The Multi-Facility Location-Allocation Problem with Polyhedral Barriers

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Abstract

In this paper we consider the problem of locating N new facilities with respect to M existing facilities in the plane and in the presence of polyhedral barriers. We assume that a barrier is a region where neither facility location nor traveling is permitted. For the resulting multi-dimensional mixed-integer optimization problem two different alternate location and allocation procedures are developed. Numerical examples show a superiority of a joint treatment of all assignment variables, including those specifying the routes taken around the barrier polyhedra, over a separate iterative solution of the assignment problem and the single-facility location problems in the presence of barriers.

Keywords: multi facility location; barriers; non-convex optimization; location allocation

1 Introduction

A commonly studied location problem is that of locating a finite set of new facilities with respect to a finite set of existing facilities. For the *generalized multi-Weber problem* (MWP), also referred to as *uncapacitated multi-facility location-allocation problem*, the problem of locating a set of N new facilities $\mathbf{x}_1, \ldots, \mathbf{x}_N \in \mathbb{R}^2$ with respect to a given set of M existing facilities $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_M\} \subset \mathbb{R}^2$ in order to minimize the total, positively weighted travel distance can be formulated as

$$
\begin{array}{lll}
\min & \displaystyle\sum_{n=1}^{N}\sum_{m=1}^{M} y_{mn} w_m d(\mathbf{x}_n, \mathbf{a}_m) & \\
\text{s.t.} & \displaystyle\sum_{n=1}^{N} y_{mn} = 1, & m = 1, \ldots, M \\
 & y_{mn} \in \{0, 1\}, & m = 1, \ldots, M,\ n = 1, \ldots, N \\
 & \mathbf{x}_n \in \mathbb{R}^2, & n = 1, \ldots, N.
\end{array}
\tag{MWP}
$$

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In this model, the binary variables y_{mn} contain the information on the assignment of existing facilities to new facilities, i.e.

$$y_{mn} = \begin{cases} 1 & \text{if } \mathbf{a}_m \text{ is assigned to } \mathbf{x}_n, \\ 0 & \text{otherwise,} \end{cases} \quad m = 1, \ldots, M,\ n = 1, \ldots, N.$$

The positive weights $w_m \in \mathbb{R}_+$, $m = 1, \ldots, M$, may model, for example, the demand of the existing facilities $\mathbf{a}_m$, $m = 1, \ldots, M$. We assume that $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is an arbitrary distance function in $\mathbb{R}^2$ that is induced by a norm $\|\bullet\|_d : \mathbb{R}^2 \to \mathbb{R}$.

The MWP was first formally stated by Cooper [1963], who also showed that it is neither convex nor concave. It may in fact have a very large number of local minima, see Eilon et al. [1971] for an example. Moreover, the problem can be interpreted as an enumeration of the Voronoi partitions of the customer set, and it was proven to be NP-hard in Megiddo and Supowit [1984]. There have been many suggestions for both heuristic and exact solution methods for the MWP. A well-known heuristic approach is the alternate location and allocation algorithm [Cooper, 1964] that alternates between a location and an allocation phase until no further improvement is made. Other heuristics include, for example, local search methods such as variable neighborhood search, tabu search and genetic algorithms, see Brimberg et al. [2000] for a comparative study. Further, in Brimberg et al. [2006] strategies to tackle large-scale MWPs are presented, that are based on a decomposition into smaller MWPs which can be treated with recent methods.

Barrier regions, in which neither locating new facilities nor traveling is permitted can be used to model, for example, lakes, mountain ranges or residential areas or, on a smaller scale, conveyor belts in an industrial plant in order to obtain a more realistic representation of distances.

Barriers were first introduced to location modeling by Katz and Cooper [1981], who consider Weber problems with the Euclidean metric and with one circular barrier. Assuming that all barrier sets are polyhedra allows the construction of a visibility graph, that can be used to compute barrier distances efficiently. There exist several heuristic and iterative algorithms for single-facility location problems that use the visibility graph for distance computations and bear some similarity to location and allocation heuristics [see Aneja and Parlar, 1994, Butt and Cavalier, 1996, McGarvey and Cavalier, 2003, among others]. For an overview about location problems with barriers we refer to Klamroth [2002].

In this paper we assume that a set of K polyhedral and pairwise disjoint barrier regions $\mathcal{B} := \bigcup_{k=1}^{K} B_k \subset \mathbb{R}^2$ with a finite set of extreme points $\mathcal{E}(\mathcal{B}) = \bigcup_{k=1}^{K} \bigcup_{l=1}^{L_k} \{\mathbf{e}_{kl}\}$ is given. The *feasible region*, where traveling as well as locating new facilities is allowed, is defined as $\mathcal{F} = \mathbb{R}^2 \setminus \operatorname{int}(\mathcal{B})$. To avoid infeasible cases we assume that $\mathcal{F}$ is connected and that all existing facilities $\mathbf{a}_1, \ldots, \mathbf{a}_M$ are located in $\mathcal{F}$. A path P connecting any two points $\mathbf{x}, \mathbf{y} \in \mathcal{F}$ while not intersecting the interior of a barrier region is denoted as feasible $\mathbf{x}$-$\mathbf{y}$-path. The length $l(P)$ of a shortest feasible $\mathbf{x}$-$\mathbf{y}$-path, measured with respect to the prescribed norm

$\|\cdot\|_d$, defines a distance function in $\mathcal{F}$, the so called barrier distance,

$$d_{\mathcal{B}}(\mathbf{x},\mathbf{y}) := \inf\{l(P): \ P \text{ feasible } \mathbf{x}\text{-}\mathbf{y}\text{-path}\}.$$

Note that the resulting distance measure $d_{\mathcal{B}}$ is in general not positively homogenous. It is, however, positive definite and symmetric and satisfies the triangle inequality. Consequently, the barrier distance $d_{\mathcal{B}}$ defines a metric on $\mathcal{F}$, see Klamroth [2002].

Using the notation introduced above, the *multi-Weber problem with barriers* (MWP$_{\mathcal{B}}$) can be formulated as

$$\begin{array}{lll} \min & \displaystyle\sum_{n=1}^{N}\sum_{m=1}^{M} y_{mn} w_m d_{\mathcal{B}}(\mathbf{x}_n,\mathbf{a}_m) & \\ \text{s.t.} & \displaystyle\sum_{n=1}^{N} y_{mn} = 1, & m=1,\ldots,M \\ & y_{mn}\in\{0,1\}, & m=1,\ldots,M,\ n=1,\ldots,N \\ & \mathbf{x}_n\in\mathcal{F}, & n=1,\ldots,N. \end{array} \tag{MWP$_{\mathcal{B}}$}$$

Since (MWP$_{\mathcal{B}}$) reduces to (MWP) if no barriers are present, it is also NP-hard according to Megiddo and Supowit [1984].

An alternative approach to model regional characteristics in the MWP is the incorporation of zone-dependent, fixed opening costs $\sum_{n=1}^{N} f(\mathbf{x}_n)$ for the new facilities in the objective function, in addition to the (unconstrained) transportation costs $\sum_{n=1}^{N}\sum_{m=1}^{M} y_{mn} w_m d(\mathbf{x}_n,\mathbf{a}_m)$. The resulting *fixed-charge location-allocation problem* was introduced in Brimberg and Salhi [2005]. Similar to the (MWP$_{\mathcal{B}}$), this problem formulation also suggests the application of alternate location and allocation heuristics, see Brimberg and Salhi [2005].

The rest of the paper is organized as follows. In the next section, geometrical properties and theoretical results for the MWP$_{\mathcal{B}}$ with general barrier distances are developed. Two variants of alternate location allocation heuristics, together with different techniques for the generation of starting solutions, are presented in Section 3. Computational results for the case that the underlying distances are Euclidean distances are presented in Section 4.

2 Problem Structure and General Properties

While several principal properties of the MWP remain valid in the presence of barriers, other properties (e.g., cluster properties) do in general not transfer to the MWP$_{\mathcal{B}}$. If not specified differently, the results presented below are true for general distance functions induced by norms and their corresponding barrier distances.

