + d_i) = w_j(l_1(X^*_{\mathcal{B}}, C_j) + d_j) = z^*_{\mathcal{B}}$$

and thus $X^*_{\mathcal{B}} \in b(w_i(C_i, d_i), w_j(C_j, d_j)) \cap C$.

Due to the optimality of $X^*_{\mathcal{B}}$ and to the fact that the weighted distance from $X^*_{\mathcal{B}}$ to all other existing facilities is strictly less than $z^*_{\mathcal{B}}$, $X^*_{\mathcal{B}}$ has to be located on an l_1-shortest permitted path connecting C_i and C_j in C. Otherwise we could move $X^*_{\mathcal{B}}$ towards a path with this property, decreasing both the distance to C_i and to C_j and thus also to Ex_i and Ex_j, contradicting the optimality of $X^*_{\mathcal{B}}$. Since there also exists an l_1-shortest permitted path connecting C_i and C_j on the network $\mathcal{N}''$, there exists a point $X_{\mathcal{N}''} \in \mathcal{N}''$ (not necessarily a node, i.e. $X_{\mathcal{N}''}$ may lie on an edge) different from $X^*_{\mathcal{B}}$ on this path (and in the cell C) such that

$$w_i(l_1(X_{\mathcal{N}''}, C_i) + d_i) = w_j(l_1(X_{\mathcal{N}''}, C_j) + d_j) = z^*_{\mathcal{B}}.$$

Thus $X_{\mathcal{N}''} \neq X^*_{\mathcal{B}}$ is also a point on the weighted bisector of C_i, d_i and C_j, d_j, i.e. $X_{\mathcal{N}''} \in b(w_i(C_i, d_i), w_j(C_j, d_j)) \cap C$. Since C is convex, all points on the line-segment

$$\overline{X^*_{\mathcal{B}} X_{\mathcal{N}''}} := \{X \,:\, X = \lambda X^*_{\mathcal{B}} + (1-\lambda) X_{\mathcal{N}''},\ \lambda \in [0,1]\}$$

connecting $X^*_{\mathcal{B}}$ and $X_{\mathcal{N}''}$ lie in C. Furthermore for $m \in \{i, j\}$ all points $X \in \overline{X^*_{\mathcal{B}} X_{\mathcal{N}''}}$ satisfy $w_m l_{1,\mathcal{B}}(X, C_m) = w_m l_1(X, C_m)$ and, due to the linearity of l_1 on C, we obtain $w_m(l_1(X, C_m) + d_m) = z^*_{\mathcal{B}}$.

Thus $X^*_{\mathcal{B}} \in \overline{X^*_{\mathcal{B}} X_{\mathcal{N}''}}$ can be moved along the line-segment $\overline{X^*_{\mathcal{B}} X_{\mathcal{N}''}}$ (which is part of the weighted bisector $b(w_i(C_i, d_i), w_j(C_j, d_j)) \cap C$) in the cell C without increasing the weighted distance to Ex_i and Ex_j until the weighted distance to some other existing facility $Ex_m \in \mathcal{E}x$ equals $z^*_{\mathcal{B}}$, $m \notin \{i, j\}$ or until it reaches the boundary of C. In the latter case, the point $X_{\mathcal{N}''}$ is an optimal solution according to Scenario 1, a case that is excluded by the assumption.

□

Let $DS(\mathcal{N}'')$ denote the set of those points in $\mathcal{N}''$ that are located on the intersection of a weighted bisector $b(w_p Ex_p, w_q Ex_q)$ of two existing facilities with the network $\mathcal{N}''$. Obviously, $DS(\mathcal{N}'')$ contains an optimal solution of $1/P/\mathcal{B}/l_{1,\mathcal{B}}/\max$ in case of Scenario 1.

Similarly, let $DS(C(\mathcal{G}))$ denote the set of points in cells $C \in C(\mathcal{G})$ that are located on the intersection of at least two weighted bisectors $b(w_i Ex_i, w_j Ex_j)$ and $b(w_j Ex_j, w_k Ex_k)$ determined by three pairwise different existing facilities $Ex_i, Ex_j, Ex_k \in \mathcal{E}x$.

