An Efficient Solution Method for Weber Problems with Barriers based on Genetic Algorithms

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Abstract

In this paper we consider the problem of locating one new facility with respect to a given set of existing facilities in the plane and in the presence of convex polyhedral barriers. It is assumed that a barrier is a region where neither facility location nor travelling are permitted. The resulting non-convex optimization problem can be reduced to a finite series of convex subproblems, which can be solved by the Weiszfeld algorithm in case of the Weber objective function and Euclidean distances. A solution method is presented that, by iteratively executing a genetic algorithm for the selection of subproblems, quickly finds a solution of the global problem. Visibility arguments are used to reduce the number of subproblems that need to be considered, and numerical examples are presented.

Key words: facility location; barriers; non-convex optimization; genetic algorithm

1 Introduction

Location problems are not only interesting and challenging from a theoretical point of view, they also have a variety of practical applications. As a typical example one may think of locating a new warehouse such that the travel costs to the suppliers and the customers of a company are minimized. In general, the quality $f(X)$ of a location $X \in \mathbb{R}^2$ for the *new facility* depends on the distances $d(X, Ex_m)$, $m = 1, \ldots, M$, between X and a finite set of *existing facilities* $\mathcal{E}x = \{Ex_1, \ldots, Ex_M\} \subset \mathbb{R}^2$. We will assume that the distance metric d is induced by a norm in $\mathbb{R}^2$. A location $X^\star$ is called optimal if it minimizes the objective value $f(X^\star)$. Throughout this paper, we consider the classical *Weber objective function* and hence seek to minimize the sum of the weighted distances between the new and the existing facilities:

$$\begin{array}{ll} \min & f(X) = \sum_{m=1}^{M} w_m d(X, Ex_m) \\ \text{s. t.} & X \in \mathbb{R}^2. \end{array} \quad (1)$$

The positive weights $w_m \in \mathbb{R}_+$, $m = 1, \ldots, M$, can be interpreted as the demand of the corresponding existing facilities, such that the objective models, in this case, the total transportation costs.

In realistic location models, various types of restrictions and constraints have to be incorporated in order to better represent the geographic reality in its geometric representation. Location problems with *forbidden regions* have been extensively studied and can be considered relatively well-solved (for an overview, see Hamacher and Nickel [1995]). On the other hand, problems involving physical *barriers* or *congestions* like mountain ranges, rivers or lakes still give rise to many open questions that are caused by the non-convexity of their objective functions (see Sarkar et al. [2004] for problems with congestions).

Barrier regions $\mathcal{B} = \bigcup_{i=1}^{N} B_i \subset \mathbb{R}^2$ impose strong restrictions on location problems since not only the location of facilities is restricted to a feasible region $\mathcal{F} = \mathbb{R}^2 \setminus \mathrm{int}(\mathcal{B})$, but also travelling is prohibited in the interior of the barriers, resulting in non-convex distance functions. To avoid infeasible cases we assume that $B_1, \ldots, B_N$ are pairwise disjoint and closed, that $\mathcal{F}$ is connected and that all existing facilities $Ex_1, \ldots, Ex_M$ are in $\mathcal{F}$. If a distance function d induced by a norm $\|\cdot\|_d$ in $\mathbb{R}^2$ is given to model unconstrained distances, the *barrier distance* $d_{\mathcal{B}}(X, Y)$ between two points $X, Y \in \mathcal{F}$ is defined as the length $l(P)$ (measured with respect to the given norm $\|\cdot\|_d$) of a shortest *feasible path* P connecting the two points and not intersecting the interior of a barrier:

$$d_{\mathcal{B}}(X, Y) := \inf\{l(P) : P \text{ feasible } X\text{-}Y \text{ path}\}.$$

Note that $d_{\mathcal{B}}$ is in general not positive homogenous, however, it is positive semidefinite and symmetric and satisfies the triangle inequality. Therefore, the barrier distance $d_{\mathcal{B}}$ defines a metric on $\mathcal{F}$. In contrast to the classical Weber problem (1), the *Weber problem with barriers*

$$\begin{array}{ll} \min & \sum_{m=1}^{M} w_m d_{\mathcal{B}}(X, Ex_m) \\ \text{s.t.} & X \in \mathcal{F} \end{array} \tag{2}$$

is in general non-convex.

Barriers were first introduced to location modeling by Katz and Cooper [1981]. The authors considered a Weber problem with the Euclidean metric and with one circular barrier. A heuristic algorithm was suggested that is based on a sequential unconstrained minimization technique for nonlinear programming problems. The problem was further analysed in Klamroth [2004] and it was shown that in the case of a single circular barrier the feasible set can be subdivided into a polynomial number of cells, on every convex subset of which the Weber objective function is convex.

Most of the work on location problems with barriers concentrates on special barrier shapes or special distance functions. Assuming that all barrier sets are polyhedra allows for the construction of a visibility graph of the existing facilities and the extreme points of the barrier polyhedra. Two nodes u, v of this graph are connected by an edge of length $d(u, v)$ if the corresponding points (existing facilities or barrier extreme points) in the plane have distance $d_B(u, v) = d(u, v)$. This visibility graph was used by Aneja and Parlar [1994] and Butt and Cavalier

[1996] for the evaluation of the objective function value at solution points in the context of heuristic and iterative algorithms. McGarvey and Cavalier [2003] embed the results of Butt and Cavalier [1996] into an application of the Big Square Small Square method [see Hansen et al., 1985, Plastria, 1992] to approximate the global optimum. In Klamroth [2001a] and Klamroth [2001b] it was shown that an optimal solution of the non-convex barrier problem can be found by solving a finite and, in the case of line barriers, polynomial number of related unconstrained subproblems, a result which will be used extensively in this paper. A generalization to multicriteria problems was discussed in Klamroth and Wiecek [2002].

From the point of view of special distance functions, rectilinear and, more general, block norm distances played a central role for the development of discretization based solution procedures. Larson and Sadiq [1983] identified an easily determined finite dominating set for rectilinear distances. This result was generalized by Batta et al. [1989] who also included forbidden regions into the model, and by Savaş et al. [2002] and Wang et al. [2002] who located finite size facilities acting as barriers themselves. For general block norm distances and polyhedral gauges, discretization results were developed in Hamacher and Klamroth [2000], Dearing et al. [2002] and Nandikonda and Nagi [2003] for Weber problems and center problems, respectively. Fekete et al. [2005] introduced Weber problems with continuous demand over some given polyhedral set, possibly with holes acting as barriers to travel, and the Manhattan metric. Kusakari and Nishizeki [1997], Choi et al. [1998] and Ben-Moshe et al. [2001] focused on computationally efficient, polynomial solution approaches for specially structured problems based on rectilinear barrier sets and distance functions. Lower and upper bounds as well as the relative accuracy of solutions for multi-facility Weber problems with the Manhattan metric, with and without barriers, were discussed in Batta and Leifer [1988].

A different approach to handle the non-convexity of the objective function can be seen in the application of global optimization methods (see Hansen et al. [1995] for an overview). A comprehensive overview about the state of the art in continuous location theory incorporating barriers is provided in Klamroth [2002].

In this paper, we make use of the decomposition approach of Klamroth [2001a] to replace the non-convex Weber problem with polyhedral barriers by a finite series of unconstrained subproblems which can be further decomposed into convex subproblems. It is shown that the number of such candidate problems can be significantly reduced by using visibility arguments. Although the method is designed for location problems in the plane $\mathbb{R}^2$ and with Euclidean distances $d(X,Y) = l_2(X,Y) = \sqrt{(Y_1 - X_1)^2 + (Y_2 - X_2)^2}$, the theoretical results will be developed for the general case that d is induced by an arbitrary norm in $\mathbb{R}^2$. To reduce the computational burden of solving a large number of such subproblems, appropriate convex subproblems are selected using a genetic algorithm and utilizing additional theoretical arguments whenever possible.

The remainder of this paper is organized as follows. In the following section, the theoretical foundations for our solution method are briefly summarized, referring to Klamroth [2001a] for most of the technical details. Section 3 discusses different possibilities of reducing the complexity of the obtained subproblems, and the general idea of the solution method is outlined in Section 4. The embedded genetic algorithm is described in some detail in Section 5, and numerical results for well-known as well as newly developed test problems are presented in Section 6.