2.1 Single-Facility Subproblems

If the assignment variables y_{mn} are fixed to constant values $\overline{y}_{mn}$, $m=1,\ldots,M$, $n=1,\ldots,N$, the MWP reduces to N single-facility location problems [see

Cooper, 1963]. A similar property also holds in the presence of barriers. In particular, the $\text{MWP}_{\mathcal{B}}$ decomposes into N Weber problems with barriers ($\text{WP}_{\mathcal{B}}(n)$), $n = 1, \ldots, N$,

$$\begin{aligned} \min \quad & \sum_{m=1}^{M} c_{mn} d_{\mathcal{B}}(\mathbf{x}_n, \mathbf{a}_m) \\ \text{s. t.} \quad & \mathbf{x}_n \in \mathcal{F}, \end{aligned} \tag{$\text{WP}_{\mathcal{B}}(n)$}$$

where

$$c_{mn} := \begin{cases} w_m & \text{if } \overline{y}_{mn} = 1, \\ 0 & \text{otherwise,} \end{cases} \quad m = 1, \ldots, M, \ n = 1, \ldots, N.$$

One approach to determine an optimal solution of ($\text{MWP}_{\mathcal{B}}$) is consequently to enumerate all feasible partitions of the set of existing facilities $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_M\}$ into so called *allocation clusters* $\mathcal{A}_n \subset \mathcal{A}$,

$$\mathcal{A}_n = \{\mathbf{a}_m \in \mathcal{A} : \ y_{mn} = 1, m = 1, \ldots, M\}, \ n = 1, \ldots, N.$$

and to solve the resulting single-facility Weber problems with barriers.

Since the number of feasible partitions of M existing facilities into N *allocation clusters*, one allocation cluster $\mathcal{A}_n$ for every new facility $\mathbf{x}_n$, equals $S(N, M)$, the Stirling number of the second kind [see Cooper, 1963] given by

$$S(N, M) = \frac{1}{N!} \sum_{n=0}^{N} \begin{pmatrix} N \\ n \end{pmatrix} (-1)^n (N - l)^M,$$

total enumeration is only applicable for small example problems.

2.2 Set Partitioning Subproblem

When keeping the location variables fixed, the MWP reduces to a set partitioning problem [see Cooper, 1963]. The same holds for the $\text{MWP}_{\mathcal{B}}$, i.e., for fixed location variables $\overline{\mathbf{x}}_n$, $n = 1, \ldots, N$, the $\text{MWP}_{\mathcal{B}}$ reduces to the set partitioning problem

$$\begin{aligned} \min \quad & \sum_{n=1}^{N} \sum_{m=1}^{M} \overline{c}_{mn} y_{mn} \\ \text{s. t.} \quad & \sum_{n=1}^{N} y_{mn} = 1, \quad m = 1, \ldots, M \\ & y_{mn} \in \{0, 1\}, \quad m = 1, \ldots, M, \ n = 1, \ldots, N, \end{aligned} \tag{SP}$$

where

$$\overline{c}_{mn} := w_m \cdot d_{\mathcal{B}}(\overline{\mathbf{x}}_n, \mathbf{a}_m), \ m = 1, \ldots, M, \ n = 1, \ldots, N.$$

Obviously, this problem is solved by simply finding for each $m \in \{1, \ldots, N\}$ an index $\overline{n}_m \in \{1, \ldots, N\}$ that solves $\min_{n=1,\ldots,N}\{\overline{c}_{mn}\}$.

Both decomposition properties outlined in Sections 2.1 and 2.2 suggest the application of a variant of the alternate location and allocation algorithm (LA) [see Cooper, 1964] also on the MWP$_\mathcal{B}$, that iterates between the solution of location problems (WP$_\mathcal{B}(n)$) and the determination of allocation clusters (SP). This approach is described in Section 3.1.

2.3 Iterative Convex Hull

A well known property of the classical Weber problem with Euclidean distances is the *convex hull property* stating that all optimal solutions have to be located within the convex hull of the existing facilities. Variations of this property also hold for Weber problems with other distance functions like, for example, rectangular distances and polyhedral gauges [see, for example, Durier and Michelot, 1994].

In the presence of barriers these results have to be modified slightly. Namely, instead of the convex hull of just the existing facilities, the *iterative convex hull* has to be considered [see Klamroth, 2002]. The iterative convex hull $\mathcal{H}(\mathcal{A}, \mathcal{B})$ of a set of existing facilities $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_M\}$ and barrier sets $\mathcal{B} = \bigcup_{k=1}^{K} B_k$ is the smallest compact convex set in $\mathbb{R}^2$ such that

$$\mathcal{A} \subset \mathcal{H}(\mathcal{A}, \mathcal{B}) \text{ and } \partial(\mathcal{H}(\mathcal{A}, \mathcal{B})) \cap \operatorname{int}(\mathcal{B}) = \emptyset.$$

Starting with the convex hull of the set of existing facilities, the iterative convex hull can be obtained by iteratively computing the convex hull of a set that is successively extended by the extreme points of barriers which intersect the boundary of the convex hull.

If the classical Weber problem with distance function d satisfies at least a *weak convex hull property*, i.e., if it has at least one optimal solution that lies in the convex hull of the existing facilities, then, according to Klamroth [2002], the Weber problem with barriers (WP$_\mathcal{B}$) has at least one optimal solution $\mathbf{x}^\star$ that is contained in the iterative convex hull of existing facilities and barrier polyhedra, i.e.

$$\mathbf{x}^\star \in \mathcal{F} \cap \mathcal{H}(\mathcal{A}, \mathcal{B}).$$

The iterative convex hull property also holds in the case of multiple new facilities since for arbitrary, feasible assignment variables, (MWP$_\mathcal{B}$) can be decomposed into N independent subproblems (WP$_\mathcal{B}(n)$), $n = 1, \ldots, N$ (cf. Section 2.1). Each of the subproblems (WP$_\mathcal{B}(n)$), $n = 1, \ldots, N$ satisfies the weak convex hull property, and thus has at least one optimal solution $\mathbf{x}_n^\star$ that satisfies

$$\mathbf{x}_n^\star \in \mathcal{F} \cap \mathcal{H}(\mathcal{A}_n, \mathcal{B}),$$

where $\mathcal{A}_n$ denotes the allocation cluster corresponding to new facility $\mathbf{x}_n^\star$, $n = 1, \ldots, N$.

Further, since the allocation clusters $\mathcal{A}_n$, $n = 1, \ldots, N$ define a partition of $\mathcal{A}$ and since $\mathcal{H}(\mathcal{A}_n^\star, \mathcal{B}) \subseteq \mathcal{H}(\mathcal{A}, \mathcal{B})$ for any subset $\mathcal{A}_n^\star \subseteq \mathcal{A}$, it holds that

$$\mathbf{x}_n^\star \in \mathcal{F} \cap \mathcal{H}(\mathcal{A}, \mathcal{B}),\ n = 1, \ldots, N.$$

Note that this property is independent of the actual assignment of existing facilities to the new facilities. In particular, for an arbitrary but fixed set of feasible assignment variables, there always exists a set of new facilities that is optimally located with respect to the assignment and that is located in the iterative convex hull.

2.4 Mixed Integer Formulation

As discussed in Klamroth [2001] and further analyzed in Bischoff and Klamroth [2006], the single-facility subproblems ($\mathrm{WP}_\mathcal{B}(n)$) as specified in Section 2.1 can be solved by partitioning the feasible region $\mathcal{F} \cap \mathcal{H}(\mathcal{A}, \mathcal{B})$ into a finite set of domains and solving the corresponding mixed integer subproblems. In this section we develop a similar decomposition for the $\mathrm{MWP}_\mathcal{B}$ and present results that motivate the application of a location and allocation procedure.

To obtain this partition of the feasible region, we distinguish those parts of $\mathcal{F}$ in which the distance from a point $\mathbf{x} \in \mathcal{F}$ is lengthened by barrier regions from those where the distance equals the underlying metric. In particular, two points $\mathbf{x}, \mathbf{y} \in \mathcal{F}$ are called *d-visible*, if $d_\mathcal{B}(\mathbf{x}, \mathbf{y}) = d(\mathbf{x}, \mathbf{y})$. The set of points that are d-visible from a point $\mathbf{x} \in \mathcal{F}$ is defined as

$$\mathrm{visible}_d(\mathbf{x}) := \big\{\mathbf{y} \in \mathcal{F} :\ d_\mathcal{B}(\mathbf{x}, \mathbf{y}) = d(\mathbf{x}, \mathbf{y})\big\}.$$

Note that the set of points that are not d-visible from $\mathbf{x}$, i.e., $\mathcal{F} \setminus \mathrm{visible}_d(\mathbf{x})$, $\mathbf{x} \in \mathcal{F}$ is the union of polyhedral sets. By computing these polyhedra, $\mathrm{visible}_d(\mathbf{x})$ can easily be obtained. See Klamroth [2002] for an extensive discussion on shortest paths and the concept of visibility.

