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Naturwissenschaftliche Fakultät I
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Integer and Nonlinear Optimization
Exercise 4

Problem 1
Prove that the following statements are equivalent (Lemma 3.2 of the lecture):

(i) $\underline{x}^1, \dots, \underline{x}^k \in \mathbb{R}^n$ are affinely independent in $\mathbb{R}^n$.

(ii) $\underline{x}^2 - \underline{x}^1, \underline{x}^3 - \underline{x}^1, \dots, \underline{x}^k - \underline{x}^1 \in \mathbb{R}^n$ are linearly independent in $\mathbb{R}^n$.

(iii) $(\underline{x}^1, -1), (\underline{x}^2, -1), \dots, (\underline{x}^k, -1) \in \mathbb{R}^{n+1}$ are linearly independent in $\mathbb{R}^{n+1}$.

Problem 2
Let $P = \{\underline{x} \in \mathbb{R}^n : A\underline{x} \leq \underline{b}\}$. Show that

(a) If $P \neq \emptyset$, then P has an inner point.

(b) If a polyhedron P is full-dimensional, then it has an interior point.

Problem 3 (Farkas' Lemma)
Prove the following variations of Farkas' Lemma:

$$\begin{array}{ll} \text{(a)} & \{\underline{x} \in \mathbb{R}^n : A\underline{x} = \underline{b}, \underline{x} \geq \underline{0}\} \neq \emptyset \quad \dot\vee \quad \{\underline{y} \in \mathbb{R}^m : \underline{y}A \geq \underline{0}, \underline{y}\,\underline{b} < 0\} \neq \emptyset. \\ \text{(b)} & \{\underline{x} \in \mathbb{R}^n : A\underline{x} = \underline{b}, \underline{x} \geq \underline{0}\} \neq \emptyset \quad \dot\vee \quad \{\underline{y} \in \mathbb{R}^m : \underline{y}A \leq \underline{0}, \underline{y}\,\underline{b} > 0\} \neq \emptyset. \\ \text{(c)} & \{\underline{x} \in \mathbb{R}^n : A\underline{x} \leq \underline{b}\} \neq \emptyset \quad \dot\vee \quad \{\underline{y} \in \mathbb{R}^m : \underline{y}A = \underline{0}, \underline{y}\,\underline{b} < 0, \underline{y} \geq \underline{0}\} \neq \emptyset. \end{array}$$

Problem 4
Let $S = \{\underline{s}^1, \dots, \underline{s}^k\} \subseteq \mathbb{R}^n$ be a finite set of points and let $\underline{y} \in \mathbb{R}^n \backslash \text{conv}(S)$. Use Farkas' Lemma to prove that there exists an inequality $\underline{\pi}\,\underline{x} \leq \underline{\pi}_0$ that separates $\underline{y}$ from $\text{conv}(S)$, that is,