2 Theoretical Background

Let $\{B_1, \ldots, B_N\} \subset \mathbb{R}^2$ be a finite set of pairwise disjoint, closed, polyhedral barriers, let $\mathcal{B} := \bigcup_{i=1}^{N} B_i$ and $\mathcal{F} := \mathbb{R}^2 \setminus \text{int}(\mathcal{B})$, and let $\mathcal{P}(\mathcal{B})$ be the finite set of barrier extreme points. Two points $X, Y \in \mathcal{F}$ are called *d-visible* with respect to a given distance metric d, $Y \in \text{visible}_d(X)$ and $X \in \text{visible}_d(Y)$, if they satisfy $d_{\mathcal{B}}(X,Y) = d(X,Y)$, i.e., if the shortest feasible path between the two points is not lengthened by barriers. More formally, the set of all d-visible points from $X \in \mathcal{F}$ is given by

$$\text{visible}_d(X) := \{Y \in \mathcal{F} :\; d_{\mathcal{B}}(X,Y) = d(X,Y)\},$$

and the corresponding set of all *not* d-visible points from $X \in \mathcal{F}$ by

$$\text{shadow}_d(X) := \{Y \in \mathcal{F} :\; d_{\mathcal{B}}(X,Y) > d(X,Y)\}.$$

A useful property that facilitates the actual determination of shortest feasible paths in the plane $\mathbb{R}^2$ is the *barrier touching property* (BTP), that is proven, for example, in Klamroth [2001a]:

Theorem 2.1 (Barrier Touching Property, BTP)**.** *Between any two points $X, Y \in \mathcal{F}$ there exists a shortest feasible path P that consists of line segments with breaking points only in extreme points of barriers.*

If there exists a shortest feasible X-Y path P such that the point $I_{X,Y} \in \mathcal{P}(\mathcal{B}) \cap \text{visible}_d(X)$ is a breaking point of P, then $I_{X,Y}$ is called an *intermediate point*. If Y is d-visible from X, we set $I_{X,Y} := Y$. Note that, due to the BTP, there exists an intermediate point for all pairs of points $X, Y \in \mathcal{F}$. The barrier distance between X and Y can hence be computed as

$$d_{\mathcal{B}}(X,Y) = d(X, I_{X,Y}) + d_{\mathcal{B}}(I_{X,Y}, Y).$$

The *visibility graph* of $\mathcal{E}x \cup \mathcal{P}(\mathcal{B})$ can be defined as a graph $G = (V(G), E(G))$ with node set $V(G) = \mathcal{E}x \cup \mathcal{P}(\mathcal{B})$. Two nodes $v_i, v_j \in V(G)$ are connected by an edge $[v_i, v_j] \in E(G)$ of length $l_{i,j}$ if the corresponding points in the plane are d-visible and if $l_{i,j} = d(v_i, v_j)$. Note that the visibility graph does not depend on the location of the new facility X. The network distances $d_G(v_i, v_j)$, i.e. the length of the shortest network paths between two nodes $v_i, v_j \in V(G)$, can thus be computed before the optimization process is started.

If the intermediate point I_{X,Ex_m} on the shortest feasible path from the new location X to an existing facility Ex_m is known, the barrier distance $d_{\mathcal{B}}(X, Ex_m)$ can be computed as

$$d_{\mathcal{B}}(X, Ex_m) = d(X, I_{X,Ex_m}) + d_G(I_{X,Ex_m}, Ex_m).$$

This observation motivated a subdivision of the Weber problem with barriers (2) into a finite series of unconstrained subproblems, defined with respect to different candidate sets for the respective intermediate points, see Klamroth [2002]. A slightly modified partition of the feasible set into so-called *candidate domains* will be used in this paper:

Definition 2.2 (Candidate Set and Candidate Domain)**.** *For a given point $X \in \mathcal{F}$, let $\mathcal{I} := (\mathcal{E}x \cup \mathcal{P}(\mathcal{B})) \cap \text{visible}_d(X)$ be the set of all existing facilities and barrier extreme points that are d-visible from X. $\mathcal{I}$ is called the* candidate set *of X. The set*

$$R := \{Y \in \mathcal{F} : (\mathcal{E}x \cup \mathcal{P}(\mathcal{B})) \cap \text{visible}_d(Y) = \mathcal{I}\} \neq \emptyset$$

is called the candidate domain *of X. The set of all candidate domains is denoted as $\mathcal{R}$.*

Lemma 2.3. *The set of all candidate domains defines a partition of the feasible region $\mathcal{F}$, since for $\mathcal{R} = \{R_1, \ldots, R_n\}$, the following holds:*

1. $\bigcup_{i=1}^{n} R_i = \mathcal{F}$
2. $R_i, R_j \in \mathcal{R},\ i \neq j \Rightarrow R_i \cap R_j = \emptyset$.

Proof.

1. Let $X \in \mathcal{F}$, and let $\mathcal{I} := (\mathcal{E}x \cup \mathcal{P}(\mathcal{B})) \cap \text{visible}_d(X)$ be the candidate set of X. Then there exists $R \in \mathcal{R}$ with $X \in R$, namely

$$R = \{Y \in \mathcal{F} :\ (\mathcal{E}x \cup \mathcal{P}(\mathcal{B})) \cap \text{visible}_d(Y) = \mathcal{I}\}.$$

2. Let $R_i, R_j \in \mathcal{R}$, $i \neq j$, be two candidate domains, and suppose that there exists a feasible point $X \in \mathcal{F}$ with $X \in R_i \cap R_j$. Let $\mathcal{I} = (\mathcal{E}x \cup \mathcal{P}(\mathcal{B})) \cap \text{visible}_d(X)$ be the candidate set of X, then

$$R_i = \{Y \in \mathcal{F} :\ (\mathcal{E}x \cup \mathcal{P}(\mathcal{B})) \cap \text{visible}_d(Y) = \mathcal{I}\} = R_j$$

in contradiction to $R_i \neq R_j$, thus proving that $R_i \cap R_j = \emptyset$.

□

Consequently, the feasible region $\mathcal{F}$ can be partitioned into pairwise disjoint candidate domains such that the candidate sets $\mathcal{I}$ remain constant within each of them. Since the number of existing facilities and barrier extreme points is finite, the number of candidate domains is also finite, and can be bounded by the cardinality of the power set $\mathcal{P}(\mathcal{E}x \cup \mathcal{P}(\mathcal{B}))$. Depending on the given distance function d, this number can be further reduced using similar arguments as in

Klamroth [2002]. This implies that the facility location problem with barriers can be reduced to a finite series of unconstrained subproblems, each of which is defined on a candidate domain $R \in \mathcal{R}$.

In order to formulate the corresponding subproblems, let $X \in \mathcal{F}$ and let $R \in \mathcal{R}$ be the candidate domain of X and $\mathcal{I}$ the candidate set of X. Wlog let $\mathcal{I} = \{I_1, \ldots, I_k\}$, $0 \leq k \leq |\mathcal{E}x \cup \mathcal{P}(\mathcal{B})|$. We define binary variables y_{im}, $m = 1, \ldots, M$, $i = 1, \ldots, k$, as

$$y_{im} = \begin{cases} 1 & \text{if } I_i \text{ is assigned to } Ex_m \text{ as intermediate point } I_{X,Ex_m}, \\ 0 & \text{otherwise.} \end{cases}$$

Then a subproblem of (2) on the candidate domain R can be formulated as

$$\begin{array}{lll} \min & \sum_{m=1}^{M} \left(\sum_{i=1}^{k} y_{im} w_m \left[d(X, I_i) + d_{\mathcal{B}}(I_i, Ex_m) \right] \right) & \\ \text{s.t.} & X \in R & \\ & \sum_{i=1}^{k} y_{im} = 1, \qquad m = 1, \ldots, M & \\ & y_{im} \in \{0,1\}, \qquad m = 1, \ldots, M, \quad i = 1, \ldots, k. & \end{array} \tag{3}$$

Note that in this formulation, the Weber objective function can be rewritten as a sum of two terms, the first of which depends on the continuous variables X and does not contain barrier distances. The second term of the sum remains constant as long as the selection of intermediate points remains unchanged:

$$\begin{array}{ll} \min & \sum_{m=1}^{M} \sum_{i=1}^{k} y_{im} w_m d(X, I_i) + \sum_{m=1}^{M} \sum_{i=1}^{k} y_{im} w_m d_{\mathcal{B}}(I_i, Ex_m) \\ \text{s.t.} & X \in R \\ & \sum_{i=1}^{k} y_{im} = 1, \quad m = 1, \ldots, M \\ & y_{im} \in \{0,1\}, \qquad m = 1, \ldots, M, \quad i = 1, \ldots, k. \end{array} \tag{4}$$

Observe that the integrality constraints for the binary variables y_{im} can be relaxed to $y_{im} \in [0,1]$, $m = 1, \ldots, M$, $i = 1, \ldots, k$. This relaxation of problem (4) is bi-convex in the sense that, for fixed y, the problem is convex in the location variables X, and for fixed X, the problem is convex in the assignment variables y. When solving problem (4) for all candidate domains, all local and consequently all global optimal solutions of the Weber problem with barriers (2) can be determined since the union of all candidate domains equals the feasible region.