Based on visibility properties, the *candidate domains* partition the feasible region as follows.

Definition 2.1 (Candidate Set and Candidate Domain).
Let $\mathcal{P} := \mathcal{A} \cup \mathcal{E}(\mathcal{B}) = \{\mathbf{p}_1, \ldots, \mathbf{p}_I\}$ be the set of all existing facilities and barrier extreme points. For a given point $\mathbf{x} \in \mathcal{F}$, let $\mathcal{P} \cap \mathrm{visible}_d(\mathbf{x})$ be the set of all existing facilities and barrier extreme points that are d-visible from $\mathbf{x}$. The set $\mathcal{P} \cap \mathrm{visible}_d(\mathbf{x})$ is called the candidate set *of $\mathbf{x}$ with index set $\mathcal{I}$,*

$$\mathcal{I} = \big\{i \in \{1, \ldots, I\} :\ \mathbf{p}_i \in \mathcal{P} \cap \mathrm{visible}_d(\mathbf{x})\big\}.$$

The set

$$R := \{\mathbf{y} \in \mathcal{F} : \mathcal{P} \cap \mathrm{visible}_d(\mathbf{y}) = \mathcal{P} \cap \mathrm{visible}_d(\mathbf{x})\} \neq \emptyset$$

is called the candidate domain *of $\mathbf{x}$.*
The finite collection of all candidate domains in $\mathcal{F}$ is denoted by $\mathcal{R}$.

The candidate domains define a partition of the feasible region into a finite number of subsets that can be bounded by the cardinality of the power set of the set of existing facilities and barrier extreme points $\mathbb{P}(\mathcal{A} \cup \mathcal{E}(\mathcal{B}))$ [see Bischoff and Klamroth, 2006].

2.4.1 The Single-Facility Case

If only one new facility must be located, the candidate domains induce a reduction of the single-facility Weber problem with barriers into a polynomial number of mixed integer programming problems, one for each candidate domain R. In each of these subproblems, the optimal location $\mathbf{x} \in R$ and the assignment of *intermediate points* to every existing facility are determined *simultaneously.* An intermediate point is a breaking point on a piecewise linear shortest feasible path from an arbitrary point $\mathbf{x} \in R$ to an existing facility $\mathbf{a}_m \in \mathcal{A}$. Due to the *barrier touching property* [Klamroth, 2002], for every facility $\mathbf{a}_m$, $m = 1, \ldots, M$, there exists at least one intermediate point $\mathbf{p}$ that is contained in the candidate set $\mathcal{P} \cap \text{visible}_d(\mathbf{x})$. Further, if $\mathbf{p} \in \mathcal{P} \cap \text{visible}_d(\mathbf{x})$ is an intermediate point from $\mathbf{x} \in R$ to $\mathbf{a}_m \in \mathcal{A}$, for the barrier distance between $\mathbf{x}$ and $\mathbf{a}_m$ holds:

$$d_{\mathcal{B}}(\mathbf{x}, \mathbf{a}_m) = d(\mathbf{x}, \mathbf{p}) + d_{\mathcal{B}}(\mathbf{p}, \mathbf{a}_m).$$

Since the barriers are polyhedra and since the set of barrier extreme points $\mathcal{E}(\mathcal{B})$ and the set of existing facilities $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_M\}$ are finite, the barrier distances $d_{\mathcal{B}}(\mathbf{p}, \mathbf{a}_m)$, $\mathbf{a}_m \in \mathcal{A}$, $\mathbf{p} \in \mathcal{P} = \mathcal{A} \cup \mathcal{E}(\mathcal{B})$ can be treated as known constants in the problem formulation. In practice, their value is determined before the actual optimization process.

Introducing variables that contain information on the assignment of intermediate points $\mathbf{p}_i \in \mathcal{P} = \{\mathbf{p}_1, \ldots, \mathbf{p}_I\}$ to existing facilities

$$z_{im} = \begin{cases} 1 & \text{if } \mathbf{p}_i \text{ is used as intermediate point to } \mathbf{a}_m, \\ 0 & \text{otherwise} \end{cases}$$

for all $m = 1, \ldots, M$, $i = 1, \ldots, I$, we obtain the following formulation for a single-facility Weber problem with barriers in one candidate domain R:

$$\begin{aligned} \min \quad & \sum_{m=1}^{M} w_m \left(\sum_{i \in \mathcal{I}} z_{im} \big(d(\mathbf{x}, \mathbf{p}_i) + d_{\mathcal{B}}(\mathbf{p}_i, \mathbf{a}_m) \big) \right) \\ \text{s.t.} \quad & \sum_{i \in \mathcal{I}} z_{im} = 1, \qquad m = 1, \ldots, M \\ & z_{im} \in \{0, 1\}, \qquad m = 1, \ldots, M, \quad i = 1, \ldots, I \\ & \mathbf{x} \in R. \end{aligned} \tag{$\mathrm{WP}_{\mathcal{B}}^{R}$}$$

2.4.2 The Multi-Facility Case

Since the definition of candidate sets and candidate domains itself is independent of the selection of appropriate allocation clusters, the $\mathrm{MWP}_{\mathcal{B}}$ can be similarly decomposed into a series of subproblems, where each of the new facilities $\mathbf{x}_n$ is restricted to one of the candidate domains R_n with candidate set $\mathcal{P} \cap \text{visible}_d(\mathbf{x}_n)$ and index set $\mathcal{I}_n = \{i \in \{1,\dots,I\} : \mathbf{p}_i \in \mathcal{P} \cap \text{visible}_d(\mathbf{x}_n)\}$, $n = 1,\dots,N$, as follows:

$$\begin{array}{lll}
\min & \displaystyle\sum_{n=1}^{N}\sum_{m=1}^{M} y_{mn} w_m \left(\sum_{i\in\mathcal{I}_n} z_{im} \big(d(\mathbf{x}_n,\mathbf{p}_i) + d_{\mathcal{B}}(\mathbf{p}_i,\mathbf{a}_m) \big) \right) & \\
\text{s.t.} & \displaystyle\sum_{n=1}^{N}\sum_{i\in\mathcal{I}_n} y_{mn}\cdot z_{im} = 1, & m = 1,\dots,M \\
& y_{mn} \in \{0,1\}, & m = 1,\dots,M,\ n = 1,\dots,N \\
& z_{im} \in \{0,1\}, & m = 1,\dots,M,\ i \in \mathcal{I}_n \\
& \mathbf{x_n} \in R_n, & n = 1,\dots,N.
\end{array} \qquad (\text{MWP}_{\mathcal{B}}^{R})$$

This problem decomposition can be seen as an extension of the reduction result of Klamroth [2001] to the case of multiple new facilities. In the following we present basic properties of the resulting mixed integer problems in order to highlight that the application of an alternate location and allocation algorithm is promising.