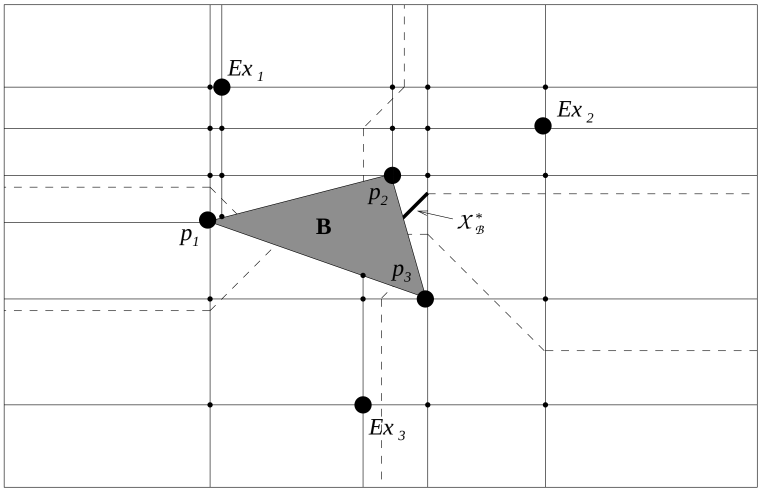

Figure 10: The dashed lines represent the bisectors $b(Ex_i, Ex_j)$ for the example problem introduced in Figure 4. Note that there exist optimal solutions on $\mathcal{N}''$, but not in an intersection of bisectors.

Consequently, Theorem 2 enables us to construct a dominating set $DS \subseteq F$, containing at least one optimal solution of $1/P/\mathcal{B}/l_{1,\mathcal{B}}/\max$. This dominating set can be defined as the union of the two sets $DS(\mathcal{N}'')$ and $DS(C(\mathcal{G}))$, i.e.

$$DS := DS(\mathcal{N}'') \cup DS(C(\mathcal{G})).$$

Before we develop an algorithm to solve $1/P/\mathcal{B}/l_{1,\mathcal{B}}/\max$ based on the dominating set DS, we will first reduce the set DS by further exploiting the particular structure of the problem.

Consider an arbitrary cell $C \in C(\mathcal{G})$ and let the *corner distance* between a point $X \in F \setminus C$ and the cell C be defined as

$$l_{1,corn}(X, C) := \min\{l_{1,\mathcal{B}}(X, C_i) \;:\; C_i \text{ is a corner point of } C\}.$$

Observe that the corner distance between an existing facility $Ex_m \in \mathcal{E}x$ and a cell $C \subseteq (F \cap R)$ can be found as

$$l_{1,corn}(Ex_m, C) = \min\{d_{\mathcal{N}''}(Ex_m, C_i) \;:\; C_i \text{ is a corner point of } C\},$$

using the network $\mathcal{N}''$. Now we can identify an existing facility $Ex^C_{\max} \in \mathcal{E}x$ with weight $w^C_{\max}$ which maximizes the weighted distance to C, i.e.

$$w^C_{\max}\, l_{1,corn}(Ex^C_{\max}, C) = \max\{w_m\, l_{1,corn}(Ex_m, C) \;:\; Ex_m \in \mathcal{E}x\}.$$

Furthermore let

$$|C| := \max\{l_1(C_i, C_j) \,:\, C_i \text{ and } C_j \text{ are corner points of } C\}$$

denote the maximal distance between two corner points of a cell $C \in C(\mathcal{G})$.

Lemma 8 *Let $\mathcal{X}^*_{\mathcal{B}}$ be the set of optimal solutions of* $1/P/\mathcal{B}/l_{1,\mathcal{B}}/\max$ *and let $z^*_{\mathcal{B}}$ be the optimal objective value of the problem. Then every optimal solution $X^*_{\mathcal{B}} \in \mathcal{X}^*_{\mathcal{B}}$ in a cell $C \in C(\mathcal{G})$ lies on the weighted bisector of two different existing facilities $Ex_i, Ex_j \in \mathcal{E}x$ satisfying*

$$w_p\, l_{1,corn}(Ex_p, C) + w_p\,|C| \geq w^C_{\max}\, l_{1,corn}(Ex^C_{\max}, C), \qquad p = i, j.$$

Proof: Recall that every optimal solution $X^*_{\mathcal{B}} \in \mathcal{X}^*_{\mathcal{B}}$ of $1/P/\mathcal{B}/l_{1,\mathcal{B}}/\max$ lies on the weighted bisector of two existing facilities $Ex_i, Ex_j \in \mathcal{E}x$ with $z^*_{\mathcal{B}} = w_p l_{1,\mathcal{B}}(X^*_{\mathcal{B}}, Ex_p)$ for $p = i, j$. Let $C \in C(\mathcal{G})$ be a cell with $X^*_{\mathcal{B}} \in C$. Then

$$z^*_{\mathcal{B}} \geq w^C_{\max}\, l_{1,corn}(Ex^C_{\max}, C).$$

On the other hand,

$$z^*_{\mathcal{B}} = w_p\, l_{1,\mathcal{B}}(X^*_{\mathcal{B}}, Ex_p) \leq w_p\, l_{1,corn}(Ex_p, C) + w_p\,|C|, \qquad p = i, j,$$

which implies the result.