3 Reduction of Candidate Sets

The main difficulty in solving the subproblems (4) lies in the determination of the binary assignment variables y_{im}, $m = 1, \ldots, M$, $i = 1, \ldots, k$, or, in other words, in the identification of the optimal intermediate points for a given candidate domain. Solution times can be improved significantly if we succeed in reducing the number of these assignment variables. In the following, visibility arguments will be applied to reduce the candidate sets for intermediate points and thus the solution space of (4).

Definition 3.1 (Projection Point). *Let $X, Y \in \mathcal{F}$ be two feasible points. If there exists a feasible X-Y path P such that $P_{X,Y} \in \mathcal{F} \cap \mathcal{P}(\mathcal{B})$ is the last point on P that is d-visible from X, then $P_{X,Y}$ is called an X-Y* projection point. *If Y is d-visible from X, we set $P_{X,Y} = Y$. The set of all X-Y projection points is denoted as $\mathcal{P}_{X,Y}$.*

Theorem 3.2 (Projection Point Property). *For two feasible points $X, Y \in \mathcal{F}$ there exists a shortest feasible X-Y path P that satisfies the following property: P satisfies the BTP with a breaking point at a projection point $P_{X,Y} \in \mathcal{P}_{X,Y}$.*

Proof. If X and Y are d-visible, then $P_{X,Y} = Y$ and the property is shown. Thus, let $Y \in \text{shadow}_d(X)$. Due to the BTP there exists a shortest feasible X-Y path P that is piecewise linear and has breaking points only at points $I \in \mathcal{P}(\mathcal{B})$. Wlog let the parametrization of the path P be given by $p : [0,1] \to \mathbb{R}^2$ with $p(0) = X$ and $p(1) = Y$. Let $\lambda := \inf\{\lambda \in [0,1] : \ p(]\lambda, 1]) \subseteq \text{shadow}_d(X)\}$. Then $\lambda < 1$ since $Y \in \text{shadow}_d(X)$, and

$$p(\lambda) \in \text{visible}_d(X) \quad \text{and} \quad p(]\lambda, 1]) \subset \text{shadow}_d(X).$$

Set $P_{X,Y} := p(\lambda) \in \mathcal{P}(\mathcal{B})$. It remains to show that $P_{X,Y}$ is a breaking point of the path P. For this purpose, let $I_n \in \mathcal{P}(\mathcal{B}) \cup \{Y\}$ be the next breaking point of the path P after $P_{X,Y}$ in direction to Y, and let $I_p \in \mathcal{P}(\mathcal{B}) \cup \{X\}$ be the previous breaking point of P before $P_{X,Y}$. (If there are no breaking points on the respective subpaths, we set $I_n := Y$ and/or $I_p := X$.) P is piecewise linear and consequently contains the line segments $[I_p, P_{X,Y}]$ and $[P_{X,Y}, I_n]$. Moreover, $I_p \in \text{visible}_d(X)$ (since P is a shortest feasible path and $P_{X,Y} \in \text{visible}_d(X)$), and hence all points on the halfline h starting at I_p and passing through $P_{X,Y}$ are d-visible from X (see Klamroth [2001a]), while $]P_{X,Y}, I_n] \subset \text{shadow}_d(X)$. This implies that the segment $]P_{X,Y}, I_n]$ must have an empty intersection with h, and hence the path P must have a breaking point at $P_{X,Y}$. □

Applying Theorem 3.2 to the subproblems (4) immediately implies the following result:

Theorem 3.3. *Let $X \in \mathcal{F}$, let $\mathcal{I}$ be the candidate set of X and let $R \in \mathcal{R}$ be the candidate domain of X. Then there exists an optimal assignment of intermediate points to existing facilities such that for all $m = 1, \dots, M$ the intermediate point on the shortest feasible path to Ex_m is an X-Ex_m projection point.*

Observe that the set of $X - Ex_m$ projection points $\mathcal{P}_{X,Ex_m}$ remains constant over the candidate domain of X, i.e., $\mathcal{P}_{X,Ex_m} = \mathcal{P}_{Ex_m} \subset \mathcal{I}$ for all $X \in R$, since the set of visible barrier extreme points and existing facilities is constant over R. Hence Theorem 3.3 suggests the consideration of different subsets of candidates for intermediate points, depending on the existing facility that is to be reached. Therefore, let $\mathcal{I}_m$ denote the candidate set for intermediate points with respect to an existing facility Ex_m, $m \in \{1, \dots, M\}$, and a given candidate domain R. An easily applicable consequence of Theorem 3.3 is thus the following corollary:

Corollary 3.4. *Let $X \in \mathcal{F}$ and $R \in \mathcal{R}$ be the candidate domain of X. Let $Ex_m \in \mathcal{E}x$ be an existing facility. If $Ex_m \in \text{visible}_d(X)$, the candidate set $\mathcal{I}_m$ can be set to $\mathcal{I}_m = \{Ex_m\}$. Otherwise the intermediate point to Ex_m can be chosen out of the reduced candidate set $\mathcal{I}_m = \mathcal{P}_{X,Ex_m}$.*

Using the following technical lemma, we will show that the candidate sets $\mathcal{I}_m$, defined with respect to an existing facility Ex_m and a candidate domain R, can be also reduced using simple ordering arguments.

Lemma 3.5. *Let $X, Y \in \mathcal{F}$ be two feasible points, and let $I_1, I_2 \in \text{visible}_d(X)$ be two feasible points that are d-visible from X. If there exists a shortest feasible I_1-Y path P such that $I_2 \in P$, then*

$$d(X, I_1) + d_{\mathcal{B}}(I_1, Y) \geq d(X, I_2) + d_{\mathcal{B}}(I_2, Y).$$

Proof.

$$\begin{array}{rlcl}
 & d_{\mathcal{B}}(X, I_1) + d_{\mathcal{B}}(I_1, I_2) & \geq & d_{\mathcal{B}}(X, I_2) \\
\overset{I_1, I_2 \in \text{visible}_d(X)}{\Leftrightarrow} & d(X, I_1) + d_{\mathcal{B}}(I_1, I_2) & \geq & d(X, I_2) \\
\Leftrightarrow & d(X, I_1) + d_{\mathcal{B}}(I_1, I_2) + d_{\mathcal{B}}(I_2, Y) & \geq & d(X, I_2) + d_{\mathcal{B}}(I_2, Y) \\
\overset{I_2 \in P}{\Leftrightarrow} & d(X, I_1) + d_{\mathcal{B}}(I_1, Y) & \geq & d(X, I_2) + d_{\mathcal{B}}(I_2, Y).
\end{array}$$

□

We hence obtain for the subproblems (4):

Theorem 3.6. *Let $X \in \mathcal{F}$ and let $R \in \mathcal{R}$ be the candidate domain of X. Let $Ex_m \in \mathcal{E}x$ be an existing facility. Let $I_1, I_2 \in \mathcal{I}_m$, $I_1 \neq I_2$ be two points of the candidate set with respect to Ex_m and R. If there exists a shortest feasible I_1-Ex_m path P such that $I_2 \in P$, then the candidate set $\mathcal{I}_m$ can be replaced by $\mathcal{I}_m \setminus \{I_1\}$.*

Proof. For all $X \in R$ we have $X \in (\text{visible}_d(I_1) \cap \text{visible}_d(I_2))$. Thus Lemma 3.5 implies that

$$d(X, I_1) + d_{\mathcal{B}}(I_1, Ex_m) \geq d(X, I_2) + d_{\mathcal{B}}(I_2, Ex_m)$$

for all $X \in R$. Two cases may occur:

1. For all $X \in R$, there is no feasible X-Ex_m path with a breaking point in I_1 that is a shortest feasible X-Ex_m path. Then I_1 is no candidate for an intermediate point in an optimal solution of the corresponding subproblem.