By rearranging the terms in the objective function of ($\text{MWP}_{\mathcal{B}}^{R}$) and defining the variables $u_{mni} = y_{mn} \cdot z_{im}$, i.e.

$$u_{mni} = \begin{cases} 1 & \text{if } y_{mn} = z_{im} = 1, \\ 0 & \text{otherwise,} \end{cases} \quad m = 1,\dots,M,\ n = 1,\dots,N,\ i \in \mathcal{I}_n,$$

we obtain the equivalent formulation

$$\begin{array}{lll}
\min & \displaystyle\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{i\in\mathcal{I}_n} u_{mni} w_m \big(d(\mathbf{x}_n,\mathbf{p}_i) + d_{\mathcal{B}}(\mathbf{p}_i,\mathbf{a}_m) \big) & \\
\text{s.t.} & \displaystyle\sum_{n=1}^{N}\sum_{i\in\mathcal{I}_n} u_{mni} = 1, & m = 1,\dots,M \\
& u_{mni} \in \{0,1\}, & m = 1,\dots,M,\ n = 1,\dots,N,\ i \in \mathcal{I}_n \\
& \mathbf{x_n} \in R_n, & n = 1,\dots,N.
\end{array} \qquad (\text{MWP}_{\mathcal{B}}^{R}\text{'})$$

Similar to the decomposition of ($\text{MWP}_{\mathcal{B}}$) into location and allocation subproblems as discussed in Sections 2.1 and 2.2, problem ($\text{MWP}_{\mathcal{B}}^{R}$') can be decomposed into a location and an allocation subproblem:

(1) If the assignment variables u_{mni} are fixed to constant values $\overline{u}_{mni}$, $m = 1,\dots,M$, $n = 1,\dots,N$, $i \in \mathcal{I}_n$, problem ($\text{MWP}_{\mathcal{B}}^{R}$'), decomposes into N Weber problems on the candidate domains R_n, $n = 1,\dots,N$,

$$\begin{array}{ll}
\min & \displaystyle\sum_{m=1}^{M}\sum_{i\in\mathcal{I}_n} c_{mni}\, d(\mathbf{x}_n,\mathbf{p}_i) \\
\text{s.t.} & \mathbf{x}_n \in R_n,
\end{array} \qquad (\text{WP}^{R}(n))$$

where $c_{mni} := w_m \cdot \overline{u}_{mni}$, i.e.

$$c_{mni} := \begin{cases} w_m & \text{if } \overline{u}_{mni} = 1, \\ 0 & \text{otherwise,} \end{cases} \quad m = 1, \dots, M,\ n = 1, \dots, N,\ i \in \mathcal{I}_n.$$

(2) For fixed location variables $\overline{\mathbf{x}}_n$, $n = 1, \dots, N$, problem (MWP$^{R}_{\mathcal{B}}$') reduces to the set partitioning problem

$$\begin{aligned} \min \quad & \sum_{n=1}^{N} \sum_{m=1}^{M} \sum_{i \in \mathcal{I}_n} \overline{c}_{mni}\, u_{mni} \\ \text{s. t.} \quad & \sum_{n=1}^{N} \sum_{i \in \mathcal{I}_n} u_{mni} = 1, \quad m = 1, \dots, M \\ & u_{mni} \in \{0, 1\}, \qquad\quad m = 1, \dots, M,\ n = 1, \dots, N,\ i \in \mathcal{I}_n, \end{aligned} \tag{SP'}$$

where

$$\overline{c}_{mni} := w_m \cdot \big(d(\overline{\mathbf{x}}_n, \mathbf{p}_i) + d_{\mathcal{B}}(\mathbf{p}_i, \mathbf{a}_m)\big),\ m = 1, \dots, M,\ n = 1, \dots, N,\ i \in \mathcal{I}_n.$$

While the allocation subproblems are similar to those in Section 2.2, the resulting location subproblems are much easier in this case. Since the barrier distances $d_{\mathcal{B}}(\mathbf{p}_i, \mathbf{a}_m)$, $i = 1, \dots, I$, $m = 1, \dots, M$ can be treated as known constants and the intermediate points $\mathbf{p}_i \in \mathcal{P}$, $i \in \{1, \dots, I_n\}$, $n = 1, \dots, N$ are specified with the given assignment variables, the barrier distances have been eliminated. Therefore, the resulting subproblem (WP$^R(n)$) is a single facility Weber problem that has a convex objective function (as a positively weighted sum of convex distances), and is restricted on the candidate domain R_n.

The number of location subproblems is much larger than in the case of the WP$_{\mathcal{B}}$, in particular, we obtain $S(M, N) \cdot M \cdot I \cdot C^N$ Weber problems, where $S(M, N)$ is the Stirling number of the second kind (cf. Section 2.1) and C denotes the number of candidate domains, which can be bounded by the cardinality of the power set $|\mathbb{P}(\mathcal{A} \cup \mathcal{E}(\mathcal{B}))|$ of the set of existing facilities and barrier extreme points [see Bischoff and Klamroth, 2006]. Consequently, due to the complexity of the proposed decomposition, a total enumeration of the location subproblems for all feasible assignments, as presented by Klamroth [2001] for the WP$_{\mathcal{B}}$, is not a viable solution method for the multi facility counterpart. Instead, a heuristic solution method, that alternately determines a low priced feasible assignments and solves the corresponding Weber subproblems seems favorable.

It is easy to show that the integrality constraints $u_{mni} \in \{0, 1\}$ of problem (SP') can be relaxed to $u_{mni} \in [0, 1]$, $m = 1, \dots, M$, $n = 1, \dots, N$, $i \in \mathcal{I}_n$, without changing the optimal objective value. That is, the continuous relaxation has an integral optimal solution which satisfies $u^\star_{mni} \in \{0, 1\}$, $m = 1, \dots, M$, $n = 1, \dots, N$, $i \in \mathcal{I}_n$. Considering this continuous relaxation of problem (SP'), which in fact always has an integer optimal solution, this decomposition implies that the objective function of problem (MWP$_{\mathcal{B}}^{R}$') is *bi-convex* in the sense that although it is non-convex, its variable set can be partitioned into two subsets of variables ($\{\mathbf{x}_1, \ldots, \mathbf{x}_N\}$ and $\{u_{mni},\ m = 1, \ldots, M,\ n = 1, \ldots, N,\ i \in \mathcal{I}_n\}$, respectively) such that, if one set of variables is fixed, the objective function is convex in the remaining variables. One of the most typical solution methods for bi-convex problems is the *alternate convex search* algorithm, which alternately solves the subproblems that are obtained when fixing the other set of variables. This approach, which results in an alternate location and allocation algorithm similar to the one for the MWP, is applied as a heuristic solution method for the MWP$_{\mathcal{B}}^{R}$'. See Section 3.2 for a detailed description of the algorithm. For further information on the optimization of bi-convex functions, we refer to Floudas and Visweswaran [1990] and Gorski et al. [2006].

2.5 Allocation Clusters

Rosing [1992] examines geometric properties of the MWP by considering the allocation clusters. He shows that under Euclidean distances the convex hulls of the N allocation clusters associated with the new facilities have pairwise disjoint interior. More precisely, let $\mathbf{x}_n^\star$, $n = 1, \ldots, N$, $y_{mn}^\star$, $m = 1, \ldots, M$, $n = 1, \ldots, N$, be an optimal solution of problem (MWP), then for arbitrary $\overline{m} \in \{1, \ldots, M\}$, $\overline{n} \in \{1, \ldots, N\}$ we have

$$\mathbf{a}_{\overline{m}} \in \operatorname{int}(\operatorname{conv}(\mathcal{A}_{\overline{n}})) \ \Rightarrow\ y_{\overline{m}n} = \begin{cases} 1 & \text{if } n = \overline{n}, \\ 0 & \text{otherwise,} \end{cases} \quad n = 1, \ldots, N.$$

This property is true, since the domains of the Voronoi diagram with generators $\mathbf{x}_n^\star$, $n = 1, \ldots, N$ are convex under Euclidean distances. See Okabe et al. [1992] for an overview about spatial tessellations by Voronoi diagrams.

Based on this result, Rosing [1992] presents a solution method that solves the single-facility location problems for all combinations of allocation clusters satisfying this property. An algorithm to enumerate these clusters is given in Harris [2003]. By selecting a set of N cheapest allocation clusters that partition the set of existing facilities, an optimal solution of problem (MWP) is found. The number of partitions of the set of existing facilities into N clusters that satisfy the given property depends on the geometry of the problem and is in general far below the Stirling number of the second kind $S(N, M)$, cf. Section 2.1.

Unfortunately, this property does in general not hold in the presence of barriers. An example is depicted in Figure 1. In this instance of problem (MWP$_{\mathcal{B}}$), two new facilities have to be located. The weights of the existing facilities $\mathbf{a_1}$ and $\mathbf{a}_5$ can be chosen sufficiently high such that one new facility is located at $\mathbf{x}_1 = \mathbf{a}_1$ while the other is located at $\mathbf{x}_2 = \mathbf{a}_5$. Then the remaining existing facilities are assigned to the new facilities $\mathbf{x}_1$ and $\mathbf{x}_2$ as illustrated in Figure 1. Note that in this simple example problem the convex hull of the iterative convex hulls of both allocation clusters, $\mathcal{A}_1 = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ and $\mathcal{A}_2 = \{\mathbf{a}_4, \mathbf{a}_5\}$ overlap. Moreover, neither $\operatorname{conv}(\mathcal{A}_1)$ nor $\operatorname{conv}(\mathcal{A}_2)$ contains a barrier polyhedron. The interested reader may find numerical values in Table 1 in order to validate the example in Figure 1.