□

Thus with respect to each cell it is sufficient to consider only those existing facilities that satisfy the distance requirement given in Lemma 8. Especially in applications with a large number of uniformly distributed existing facilities, this result leads to a significant reduction of intersection points of weighted bisectors that have to be considered.

Summarizing the results above, we get the following dominating set for an optimal solution of $1/P/\mathcal{B}/l_{1,\mathcal{B}}/\max$:

Theorem 3 *Let DS be a set of points in F consisting, in all cells $C \subseteq R \cap F$, of*

(i) the intersection points of the network $\mathcal{N}''$ with the weighted bisector determined by two different existing facilities $Ex_i, Ex_j \in \mathcal{E}x$, and

(ii) the intersection points of at least two weighted bisectors determined by three pairwise different existing facilities Ex_i, Ex_j, Ex_k,

where only those existing facilities $Ex_p \in \mathcal{E}x$ are considered in (i) and (ii) that satisfy

$$w_p \, l_{1,corn}(Ex_p, C) + w_p \, |C| \geq w_{\max}^C \, l_{1,corn}(Ex_{\max}^C, C).$$

Then DS contains at least one optimal solution of $1/P/\mathcal{B}/l_{1,\mathcal{B}}/\max$.

Note that the dominating set *DS* of Theorem 3 is in general not finite since the set of intersection points of two weighted bisectors with respect to rectilinear distance is not necessarily a finite set.

4 Algorithmic Consequences

As was shown in the previous sections, in case of the center objective function it is not sufficient to consider intersection points of the grid $\mathcal{G}$ as it was the case for the median objective function. Moreover, the problem is not equivalent to a network location problem on $\mathcal{N}''$ even though this was proven for the corresponding median problem [16]. Therefore it is necessary to consider additional points in the intersections of specific weighted bisectors between existing facilities in order to find an optimal solution of the center problem with barriers. An example problem where the unique optimal solution lies in the intersection of weighted bisectors and not on the grid $\mathcal{N}''$ is given in Figure 11. Observe also that for example the bisector $b(Ex_1, Ex_3)$ has a breakpoint in the interior of a cell in this example.

The following outline of an algorithm for the solution of center problems with barriers and rectilinear distances is based on the construction of intersections of weighted bisectors between pairs of existing facilities, yielding a dominating set *DS* for an optimal solution of $1/P/\mathcal{B}/l_{1,\mathcal{B}}/\max$ as discussed above. Observe that the determination of the individual candidate sets $DS(\mathcal{N}'')$ and $DS(C(\mathcal{G}))$ can be combined since both candidate sets use the same weighted bisectors between pairs of existing facilities.

Algorithm 1 (Bisector Algorithm for $1/P/\mathcal{B}/l_{1,\mathcal{B}}/\max$)

Input: *Location problem* $1/P/\mathcal{B}/l_{1,\mathcal{B}}/\max$.

Step 1: *Construct $\mathcal{G}$, R and $\mathcal{N}''$.*

Step 2: *Find a dominating set DS by determining the weighted bisectors $b(w_i Ex_i, w_j Ex_j) \cap R$ between all pairs of existing facilities, and the intersection of*

> *(a) all pairs of weighted bisectors, determined by three existing facilities at a time*
>
> *(b) and of weighted bisectors with the network $\mathcal{N}''$.*

Step 3: *From the candidate solutions determined in Steps 2(a) and (b), find $z^*_{\mathcal{B}} := \min\{f_{\mathcal{B}}(X) \,:\, X \in DS\}$ and $\mathcal{X}_{\mathcal{B}} := \arg\min\{f_{\mathcal{B}}(X) \,:\, X \in DS\}$.*

Output: *A subset $\mathcal{X}_{\mathcal{B}}$ of the set of optimal solutions $\mathcal{X}^*_{\mathcal{B}}$ of $1/P/\mathcal{B}/l_{1,\mathcal{B}}/\max$ and the optimal objective value $z^*_{\mathcal{B}}$.*