2. There exists $X \in R$ and a feasible X-Ex_m path with breaking point I_1 that is a shortest feasible X-Ex_m path. Then the path whose subpath from I_1 to Ex_m is replaced by the shortest feasible I_1-Ex_m path that has a breaking point in I_2 is also a shortest feasible X-Ex_m path. Hence, I_2 can be chosen equivalently as assigned intermediate point instead of I_1.

In both cases the candidate set $\mathcal{I}_m$ can be replaced by $\mathcal{I}_m \setminus \{I_1\}$. □

The solution method presented in the followings section makes use of Corollary 3.4 and of Theorem 3.6 to reduce the candidate sets $\mathcal{I}_m$, $m = 1, \ldots, M$, on a given candidate domain R. While the identification of projection points generally requires additional computational effort, Theorem 3.6 is easily applicable since for $I \in \mathcal{P}(\mathcal{B})$ the network distances $d_G(I, Ex_m)$, $m = 1, \ldots, M$, in the visibility graph are computed using shortest paths algorithms like the Algorithm of Dijkstra. Consequently, the intermediate nodes (corresponding to possible breaking points) of the shortest network paths can be determined with no additional costs. A barrier extreme point I that is d-visible from all points in a given candidate domain can be deleted from the candidate set $\mathcal{I}_m$ if there exists at least one other barrier extreme point on the shortest I-Ex_m path that is d-visible from all points in R.

The numerical experiments that are summarized in Section 6 show, that the determination of individual candidate sets $\mathcal{I}_m = \{I_{i_1^m}, \ldots, I_{i_{k_m}^m}\} \subseteq \mathcal{I}$ for each of the existing facilities $Ex_m \in \mathcal{E}x$ allows in most examples a considerable reduction of the number of binary variables in the subproblems (4), which can now be written as

$$
\begin{array}{lll}
\min & \sum_{m=1}^{M} \sum_{j=1}^{k_m} y_{jm} w_m d(X, I_{i_j^m}) + \sum_{m=1}^{M} \sum_{j=1}^{k_m} y_{jm} w_m d_{\mathcal{B}}(I_{i_j^m}, Ex_m) & \\
\text{s.t.} & X \in R & \\
& \sum_{j=1}^{k_m} y_{jm} = 1, \quad m = 1, \ldots, M & \\
& y_{jm} \in \{0, 1\}, \qquad m = 1, \ldots, M, \quad j = 1, \ldots, k_m. &
\end{array}
\tag{5}
$$

4 The Solution Method

The solution method for Weber problems with polyhedral barriers and Euclidean distances presented in this section combines the decomposition into subproblems of type (5) as described in the previous sections with an iterative solution procedure similar to the *FORBID* algorithm presented in Butt and Cavalier [1996]. While *FORBID* works with a partition of the feasible region into *regions of constant intermediate points*, i.e., into regions where the intermediate points on shortest feasible paths to *all* existing facilities remain constant, we suggest the utilization of the (in general much larger) candidate domains where only visibility properties remain unchanged. Analogous to *FORBID*, the method allows a computation across the domains and does not require the explicit computation of the candidate domains.

Appropriate assignments of intermediate points in the respective candidate domains are found by solving relaxed versions of the corresponding subproblems (5): In iteration f, $f \in \mathbb{N}$, for a given feasible point X^f the subproblem for the candidate domain R^f of X^f is solved, where the constraint $X \in R^f$ is relaxed in order to obtain an unconstrained problem. Let the optimal solution of the subproblem in iteration f be denoted by X^{f+1}. Two cases may occur:

1. $X^{f+1} \in R^f$. Then X^{f+1} is a global minimizer of the corresponding subproblem and thus at least a local minimizer of the location problem (2).

2. $X^{f+1} \notin R^f$. In this case further subproblems have to be solved.

The overall solution method can be outlined as follows:

Algorithm 4.1.

Input: *Weber problem with polyhedral barriers (2) in* $\mathbb{R}^2$ *with distance metric* d*, feasible solution* $X^0 \in \mathcal{F}$.

Step 0: *Set* $f := 0$.
Compute the barrier distances in the visibility graph G *of* $\mathcal{E}x \cup \mathcal{P}(\mathcal{B})$.
Compute the subset $\mathcal{I}^f \subseteq \mathcal{E}x \cup \mathcal{P}(\mathcal{B})$ *that is d-visible from* X^f.

Step 1: *Determine appropriately reduced candidate sets* $\mathcal{I}_m \subseteq \mathcal{I}^f$, $m = 1, \ldots, M$.
Set $k_m := |\mathcal{I}_m| \leq |\mathcal{I}^f|$.

Step 2: *Solve the mixed-integer optimization problem*

$$\begin{aligned} \min \quad & \sum_{m=1}^{M} \sum_{j=1}^{k_m} y_{jm} w_m d(X, I_{i_j^m}) + \sum_{m=1}^{M} \sum_{j=1}^{k_m} y_{jm} w_m d_G(I_{i_j^m}, Ex_m) \\ \text{s.t.} \quad & \sum_{j=1}^{k_m} y_{jm} = 1, \quad m = 1, \ldots, M, \\ & y_{jm} \in \{0, 1\}, \quad m = 1, \ldots, M, \quad j = 1, \ldots, k_m. \end{aligned} \tag{6}$$

Let the solution be denoted by (X^{f+1}, y^{f+1}).

Step 3: *If* $X^{f+1} \in \text{int}(B)$, $B \in \mathcal{B}$, *solve the continuous optimization problem*

$$\begin{aligned} \min \quad & \sum_{m=1}^{M} \sum_{j=1}^{k_m} y_{jm}^{f+1} w_m d(X, I_{i_j^m}) \\ \text{s.t.} \quad & X \in \partial(B). \end{aligned} \tag{7}$$

Set X^{f+1} *to the solution of this problem.*

Setp 4: *Determine the subset* $\mathcal{I}^{f+1} \subseteq (\mathcal{E}x \cup \mathcal{P}(\mathcal{B}))$, *that is d-visible from* X^{f+1}.

Setp 5: *If* $\mathcal{I}^{f+1} \neq \mathcal{I}^f$, *set* $f := f + 1$ *and go to Step 1.*
Else set $X^\star := X^{f+1}$ *and terminate.*

Output: $X^\star$, *a local optimal solution of (2).*

In this algorithm, similar to *FORBID*, comparably few optimization problems have to be solved. In contrast to *FORBID* where pure location problems and pure allocation problems (allocating intermediate points to the existing facilities) are solved iteratively, location and allocation problems are interwoven in Algorithm 4.1 in the form of the mixed integer formulations of subproblems (6). This yields a more global approach, while still using the geometry of the problem to split the non-convex problem (2) into smaller, easier tractable subproblems. The disadvantage of solving mixed integer programming problems instead of purely continuous and/or convex subproblems can be counteracted by the reduction of the candidate set, c.f. Section 3.

Algorithm 4.1 was implemented for the case of Euclidean distances $d = l_2$. In this case, the mixed integer programming problems (6) are approximately solved using a dexterous combination of a genetic algorithm (for the determination of the assignment variables y) and the Weiszfeld procedure (for the corresponding determination of the location variables X). More precisely, the two solution methods are combined to solve the subproblems (6) as follows:

- A genetic algorithm is used to determine appropriate intermediate points from the candidate sets $\mathcal{I}_m$ to the existing facilities Ex_m, $m = 1, \ldots, M$,

- Three different variations of the Weiszfeld procedure are applied in different stages of the solution method. Firstly, improved locations X are computed by single Weiszfeld iterations before evaluating the quality of every considered assignment of intermediate points in each generation step of the genetic algorithm. Secondly, every run of the genetic algorithm is followed by one Weiszfeld procedure to optimize the location variables X with respect to the optimized assignment of intermediate points. Thirdly, a variation of the Weiszfeld procedure is used if the optimized location X in Step 3 lies in the interior of a barrier $B \in \mathcal{B}$. Then the best solution on $\partial(B)$ is determined by solving problem (7) on every line segment $b \in \partial(B)$, and selecting the best solution found. The optimal location on a line segment can be found by optimizing only one location variable of $X = (x_1, x_2)$ and determining the other variable such that X lies on the line containing the segment b. Since problem (7) is convex on every line, in case an iterate X lies outside the segment b the corresponding (closest) end point of b is chosen as optimal solution of this subproblem.