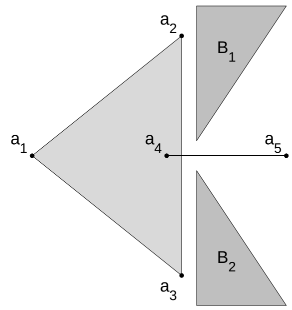

Figure 1: In contrast to the MWP under Euclidean distances, an optimal solution of the $\text{MWP}_{\mathcal{B}}$ may contain overlapping allocation clusters.

$a_1{=}(0,10)$	$w_1{=}10^{10}$	$B_1{=}\{(11,11),(11,20),(17,20)\}$
$a_2{=}(10,18)$	$w_2{=}1$	$B_2{=}\{(11,9),(11,0),(17,0)\}$
$a_3{=}(10,2)$	$w_3{=}1$	$d_{\mathcal{B}}(a_1,a_2){=}12.8062 \quad < \quad 13.1538{=}d_{\mathcal{B}}(a_5,a_2)$
$a_4{=}(9,10)$	$w_4{=}1$	$d_{\mathcal{B}}(a_1,a_3){=}12.8062 \quad < \quad 13.1538{=}d_{\mathcal{B}}(a_5,a_3)$
$a_5{=}(17,10)$	$w_5{=}10^{10}$	$d_{\mathcal{B}}(a_1,a_4){=}\ 9.0000 \quad > \quad 8.0000{=}d_{\mathcal{B}}(a_5,a_4)$

Table 1: Numerical values to the example presented in Figure 1.

Therefore, a solution method similar to the one proposed by Rosing [1992] can not be used for ($\text{MWP}_{\mathcal{B}}$). That is, no general geometric property is known in the presence of barriers that can be applied to reduce the number of allocation clusters to consider.

However, the cluster property can be transferred to those regions of a problem ($\text{MWP}_{\mathcal{B}}$) where no barrier regions interfere. If the convex hull of the union of a set of allocation clusters $\{\mathcal{A}_n,\ n \in \mathcal{N}\}$ with $\mathcal{N} \subseteq \{1,\dots,N\}$ does not contain any barrier polyhedra, the corresponding subproblem of locating $|\mathcal{N}|$ new facilities with respect to the existing facilities $\bigcup_{n\in\mathcal{N}} \mathcal{A}_n$ is an unconstrained MWP for which the property holds. Consequently, the number of partitions of the existing facilities into N clusters that need to be evaluated in order to exactly solve problem (MFLAPwb) can be reduced at least in those regions that are not intersected by any barriers.

In the alternate location and allocation algorithm developed in this paper, this cluster property is automatically fulfilled after the first iteration, since the partition variables are set to minimize optimization problem SP with respect to the fixed location variables.

3 Alternate Location-Allocation Heuristic

Since their introduction by Cooper [1964] alternate location and allocation heuristics (LA) that are based on the alternating solution of location and allocation subproblems have been among the most successful heuristics for the MWP. In the case of the $\text{MWP}_{\mathcal{B}}$, the subproblem decompositions presented in Sections 2.1 and 2.2 on one hand, and the mixed-integer formulations presented in Section 2.4 on the other hand, suggest two different extensions of LA to multi-facility Weber problems with barriers. Namely, the determination of intermediate points and thus the routes taken around the barriers can either be included in the location step, leading to an Alternate Location-with-Barriers Allocation Algorithm ($\text{L}_{\mathcal{B}}\text{A}$), or can be integrated into the allocation step, hence implying an Alternate Location Allocation-with-Routes Algorithm ($\text{LA}_{\mathcal{P}}$).

3.1 Alternate Location-with-Barriers Allocation

Given a feasible starting solution for the location variables $\mathbf{x}_1, \ldots, \mathbf{x}_N \in \mathcal{F}$, the Alternate Location-with-Barriers Allocation Algorithm ($\text{L}_{\mathcal{B}}\text{A}$) iterates between the determination of allocation clusters by solving problem (SP), and the consecutive solution of N Weber problems with barriers ($\text{WP}_{\mathcal{B}}(n)$), $n = 1, \ldots, N$, one for each allocation cluster:

Algorithm 3.1 ($\text{L}_{\mathcal{B}}\text{A}$).

Input: *Starting locations* $\mathbf{x}_1, \ldots, \mathbf{x}_N \in \mathcal{F}$

`REPEAT`

Allocation Step: *For* $m = 1, \ldots, M$*, set*

$$\overline{n} := \operatorname*{minarg}_{n=1,\ldots,N} \{d_{\mathcal{B}}(\mathbf{x}_n, \mathbf{a}_m)\} \quad \textit{and} \quad y_{mn} := \begin{cases} 1 & \textit{if } n = \overline{n}, \\ 0 & \textit{otherwise,} \end{cases} \quad n = 1, \ldots, N$$

Location Step: *For* $n = 1, \ldots, N$*, set*

$$c_{mn} := \begin{cases} w_m & \textit{if } y_{mn} = 1, \\ 0 & \textit{otherwise,} \end{cases} \quad m = 1, \ldots, M,$$

and solve

$$\begin{aligned} \min \quad & \sum_{m=1}^{M} c_{mn} d_{\mathcal{B}}(\mathbf{x}_n, \mathbf{a}_m) \\ \text{s.t.} \quad & \mathbf{x}_n \in \mathcal{F} \end{aligned} \qquad (\text{WP}_{\mathcal{B}}(n))$$

`UNTIL` *a stopping criterion is satisfied.*

The actual performance of $L_{\mathcal{B}}A$ vitally depends on the quality of the initial location solution $\mathbf{x}_1, \ldots, \mathbf{x}_N$. Different possibilities for its selection are discussed in Section 3.4 below. Moreover, the solution of the non-convex Weber problems with barriers ($\mathrm{WP}_{\mathcal{B}}(n)$) in the location steps is in general much more time-consuming than the solution of unconstrained Weber problems. We therefore suggest the application of a heuristic based on a genetic algorithm for the solution of ($\mathrm{WP}_{\mathcal{B}}(n)$), see Bischoff and Klamroth [2006] for details.

Note that, in contrast to the total enumeration technique mentioned in Section 2.1, the $L_{\mathcal{B}}A$ does not necessarily find the global optimal solution of problem ($\mathrm{MWP}_{\mathcal{B}}$), even if the location subproblems ($\mathrm{WP}_{\mathcal{B}}(n)$) are solved exactly. However, since enumerating all potentially feasible allocation clusters is generally computationally too expensive, it offers an efficient alternative that frequently finds very good solutions in practice.

3.2 Alternate Location Allocation-with-Routes

Based on the integrated formulation ($\mathrm{MWP}_{\mathcal{B}}^{R}$') and on its decomposition into location and allocation subproblems presented in Section 2.4, the Alternate Location Allocation-with-Routes Algorithm ($\mathrm{LA}_{\mathcal{P}}$) iterates between the combined determination of allocation clusters and routes taken around the barriers to the existing facilities by solving problem (SP'), and relaxations of the location problems ($\mathrm{WP}^R(n)$), $n = 1, \ldots, N$:

Algorithm 3.2 ($\mathbf{LA}_{\mathcal{P}}$).