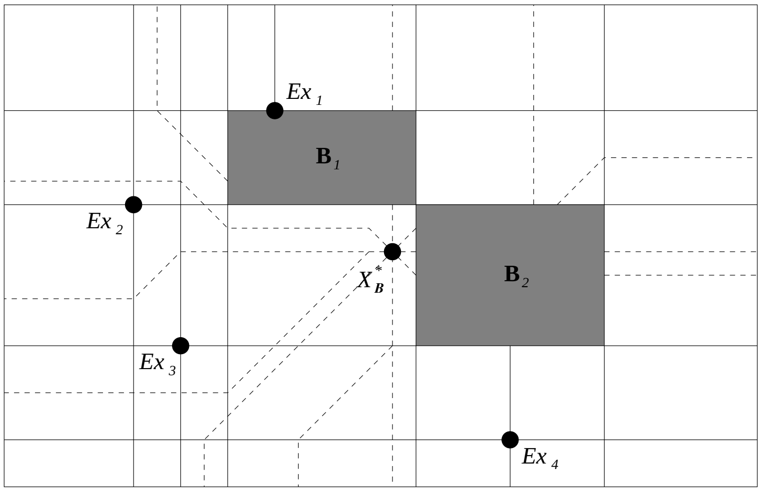

Figure 11: An example problem with four existing facilities having equal weights $w_m = 1$, $m = 1, \ldots, 4$. The weighted bisectors of all pairs of existing facilities are represented by dashed lines.

In Step 1 of Algorithm 1 the fundamental data structures are implemented, and a network $\mathcal{N}''$ of the same asymptotic size as the network used in [16] for the corresponding median problem is constructed. In particular, the number of vertices, edges and cells of the original network $\mathcal{N}$ can be bounded by $O((|\mathcal{E}x| + |\mathcal{P}(\mathcal{B})|)^2)$ using simple planarity arguments. Since $\mathcal{N}' \subseteq \mathcal{N}$ and since the number of edges added in the transition from $\mathcal{N}'$ to $\mathcal{N}''$ can be bounded by a constant within each cell, the same bounds also hold for the extended network $\mathcal{N}''$.

Different from the case of the median objective function, the additional determination of weighted bisectors is needed in the case of the center objective function in Step 2 of the algorithm. These bisectors can be found adapting an algorithm of Mitchell [19] which is based on the determination of l_1-shortest permitted paths in the presence of polyhedral barriers. The presented algorithm extends a "continuous Dijkstra" technique of propagating a "wavefront" from a given source node s towards a termination node t. The propagation is implemented based on "dragged segments", that is, line segments that make up portions of the wavefront and that are dragged in south-east, north-east, north-west and south-west direction, respectively, according to the unit ball of the Manhattan metric. Mitchell [19] showed that this approach can also be used for the determination of bisectors between pairs of points. In particular, the bisector $b(Ex_i, Ex_j)$ between two existing facilities can be constructed in time $O(|\mathcal{P}(\mathcal{B})| \cdot \log |\mathcal{P}(\mathcal{B})|)$ by initiating one wavefront at each of the two facilities. If the weight of the respective facility is associated with its wavefront (and with each of the corresponding dragged segments whose data structure includes the information about the segments distance from its source node), this algorithm can be easily adapted to handle the more general case of a weighted bisector $b(w_iEx_i, w_jEx_j)$ with positive weights w_i, w_j.

The number of intersections of weighted bisectors determined by three existing facilities at a time is bounded by $O(|\mathcal{E}x|^3)$, where each intersection may consist of a set of points, and the number of intersections of weighted bisectors between two existing facilities and the network $\mathcal{N}''$ is bounded by $O(|\mathcal{E}x|^2)$. Observe that all weighted bisectors are piecewise linear, and that in the case that an intersection consists of a set of points, a "best candidate" solution within this set can be determined by minimizing the weighted distance to only one of the facilities defining the set. Restricting this search to cells and using the piecewise linearity of $l_{1,\mathcal{B}}$ as well as its description based on cell corners, this remains a simple task that can be solved as part of Step 3 of the algorithm.

We can conclude that the overall complexity of Algorithm 1 remains polynomial even if all of the $O(|\mathcal{E}x|^3)$ intersection sets are examined. Note that the reductions of the dominating set due to Theorem 1 and Lemma 8 are not yet reflected in this discussion.

5 Conclusions

In this paper a dominating set result is developed for center location problems with Manhattan distances where polyhedral barriers restrict traveling in the plane. A polynomial time algorithm to solve this non-convex optimization problem is suggested which is based on this dominating set result.

All results can be extended to barrier center problems with respect to arbitrary block norms having four fundamental vectors using an appropriate linear transformation of the coordinate system.

This paper can be seen as a continuation of earlier work on the discretization of planar location problems which has proven to be a powerful method in location theory. Future research includes the analysis of level curves for barrier problems which will be helpful, for instance, in dealing with multi-criteria location problems with barriers.

It is an interesting open question whether the wave-front approach used in [19] could be further combined with the ideas discussed in the previous sections. A first promising result in this direction was obtained in the two recent masters projects [7] and [25]. In these projects, prototype algorithms were implemented that allow the solution of example problems with the Manhattan metric and with the Euclidean metric. This interesting approach will be further developed in future research.

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