This approach is repeated until the optimal location variables X are located in the same candidate domain as in the previous iteration, i.e. if no improved assignment of intermediate points is found. The details of the implemented genetic algorithm are described in the following section.

Since it can not be guaranteed that Algorithm 4.1 terminates with a global optimal solution of problem (2), it may be restarted with different starting solutions in order to improve the quality of the solution.

5 Application of the Genetic Algorithm

Genetic algorithms are inspired by the process of evolution and were first introduced by Holland [1975]. They search the solution space in parallel, working with whole sets of feasible solutions, the so called *population*, rather than with individual solutions. By iterating between consecutive *generations*, they try to generate better solutions by combining and/or modifying good solutions, in order to gradually progress towards the optimal solution.

Genetic algorithms have been only rarely applied to location problems. With respect to the assignment part, the implementation suggested here has some similarity to a recent method presented in Topcuoglu et al. [2005] for the solution of uncapacitated, discrete hub location problems. Different from their approach, the representation of solutions in our method is exclusively based on an encoding of the assignment information. A corresponding location solution is only implicitly used for the evaluation of the fitness of the respective assignment.

The genetic algorithm is applied in Step 2 of Algorithm 4.1 for the solution of the subproblems (6). With the notation used in Algorithm 4.1, the genetic algorithm solves the following optimization problem in each iteration step:

$$\begin{aligned} \min \quad & f(I_{i_1}, \ldots, I_{i_M}) = \textstyle\sum_{m=1}^{M} l_2\,(X, I_{i_m}) + d_G(I_{i_m}, Ex_m) \\ \text{s.t.} \quad & I_{i_m} \in \mathcal{I}_m,\ m = 1, \ldots, M, \end{aligned} \tag{8}$$

Here, the optimization variable I_{i_m} denotes the candidate from the set $\mathcal{I}_m$ that is chosen as an intermediate point to the existing facility Ex_m, where the candidate sets $\mathcal{I}_m$, $m = 1, \ldots, M$, determined in step 1 of Algorithm 4.1, are reduced according to the results presented in Section 3. If, for example, Ex_m, $m \in \{1, \ldots, M\}$, is d-visible from X, then $\mathcal{I}_m = \{Ex_m\}$ (c.f. Corollary 3.4). In this case, the variable I_m is fixed to $I_m = Ex_m$. If Ex_m is not d-visible from X, the numerical results obtained by Theorem 3.6 are applied to reduce the candidate set and obtain subproblems that can be solved efficiently.

Besides the coded assignment solution, every individual additionally consists of a *test point* which represents the location variable $X \in \mathbb{R}^2$ in problem (8). For a constant test point X (that is identical for all individuals) the optimal solution of this problem identifies the intermediate points from X to the existing facilities. However, since we seek to find the optimal $X^\star$-Ex_m intermediate points, $m = 1, \ldots, M$, where $X^\star$ is an optimal solution of (6) and in general unknown, the test point has to be chosen and changed in an appropriate way. For this purpose, the test points of the individuals are varied in each generation step as follows:

- In the initial population, the test point of every individual Ind is set to the solution obtained in iteration f of Algorithm 4.1, $X_{Ind} := X^f$.

- Before evaluating the fitness of each individual in the beginning of a generation step of the genetic algorithm, the test point of each individual Ind is improved separately with respect to its intermediate points $(I_{i_1}, \ldots, I_{i_M})_{Ind}$ by executing one single iteration of the Weiszfeld algorithm, i.e., $X_{Ind} := W(X_{Ind})$. Therefore, improved test points X_{Ind} are obtained with little computational expense in each generation.

- But also during the *recombination phase* the test points are varied. If two individuals Ind_1 and Ind_2 are selected for recombination, their test points X_{Ind_1} and X_{Ind_2} are varied by *intermediate recombination*:

  $$\begin{aligned} X_{Ind_1} &= p_X \cdot X_{Ind_1} + (1 - p_X) \cdot X_{Ind_2} \\ X_{Ind_2} &= p_X \cdot X_{Ind_2} + (1 - p_X) \cdot X_{Ind_1}, \end{aligned}$$

  where $p_X \in [0.5, 1]$ is a given input parameter.

During the other stages of the generation steps of the genetic algorithm, the test points remain unchanged. Note that they are thus changed in a deterministic way for given assignments of intermediate points. The assignments of intermediate points, i.e. the actual optimization variables, are processed as follows:

Initialization: Individuals are initialized with random values $I_{i_m} \in \mathcal{I}_m$, $m = 1, \ldots, M$.

Fitness Values:

1. First the function value

$$f_{Ind} = f((I_{i_1}, \ldots, I_{i_M})_{Ind}, X_{Ind}) = \sum_{m=1}^{M} l_2\left(X, I_{i_m}\right) + d_G(I_{i_m}, Ex_m)$$

   is computed for all individuals Ind by means of their corresponding test points X_{Ind}. Note that $f((I_{i_1}, \ldots, I_{i_M}), X)$ does in general not correspond to the objective function value of X in the location problem (5).

2. We apply a fitness function called *linear ranking* to map the values of f_{Ind} to the actual fitness values.

Selection: The stochastic universal sampling method as well as the tournament selection technique have been applied. Since no significant difference was observed, the computational results presented in Section 6 are based on stochastic universal sampling.

Recombination: In contrast to the test points, the assignments $(I_{i_1}, \ldots, I_{i_M})$ of each pair of selected individuals is recombined using *uniform crossover*.

Mutation: To make sure that every valid solution can be reached a mutation operator is applied. With a certain probability every gene, i.e. every intermediate point $I_{i_m} \in \mathcal{I}_m$, is changed to a random but valid point $I'_{i_m} \in \mathcal{I}_m$.

Reinsertion: Subsequently, the individuals are reinserted in the population. The user can decide whether the whole population shall be replaced, or whether the fittest individuals of the previous generation shall survive (which implies elitist selection), and whether all new individuals shall be incorporated into the population or only the best.

Termination: One stopping criterion is an upper bound on the number of generations, the other an upper bound of identical individuals representing the best known solution.

Once the genetic algorithm terminates, the Weiszfeld algorithm is applied to the best assignment of intermediate points $(I_{i_1}, \ldots, I_{i_M})_{Ind}$, using the corresponding test point X_{Ind} as starting solution.

6 Computational Experience

All of the computations presented in this section have been run on an Intel Pentium 4, 2.66Ghz, 512MB RAM computer. The solution method was implemented in Matlab. All results were found by one single computation. Since genetic algorithms have a stochastic component, the results may vary if the computations are repeated. All problems considered are single facility Weber problems of type (2) with polyhedral - or polyhedral approximated - barriers and Euclidean distances.