Input: *Starting locations* $\mathbf{x}_1, \ldots, \mathbf{x}_N \in \mathcal{F}$ *and the index sets* $\mathcal{I}_n$*,* $n = 1, \ldots, N$ *of their candidate sets* $\mathcal{P} \cap \mathrm{visible}_d(\mathbf{x}_n)$

`REPEAT`

Allocation Step: *For* $m = 1, \ldots, M$*, set*

$$(\overline{n}, \overline{i}) := \operatorname*{minarg}_{n=1,\ldots,N} \operatorname*{minarg}_{i \in \mathcal{I}_n} \left\{ d(\mathbf{x}_n, \mathbf{p}_i^n) + d_{\mathcal{B}}(\mathbf{p}_i^n, \mathbf{a}_m) \right\}$$

and

$$u_{mni} := \begin{cases} 1 & \textit{if } (n,i) = (\overline{n}, \overline{i}), \\ 0 & \textit{otherwise,} \end{cases} \quad n = 1, \ldots, N, \; i \in \mathcal{I}_n$$

Location Step: *For* $n = 1, \ldots, N$*, set*

$$c_{mni} := \begin{cases} w_m & \textit{if } u_{mni} = 1, \\ 0 & \textit{otherwise,} \end{cases} \quad m = 1, \ldots, M, i \in \mathcal{I}_n,$$

and solve

$$\begin{array}{ll} \min & \displaystyle\sum_{m=1}^{M} \sum_{i \in \mathcal{I}_n} c_{mni} \, d(\mathbf{x}_n, \mathbf{p}_i^n) \\ \text{s.t.} & \mathbf{x}_n \in \mathcal{F}, \end{array} \qquad (\mathrm{WP}(n))$$

and determine the index sets $\mathcal{I}_n$, $n = 1, \ldots, N$ of the candidates sets $\mathcal{P} \cap \text{visible}_d(\mathbf{x}_n)$ with respect to the new locations $\mathbf{x}_n$,

`UNTIL` *a stopping criterion is satisfied.*

As in the case of the $\text{L}_\mathcal{B}\text{A}$, the selection of an appropriate starting solution influences the efficiency and the solution quality of $\text{LA}_\mathcal{P}$. We implemented $\text{LA}_\mathcal{P}$ with the same termination criteria and variants for obtaining initial solutions as the $\text{L}_\mathcal{B}\text{A}$, see Sections 3.3 and 3.4 for details.

Note that the allocation steps in both algorithms have the same complexity. The location step in contrast is significantly easier for the $\text{LA}_\mathcal{P}$, since it only involves the solution of N Weber problems which can be realized using the Weiszfeld algorithm [Weiszfeld, 1937]. However, relaxing the constraints $\mathbf{x}_n \in R_n$, $n = 1, \ldots, N$ to $\mathbf{x}_n \in \mathbb{R}^2$ in (WP(n)) may lead to an underestimation of the actual distances in the location step and / or to infeasible location solutions. While infeasibility is avoided by moving infeasible locations to the optimal point on the boundary of the barrier set containing them (using Weiszfeld iterations on the respective barrier boundaries), the distances to the existing facilities are only later corrected in the next allocation step.

Furthermore, it is possible that from an optimal solution of the relaxed problem (WP(n)) the corresponding intermediate points are not visible anymore. Therefore, monotone decreasing objective values can not be guaranteed. Convergence of the algorithm is only ensured if in all iterations the intermediate points are d-visible from the optimal locations of the relaxed problem. Obviously, this property is satisfied if no barrier intersects the convex hull of the intermediate points. Despite this theoretical shortcoming, we did not observe difficulties concerning the convergence of this solution method in practice.

This approach can also be interpreted as a generalization of the iterative solution heuristic for single-facility Weber problems with polyhedral barriers as suggested by Butt and Cavalier [1996] to the multi-facility case.

3.3 Termination Criteria

A natural stopping condition for both versions of the alternate location and allocation heuristic, $\text{L}_\mathcal{B}\text{A}$ and $\text{LA}_\mathcal{P}$, is that the same set of allocation clusters is determined in two consecutive iterations of the allocation step. In this case, a local minimum is found, and neither location nor allocation variables would be changed in the following iterations. Additionally, an upper bound on the number of iterations has been set, that never was reached in practice.

3.4 Starting Solutions

We have implemented both variants of the LA presented above with a random multi-start approach (Section 3.4.1) as well as using an optimized solution of the problem relaxation to a p-median problem as a starting solution. The network of the p-median problem is defined by the visibility graph of the existing facilities and barrier extreme points (Section 3.4.2). For a numerical comparison of both approaches, see Section 4.

3.4.1 Random Starting Solutions

One possibility to increase the chances of convergence to the global minimum with alternate location and allocation algorithms is to repeat the algorithm multiple times with different initial values. In the end, the best solution of all computations is returned.

Since in our implementation both LA variants start with an allocation step, the initial locations are a randomly chosen subset of the set of existing facilities and barrier extreme points, i.e. $\{\overline{\mathbf{x}}_1, \ldots, \mathbf{x}_N\} \subseteq \mathcal{A} \cup \mathcal{E}(\mathcal{B})$. Restricting the starting locations to this set is convenient since such initial values are feasible and their determination requires almost no computational cost. Moreover, the resulting location solutions are automatically contained in the iterative convex hull of the problem, and since optimal new facilities often coincide with existing facilities (or are located in close proximity), this technique yields good initial values in practice. However, this restriction may in some cases make it impossible to reach the global optimum of the problem.

3.4.2 p-Median Based Starting Solutions

When choosing initial values randomly, there may always be a number of starting solutions that result in local optimal solutions with function values far away from the global minimum. To avoid these redundant computations, different selection criteria for the starting solutions may be applied. For example, *p-median based* algorithms first transform the continuous MWP into a network location problem with a node set that is given by the existing facilities. The optimal, or at least a sufficiently good solution of the p-median problem on this network (with $p = N$) is then used as a starting solution for the LA.

Applied to the $\text{MWP}_\mathcal{B}$, the corresponding network location problem additionally has to represent the barrier distances and is therefore based on the *visibility graph* of the existing facilities and the barrier extreme points. The visibility graph of $\mathcal{P} = \mathcal{A} \cup \mathcal{E}(\mathcal{B}) = \{\mathbf{p}_1, \ldots, \mathbf{p}_I\}$ is a graph $G = (V, E)$ with node set $V = \{1, \ldots, I\}$, where node $l \in V$ corresponds to the point $\mathbf{p}_l \in \mathcal{P}$, $l = 1, \ldots, I$. Every pair of nodes $i, j \in V$ for which the corresponding points $\mathbf{p}_i$ and $\mathbf{p}_j$ are d-visible in $\mathcal{F}$ is connected by an edge $[i, j] \in E$ of length $l_{i,j} = d(\mathbf{p}_i, \mathbf{p}_j)$. Consequently, the length $d_G(i, j)$ of a shortest network path between two arbitrary nodes $i, j \in V$ corresponds to the barrier distance $d_\mathcal{B}(\mathbf{p}_i, \mathbf{p}_j)$ between the corresponding points in the plane.

The p-median problem [Hakimi, 1964, 1965] on G is to find a set of $p = N$ nodes $\{v_1, \ldots, v_N\} \subseteq V$ that minimize the total transportation cost to the nodes $\{a_1, \ldots, a_M\} \subseteq V$ that correspond to the existing facilities $\{\mathbf{a}_1, \ldots, \mathbf{a}_M\} = \mathcal{A}$

in the plane:

$$\begin{aligned}
\min \quad & \sum_{n=1}^{N}\sum_{m=1}^{M} y_{mn} w_m d_G(v_n, a_m) \\
\text{s. t.} \quad & \sum_{n=1}^{N} y_{mn} = 1, \quad m = 1, \dots, M \\
& y_{mn} \in \{0,1\}, \quad m = 1, \dots, M,\ n = 1, \dots, N \\
& v_n \in V, \qquad\quad n = 1, \dots, N.
\end{aligned} \tag{p-med}$$

A solution y_{mn}, $m = 1, \dots, M$, $n = 1, \dots, N$ and v_n, $n = 1, \dots, N$ of the p-median problem can be easily transferred to a starting solution for (MWP$_{\mathcal{B}}$) by setting $\mathbf{x}_n := \mathbf{p}_n$, $n = 1, \dots, N$.

Since the p-median problem is itself NP-hard [see Garey and Johnson, 1979] and since we are only interested in finding a good starting solution for our problem, we did not apply an exact solution method for its solution. Instead, we applied a tabu search procedure developed by Rolland et al. [1996]. Obviously, starting with the optimal solution of the p-median solution both methods not necessarily converge to the global optimum of the corresponding (MWP$_{\mathcal{B}}$), and it may even happen that a suboptimal solution of the relaxation possibly reaches better results in the original problem. Therefore, a randomized search method that yields different very good starting solutions is not only more efficient but also a helpful component for diversification when performing multiple starts.

Even though starting with an optimal p-median solution does not guarantee the LA to converge to the global minimum of (MWP$_{\mathcal{B}}$), starting with a good p-median based solution has turned out to be efficient in the sense of quick convergence and effective in finding optimal or near-optimal solutions even for large-scale problems.