6.1 The First Example Problem from Katz and Cooper [1981]

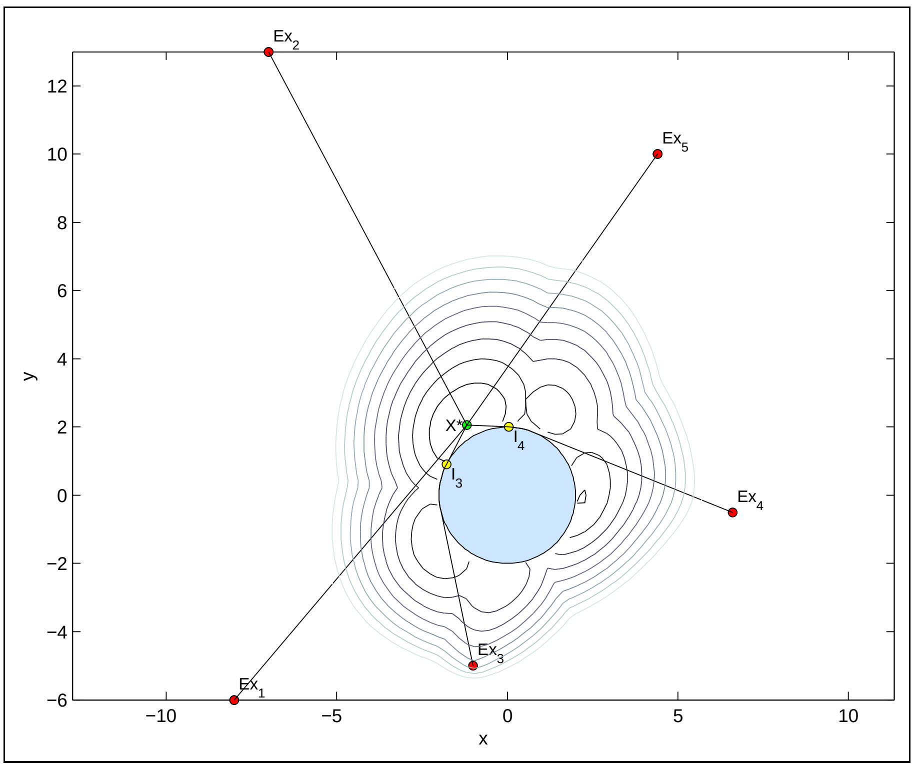

Figure 1: The first example problem from Katz and Cooper [1981], the level curves of the objective function and the computed solution. The circular barrier was approximated by an 128-sided equilateral polygon from the inside.

Problem data:

- Five existing facilities with weights $w_i = 1$, $i = 1, \ldots, 5$, and coordinates

$(-8.0, -6.0)$, $(-7.0, 13.0)$, $(-1.0, -5.0)$, $(6.6, -0.5)$, $(4.4, 10.0)$.

- One circular barrier with center at (0,0) and radius 2.

As our solution method only works with polyhedral barriers, the circle has been approximated by an equilateral polygon with up to 512 extreme points, both from the outside and from the inside. The results are presented in the table below. In the case of 16 extreme points, the solution is identical to the one found by Butt and Cavalier [1996]. For the improved approximations with up to 512 extreme points, the solution found in Klamroth [2001a] for the original circle could be validated.

$\|\mathcal{P}(\mathcal{B})\|$	$X^\star$	$f(X^\star)$	# iterations	time (s)
approx. from outside:				
16	(-1.201580, 2.077647)	48.281797	2	0.078
32	(-1.190873, 2.067660)	48.261460	3	0.141
64	(-1.185968, 2.062756)	48.256464	5	0.250
128	(-1.186446, 2.060556)	48.255225	4	0.250
256	(-1.186174, 2.060530)	48.254917	5	0.510
512	(-1.186063, 2.060519)	48.254840	5	0.922
approx. from inside:				
512	(-1.186050, 2.060516)	48.254802	5	1.031
256	(-1.185953, 2.060508)	48.254764	5	0.485
128	(-1.185897, 2.060503)	48.254609	4	0.265
64	(-1.186927, 2.058351)	48.253988	4	0.203
32	(-1.181308, 2.057875)	48.251504	3	0.125
16	(-1.181308, 2.057875)	48.241865	2	0.078

Observe that, particularly for this example problem, a drastic reduction of the size of the candidate set based on Theorem 3.6 is possible. If an existing facility Ex_m, $m \in \{1, \ldots, 5\}$, is l_2-visible from the currently considered solution X^f (and from the corresponding candidate domain R^f), then the candidate set $\mathcal{I}_m$ contains only the existing facility Ex_m itself. If, on the other hand, the existing facility Ex_m is behind the polyhedral approximation of the circular barrier (as viewed from X^f), then $|\mathcal{I}_m| = 2$ since it is sufficient to choose the intermediate point out of those two extreme points of the barrier that are l_2-visible from X^f and located on $\partial(\text{shadow}_{l_2}(X^f))$. In both cases, the cardinalities of the candidate sets are very low, and consequently only a small number of individuals is needed in the genetic algorithm for an exhaustive search of the solution space.

The above computation times do not include the preprocessing phase where, among others, network distances and shortest paths in the visibility graph have to be computed. Depending on the accuracy of the polyhedral approximation of the circular barrier, this took between $0.031s$ (for the approximation with 16 extreme points) and $27.328s$ (for 512 extreme points).

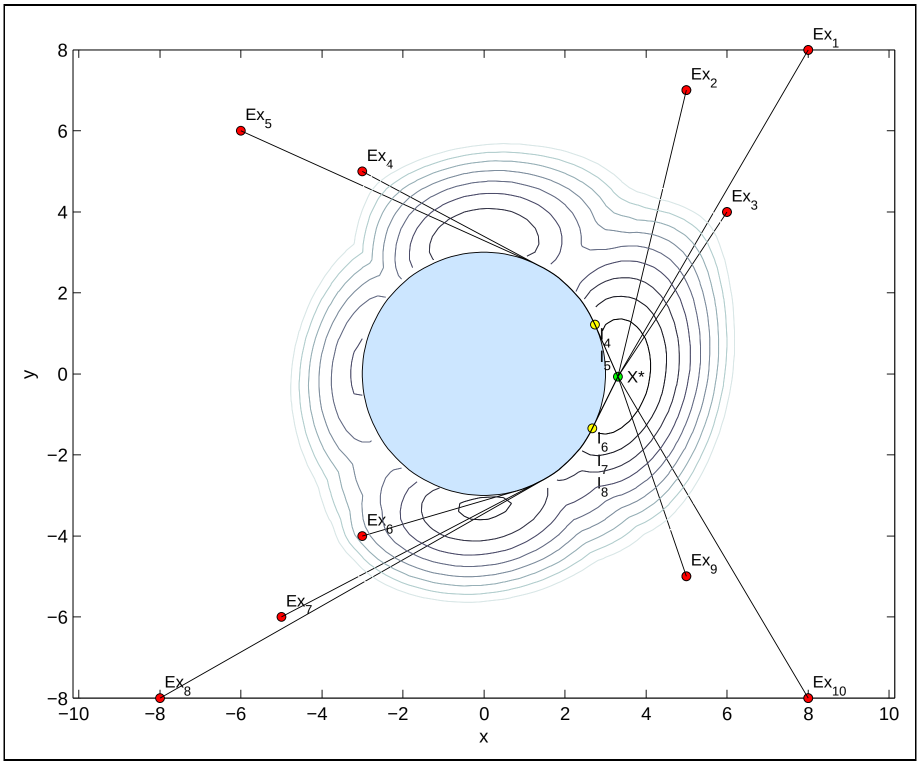

Figure 2: The second example problem from Katz and Cooper [1981], level curves and the computed solution. The circular barrier is approximated by an 128-sided equilateral polygon from the inside.

6.2 The Second Example Problem from Katz and Cooper [1981]

Problem data:

- Ten existing facilities with weights $w_i = 1$, $i = 1, \ldots, 10$ and coordinates $(8, 8)$, $(5, 7)$, $(6, 4)$, $(-3, 5)$, $(-6, 6)$, $(-3, -4)$, $(-5, -6)$, $(-8, -8)$, $(5, -5)$, $(8, -8)$.
- One circular barrier with center at (0,0) and radius 3.

Katz and Cooper [1981] report eight local minima with function values from 76.558 to 79.225.

To apply the solution method to this location problem, the circular barrier was approximated by 16- and 128-sided equilateral polygon from the outside and the inside, respectively. The results are present in the table below. Additionally, we performed a grid point computation for the example problems with about

10^4 points, i.e. with a distance of less than 0.2 between two adjacent grid points in this example. When approximating the circle by a 128-sided polygon, the minimal function values of the grid points were 88.36 when approximating from the inside and 88.39 when approximating from the outside, respectively. These values validate the solutions indicated in the table, but contradict the results in Katz and Cooper [1981].

$\|\mathcal{P}(\mathcal{B})\|$	$X^\star$	$f(X^\star)$	# iterations	time (s)
approx. from outside:				
16	(3.324784, -0.085586)	88.468917	2	0.156
128	(3.307095, -0.067167)	88.325077	4	0.453
approx. from inside:				
128	(3.305932, -0.067746)	88.321938	4	0.422
16	(3.303454, -0.062217)	88.249042	1	0.157

6.3 The Numerical Example from Aneja and Parlar [1994]

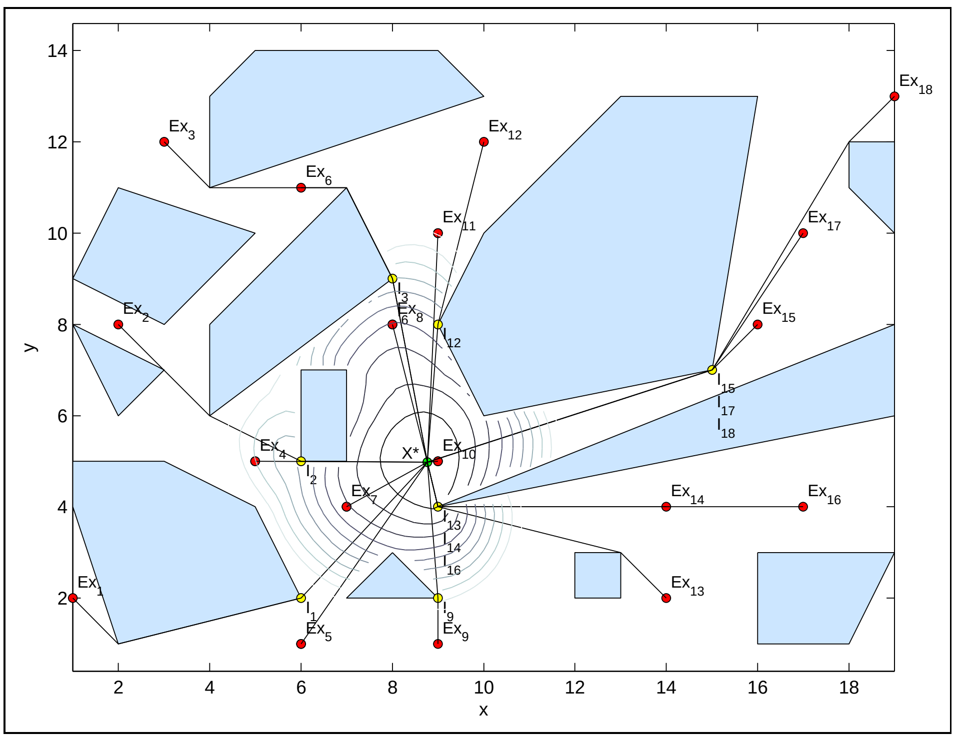

Figure 3: The example problem from Aneja and Parlar [1994] with all barriers, level curves and the computed solution. Note that the non-convex barriers are replaced by their convex hulls.

Problem data:

- 18 existing facilities with weights $w_i = 1$, $i = 1, \ldots, 18$:

$$\begin{array}{cccccc} (\ 1,2) & (\ 2,8) & (\ 3,12) & (\ 5,5) & (\ 6,\ \ 1) & (\ 6,11) \\ (\ 7,4) & (\ 8,8) & (\ 9,\ \ 1) & (\ 9,5) & (\ 9,10) & (10,12) \\ (14,2) & (14,4) & (16,\ \ 8) & (17,4) & (17,10) & (19,13) \end{array}$$

- Barriers: Polyhedral sets with (clockwise sorted) extreme points:

$$\begin{array}{ll} B_1 & : \big((1,5),(3,5),(4,3),(5,4),(6,2),(2,1)\big) \\ B_2 & : \big((1,8),(3,7),(2,6)\big) \\ B_3 & : \big((1,9),(2,11),(5,10),(3,8)\big) \\ B_4 & : \big((4,6),(4,8),(7,11),(8,9)\big) \\ B_5 & : \big((4,11),(4,13),(5,14),(9,14),(10,13)\big) \\ B_6 & : \big((6,5),(6,7),(7,7),(7,5)\big) \\ B_7 & : \big((7,2),(8,3),(9,2)\big) \\ B_8 & : \big((9,8),(10,10),(13,13),(16,13),(15,7),(10,6),(12,9)\big) \\ B_9 & : \big((12,2),(12,3),(13,3),(13,2)\big) \\ B_{10} & : \big((9,4),(19,8),(19,6)\big) \\ B_{11} & : \big((16,1),(16,3),(19,3),(18,1)\big) \\ B_{12} & : \big((18,11),(18,12),(19,12),(19,10)\big). \end{array}$$

Aneja and Parlar [1994] solved this location problem (and also similar ones that are defined by omitting certain barriers in the above example) using simulated annealing. Butt and Cavalier [1996] also examined this example problem (with all twelve barrier sets), and verified the solution found by Aneja and Parlar [1994]. Even though this example problem contains two non-convex barriers (B_1 and B_8), solution methods designed for problems with convex barriers can be applied since, as was already shown in Butt and Cavalier [1996], the two non-convex barriers B_1 and B_8 can be replaced by their convex hulls without changing the optimal solution.

Due to the size of the problem with comparably many barrier sets, the candidate sets $\mathcal{I}_m$ that are used during the solution process are much larger than in the previous example. Consequently, in order to search the solution space effectively, a larger set of individuals has to be used in this case. Even though a relatively bad initial solution at $X = (0,0)$ was used analogous to Aneja and Parlar [1994], Algorithm 4.1 converged to the global optimum within only a few iterations. The solutions found are identical to those determined in Aneja and Parlar [1994], at least up to the fourth digit:

barriers	$X^\star$	$f(X^\star)$	# iterations	time (s)
B_1-B_{12}	(8.7667, 4.9797)	119.1387	4	34.485
B_1-B_{10}	(8.7667, 4.9797)	119.1047	4	36.157
B_1-B_8	(9.1873, 5.4860)	116.3976	4	21.219
B_1-B_6	(9.2658, 6.2527)	114.5610	4	10.781
B_1-B_4	(9.2173, 6.1528)	113.7656	3	2.547
B_1-B_2	(9.0372, 6.1150)	111.6889	3	0.938
$\emptyset$	(8.9127, 6.3554)	110.0068	1	0.282

6.4 A new Series of Example Problems

In order to benchmark our implementation, a series of test problems that are computationally difficult for Algorithm 4.1 was constructed with the following problem data:

- m existing facilities, uniformly distributed on a circle with center $(0,0)$ and radius 10, with weights $w_i = 1$, $i = 1, \ldots, m$.
- Six equilateral polyhedra, each of them having k extreme points and approximating one of the following circular barriers from the inside:

circle	B_1	B_2	B_3	B_4	B_5	B_6
center	(5,-2)	(5, 3)	(-5, 2)	(-3,-4)	(0, 6)	(2,-5)
radius	1.0	1.2	1.4	1.6	1.8	2.0

This example problem has been solved for different values of m and k:

- 5 existing facilities and 10-sided barriers:

# individuals	determined solution $X^\star$	$f(X^\star)$	iterations	time (s)
10	(-0.67627,-0.098415)	50.4206	8	2.187
50	(-0.67627,-0.098415)	50.4206	3	7.312
100	(-0.67627,-0.098415)	50.4206	3	14.609

  More iterations are needed if fewer individuals are used. However, the computation time for the genetic algorithm is considerably smaller in this case.

- 10 existing facilities and 20-sided barriers:

# individuals	determined solution $X^\star$	$f(X^\star)$	iterations	time (s)
10	- termination after 100 iterations -			100.235
50	(0.44115,0.49830)	101.0068	4	25.547
100	(-0.44642,0.052173)	100.5583	3	49.859

  Grid point evaluations over the whole feasible set imply that the problem has only one global minimum and a smooth and simple structure in its neighborhood. Nevertheless, Algorithm 4.1 is relatively inefficient in this example. It even fails completely in the 10 individuals case, due to the large solution space with many candidates for intermediate points.