4 Computational Results

In this section the two heuristics and the two different starting solution techniques presented in Section 3 are compared. The algorithms have been implemented in Matlab (Release 14) and were evaluated on a Sun Fire V20z machine with two AMD Opteron 2.4GHz CPUs and 8GB memory.

Since the single-facility barrier location problems that occur in the L$_{\mathcal{B}}$A are solved with a solution method that is based on genetic algorithms for which random decisions are a fundamental tool, and since the starting solutions are obtained either with a tabu-search heuristic or are totally random-based, repeated computations may lead to different results.

Both methods have been tested and compared with a large set of location problems. In order to give the reader an outline of the performance, the computational results of a well-known facility location problem with barriers introduced by Aneja and Parlar [1994] have been selected to be presented as sample data. This location problem consists of 18 existing facilities with weights

$w_m = 1$, $m = 1, \ldots, 18$, and twelve barrier regions. The coordinates of the existing facilities and barrier extreme points are specified in Table 2.

Table 2: Problem data of the example problem from Aneja and Parlar [1994]

Existing facility locations:

$\mathbf{a}_1 = (1, 2)$	$\mathbf{a}_2 = (2, 8)$	$\mathbf{a}_3 = (3, 12)$	$\mathbf{a}_4 = (5, 5)$
$\mathbf{a}_5 = (6, 1)$	$\mathbf{a}_6 = (6, 11)$	$\mathbf{a}_7 = (7, 4)$	$\mathbf{a}_8 = (8, 8)$
$\mathbf{a}_9 = (9, 1)$	$\mathbf{a}_{10} = (9, 5)$	$\mathbf{a}_{11} = (9, 10)$	$\mathbf{a}_{12} = (10, 12)$
$\mathbf{a}_{13} = (14, 2)$	$\mathbf{a}_{14} = (14, 4)$	$\mathbf{a}_{15} = (16, 8)$	$\mathbf{a}_{16} = (17, 4)$
$\mathbf{a}_{17} = (17, 10)$	$\mathbf{a}_{18} = (19, 13)$		

Barrier region extreme points:

$$
\begin{aligned}
\mathcal{E}(B_1) &= \{(1,5),(3,5),(4,3),(5,4),(6,2),(2,1)\} \\
\mathcal{E}(B_2) &= \{(1,8),(3,),(2,6)\} \\
\mathcal{E}(B_3) &= \{(1,9),(2,11),(5,10),(3,8)\} \\
\mathcal{E}(B_4) &= \{(4,6),(4,8),(7,11),(8,9)\} \\
\mathcal{E}(B_5) &= \{(4,11),(4,13),(5,14)(9,14),(10,13)\} \\
\mathcal{E}(B_6) &= \{(6,5),(6,7),(7,7),(7,5)\} \\
\mathcal{E}(B_7) &= \{(7,2),(8,3),(9,2)\} \\
\mathcal{E}(B_8) &= \{(9,8),(10,10),(13,13),(16,13),(15,7),(10,6),(12,9)\} \\
\mathcal{E}(B_9) &= \{(12,2),(12,3),(13,3),(13,2)\} \\
\mathcal{E}(B_{10}) &= \{(9,4),(19,8),(19,6)\} \\
\mathcal{E}(B_{11}) &= \{(16,1),(16,3),(19,3),(18,1)\} \\
\mathcal{E}(B_{12}) &= \{(18,11),(18,12),(19,12),(19,10)\}
\end{aligned}
$$

Note that barriers B_1 and B_8 are not convex, but, since no existing facilities are contained in their convex hulls, they can be replaced by their convex hulls before applying the solution methods in order to reduce the number of candidates of intermediate points. Also Butt and Cavalier [1996] and Bischoff and Klamroth [2006] performed this substitution before further examining the problem.

Originally, Aneja and Parlar [1994] solved this problem as a single-facility problem with a simulated annealing heuristic and obtained the solution $\mathbf{x}^\star = (8.76, 4.97)$, with an objective function value $f(\mathbf{x}^\star) = 119.13$. This result is confirmed in Butt and Cavalier [1996] and Bischoff and Klamroth [2006] as best-known solution. We now solved this location problem for the case of $N = 1, \ldots, 18$ new facilities. In the special case of $N = 1$ the aforementioned solution was obtained as well.

The computational results of $\mathrm{L}_{\mathcal{B}}\mathrm{A}$ and $\mathrm{LA}_{\mathcal{P}}$ are summarized in Table 3 and Table 4, respectively. Both algorithms have been examined with the starting solution techniques described in Section 3.4. When applying the random multistart technique (see Section 3.4.1), the best solution obtained in ten executions of the algorithms, each starting with a random initial solution, was selected. To give a more representative comparison with respect to the randomized search subroutines, every problem has been solved five times with both algorithms and both starting solution techniques.

In the tables below, f_{low} denotes the lowest function value found in any computation of the corresponding multi-facility location problem. f_{avr} stands for the average of the function values in the five computations. Δf is the relative difference between the lowest function value f_{low} and the average function value f_{avr},

$$\Delta f := \frac{f_{\text{avr}} - f_{\text{low}}}{f_{\text{low}}}.$$

In the column $t(s)$, the average CPU time in seconds of the five computations is displayed. Table 5 shows how many of the five computations converged to the lowest function value f_{low}.

		p-median based			random multi-start		
N	f_{low}	f_{avr}	Δf	$t(s)$	f_{avr}	Δf	$t(s)$
1	119.1387	119.1387	0.00	2.88	119.1387	0.00	66.47
2	90.3821	95.2783	5.42	1.46	93.4821	3.43	62.75
3	66.0557	66.0557	0.00	3.83	66.6473	0.90	55.08
4	49.5569	49.5569	0.00	2.95	49.5569	0.00	46.02
5	41.4761	42.4428	2.33	3.19	42.1940	1.73	38.11
6	34.6326	34.6326	0.00	2.65	34.8281	0.56	29.09
7	29.8940	30.0721	0.60	1.51	30.1504	0.86	23.86
8	25.9033	26.0814	0.69	0.70	26.4410	2.08	19.83
9	22.2530	22.4311	0.80	0.69	23.3327	4.85	17.67
10	19.0169	20.1950	6.20	0.74	20.4195	7.38	15.40
11	16.7808	16.7808	0.00	0.69	17.1288	2.07	13.28
12	13.7808	13.7808	0.00	0.75	14.2387	3.32	12.72
13	10.9443	10.9443	0.00	0.78	11.6485	6.43	11.97
14	8.7082	8.7082	0.00	0.83	9.0283	3.68	13.41
15	6.4721	6.4721	0.00	0.88	6.4721	0.00	12.44
16	4.2361	4.2361	0.00	0.93	4.2361	0.00	13.84
17	2.0000	2.0000	0.00	0.97	2.0472	2.36	14.89
18	0.0000	0.0000	0.00	1.02	0.0000	0.00	15.53

Table 3: Computational results of $\text{L}_{\mathcal{B}}\text{A}$ for the example problem

The results show a clear superiority of the p-median based starting solution to the random multi-start technique with ten starts. When starting with a good solution of the p-median problem, solutions with lower function values are obtained more often, the best known function values are reached more often and the computation times are drastically reduced since the computation time to determine a good solution of the p-median problem is negligible with respect to the time that is needed for the location and allocation procedure. There-

		p-median based			random multi-start		
N	f_{low}	f_{avr}	Δf	$t(s)$	f_{avr}	Δf	$t(s)$
1	119.1387	119.1387	0.00	0.17	121.9215	2.34	0.38
2	90.3821	90.8684	0.54	0.25	92.3281	2.15	1.15
3	66.0557	66.0557	0.00	0.40	69.0292	4.50	1.85
4	49.5569	49.8363	0.56	0.46	54.8103	10.60	2.51
5	41.4761	42.6511	2.83	0.44	43.2317	4.23	3.21
6	34.6326	34.6326	0.00	0.69	36.6098	5.71	3.80
7	29.8940	29.8940	0.00	0.62	32.0692	7.28	3.92
8	25.9033	25.9033	0.00	0.61	27.1249	4.72	4.15
9	22.2530	22.2530	0.00	0.66	23.7588	6.77	4.48
10	19.0169	20.0170	5.26	0.65	20.8687	9.74	5.25
11	16.7808	16.7808	0.00	0.62	18.2901	8.99	5.39
12	13.7808	13.7808	0.00	0.66	14.4354	4.75	5.35
13	10.9443	10.9443	0.00	0.70	12.2952	12.34	5.88
14	8.7082	8.7082	0.00	0.74	9.4213	8.19	5.98
15	6.4721	6.4721	0.00	0.78	6.9193	6.91	6.23
16	4.2361	4.2361	0.00	0.82	4.4249	4.46	6.46
17	2.0000	2.0000	0.00	0.85	2.0944	4.72	6.86
18	0.0000	0.0000	0.00	0.90	0.0000	0.00	6.09

Table 4: Computational results of $\text{LA}_{\mathcal{P}}$ for the example problem

fore, since the algorithms are executed ten times in the case of the multi-start technique, also the computation time increases by about a factor of ten.