- 20 existing facilities and 20-sided barriers:

# individuals	determined solution $X^\star$	$f(X^\star)$	iterations	time (s)
10	- termination after 100 iterations -			450.547
50	(-0.56934, 0.16814)	202.0180	16	493.625
100	(0.17331,-0.061936)	202.1580	3	160.453

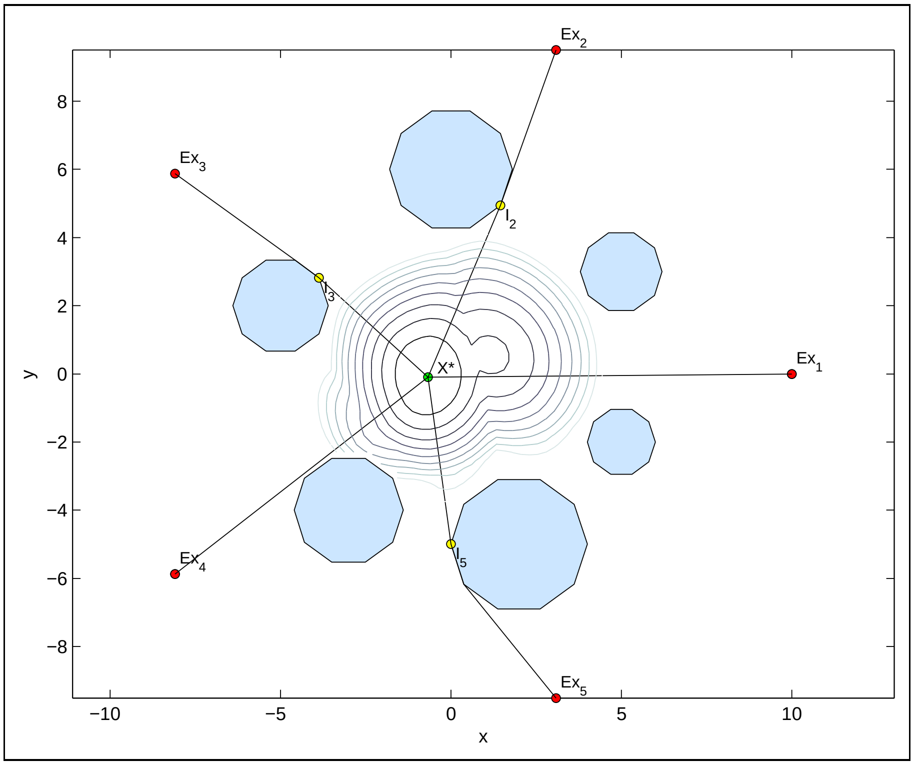

Figure 4: The example problem for $m = 5$ and $k = 10$.

As in the previous example, the solution method cannot find a solution if the genetic algorithm is run with only 10 individuals. It still works satisfactorily with 100 individuals.

- 40 existing facilities and 5-sided barriers:

# individuals	determined solution $X^\star$	$f(X^\star)$	iterations	time (s)
10	- termination after 100 iterations -			1278.766
50	(-0.028532,-0.022809)	402.9187	3	212.1
100	(-0.028532,-0.022809)	402.9187	2	294.6

This example turns out to be easier to solve than the previous ones. The method needs surprisingly few iterations to find a good solution in this case. Although each individual has 40 genes, each gene can take on only a relatively small number of feasible values since the barrier sets have only a small number of extreme points.

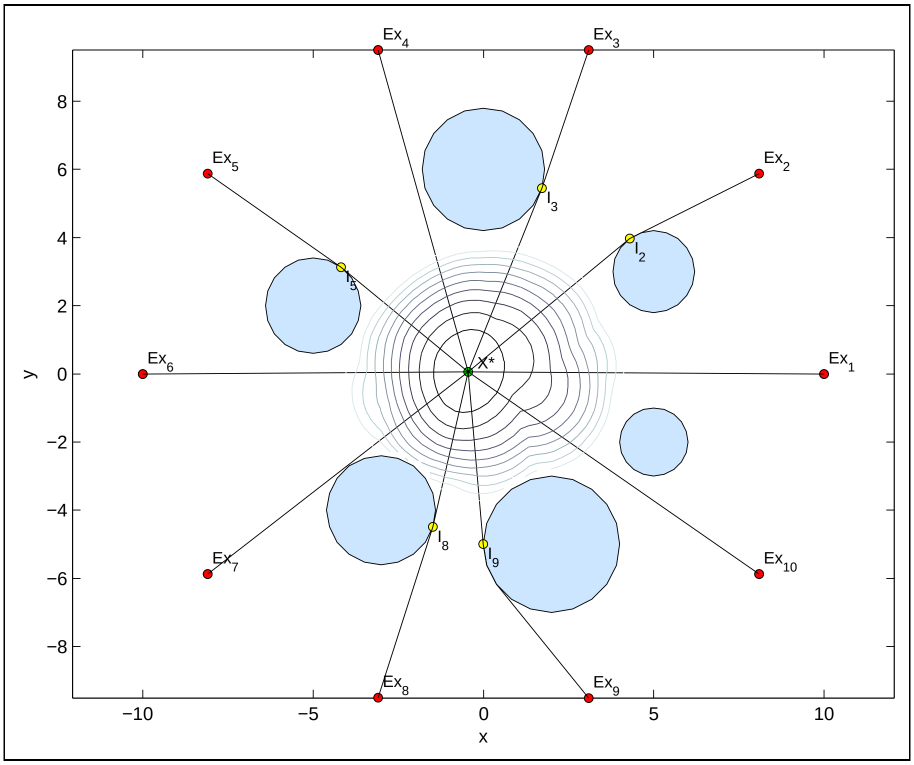

Figure 5: The example problem for $m = 10$ and $k = 20$.

7 Conclusions

We developed a solution method for single-facility Weber problems with polyhedral barriers that is based on a partition of the feasible region into candidate domains which are defined based on visibility properties. This partition allows a reduction of the non-convex problem into a finite series of mixed integer programming problems that consist of the simultaneous determination of an optimal location in the selected candidate domain *and* of an optimal assignment of appropriate intermediate points on the paths to the existing facilities. Possible ways of reducing the complexity of these subproblems are discussed, and a genetic algorithm is presented that approximates the optimal solution in the case of Euclidean distances. Numerical experiments show that the resulting algorithm is efficient and achieves a very high solution quality in all of the considered test problems.

Moreover, further numerical tests showed that the algorithm performed very well as a subroutine in alternate location and allocation algorithms for multi-facility location-allocation problems with polyhedral barriers.

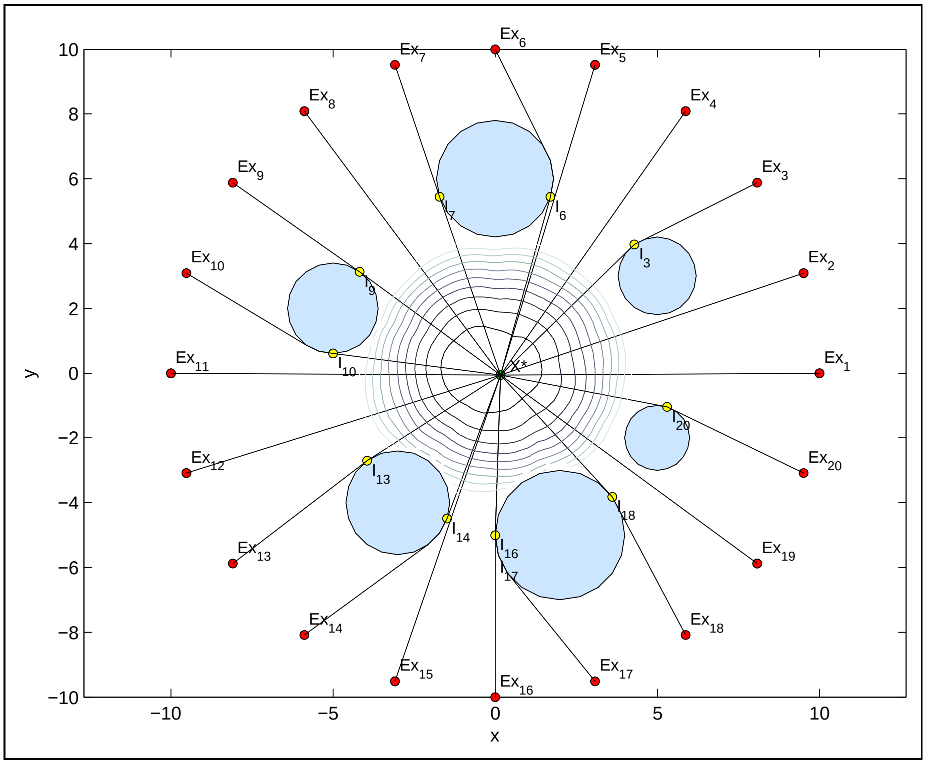

Figure 6: The example problem for $m = 20$ and $k = 20$.

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