We also observed that a joint treatment of all assignment variables, including those specifying the routes taken around the barrier polyhedra as done in the $\text{LA}_{\mathcal{P}}$, is superior to the $\text{L}_{\mathcal{B}}\text{A}$ where the assignment problem and the barrier location problems are solved separately. Obviously, the computation time of $\text{L}_{\mathcal{B}}\text{A}$ is higher since, while the allocation step has more or less the same complexity in both algorithms, the subproblem in the location step of $\text{L}_{\mathcal{B}}\text{A}$ is more time-consuming.

In the two special cases $N = 1$ and $N = 18$ the best-known solution was obtained in almost all computations with any algorithm and starting solution. This is due to the fact that in both cases the optimal assignments are known: If $N = 1$, every existing facility is assigned to the same new facility, if $N = 18$, every existing facility is assigned to a different one. Only the $\text{LA}_{\mathcal{P}}$ together with random multi-start converged to a suboptimal solution for $N = 1$ in two of the five trials. Although the allocation of existing facilities to the new facility is trivial in this case, the determination of the intermediate points and the assignment of intermediate points leads to different results.

Further, note that whenever the best known solution has been determined starting with a p-median starting solution, it has been obtained in all five trials.

	$L_{\mathcal{B}}A$		$LA_{\mathcal{P}}$	
N	p-median	random	p-median	random
1	5	5	5	3
2	0	0	0	1
3	5	4	5	2
4	0	1	0	0
5	0	0	0	1
6	0	2	0	0
7	0	1	0	0
8	0	0	0	1
9	0	0	5	0
10	0	0	0	1
11	5	3	5	0
12	5	0	5	2
13	5	1	5	0
14	5	2	5	1
15	5	5	5	1
16	5	5	5	1
17	5	4	5	3
18	5	5	5	5

Table 5: Number of runs in which the best known solution has been obtained

Although the p-median heuristic may generate different very good solutions, it actually found the same, probably optimal solutions of the p-median subproblem in most of the runs. Starting with the same initial solution, the $LA_{\mathcal{P}}$ and $L_{\mathcal{B}}A$ either converged to the best known solution of the $MWP_{\mathcal{B}}$ in all five runs or in none of them.

While the computation time of $L_{\mathcal{B}}A$ decreases with an increasing number of new facilities, the time needed by $LA_{\mathcal{P}}$ decreases. This is due to the fact that on the one hand the effort increases in the assignment step with the number of new facilities available, on the other hand the location subproblems become smaller and thus easier to solve. Since $L_{\mathcal{B}}A$ solves single-facility location problems with barriers the computation time depends mostly on the difficulty of the location problems, while $LA_{\mathcal{P}}$ determines the assignment of existing facilities together with the intermediate points which results in a longer computation time if the number of new facilities is large. In the special case of $N = 18$ all location subproblems are trivial since every new facility is located at one existing facility only.

In 11 of the 18 location problems, $LA_{\mathcal{P}}$ together with the p-median starting solution converged to the solution with the lowest function value in all five computations. This most effective combination of algorithm and starting solution technique additionally had an average computation time of less than one second for every problem.

In Figure 2 the example problem together with the best locations and the corresponding allocation clusters in the case of three new facilities is displayed.

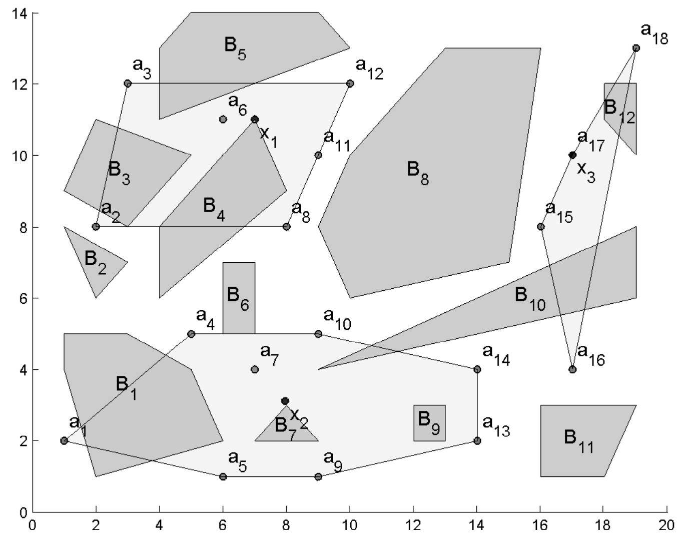

Figure 2: The example problem, the best known locations and the corresponding allocation clusters for three new facilities

Figure 3 shows a plot of the lowest objective function values with respect to the number of new facilities located. It can be seen that for a small number of new facilities there is a big difference between the function values if an additional new facility is available. On the other hand, if N, the number of new facilities is large as compared to the number of existing ones, the total transportation costs are not that drastically reduced by increasing N further. A similar behavior is observed also in the unconstrained case, i.e., if all barrier polyhedra are removed from the example problem. Both situations (with and without barriers) are displayed in Figure 3 in a direct comparison. As was to be expected, the function values of the unconstrained problem are lower than those of the constrained problem particularly for relatively small values of N. If N is relatively large as compared to the number of existing facilities, smaller allocation clusters are obtained and less barriers interfere. As soon as the distances are not lengthened by barriers, the same objective values can be reached in both problems.

The trade-off analysis between the number of new facilities and the corresponding reduction of transportation costs points to an extended or multicriteria model of the generalized multi-Weber problem (with and without barriers) in which the number of new facilities is not predefined, but either considered as an additional objective function, or interpreted as an additional parameter in the optimization problem associated with a cost function.

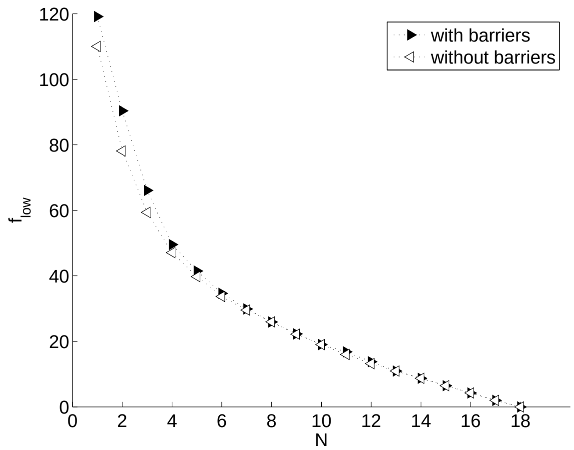

Figure 3: The lowest function values for the example problem of Aneja and Parlar [1994] with and without barrier regions with respect to the number of new facilities

5 Conclusion

We developed two alternate location and allocation heuristics for multi-facility location-allocation problems with barriers. Both solution methods iteratively decompose the (MWP$_{\mathcal{B}}$) into single-facility problems and a set partitioning problem. While in the allocation step of one method only optimal assignments are specified, in the other additionally the paths taken around the barriers are optimized. A numerical comparison shows the superiority in efficiency and effectiveness of a joint treatment of all discrete variables in the allocation step.

The numerical results show that the developed algorithms are suitable for the solution of reasonably sized multi-facility location-allocation problems with barriers, both with regard to computation time and solution quality. This makes this non-convex and numerically difficult, NP-hard problem class accessible for an efficient heuristic solution. Future research should focus on alternative solution methods, including the development of exact algorithms, for example, based on branch and bound. Moreover, the bi-convex structure of the problem as discussed in Sections 2.1 and 2.2 could be exploited further, for example, using ideas from Gorski et al. [2006]. Another interesting extension of the problem is a bicriteria analysis of the trade-off between the number of new facilities and the transportation cost as mentioned in Section 4 above.

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