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Solution to Exercise 4.4

- In a non-degenerate pivot-operation, the simplex method moves from one extreme point to an adjacent extreme point of P.
- Let $\underline{x}$ be a bfs ($\Leftrightarrow$ extreme point)

$$\underline{x} = (x_1, \ldots, x_m, \ 0, \ldots, 0)^T$$

and let

$$\underline{y} = (y_1, \ldots, y_{m-1}, \ 0, \ y_{m+1}, 0, \ldots, 0)^T$$

be a bfs after one pivot operation.

To show:
$\underline{x}$, $\underline{y}$ are adjacent extreme points of P or (by Definition 4 on the exercise sheet): $[\underline{x}, \ \underline{y}]$ is an edge of the (convex) polytope P, i.e.

$$[\underline{x}, \ \underline{y}] = \{\underline{z} \in \mathbb{R}^n \mid \underline{z} = \lambda \underline{x} + (1-\lambda)\underline{y}, \lambda \in [0, \ 1]\}$$

Proof:
Let $\hat{\underline{z}} \in [\underline{x}, \ \underline{y}]$
$\Rightarrow \hat{\underline{z}} = \lambda \underline{x} + (1-\lambda)\underline{y} = (\lambda x_1 + (1-\lambda)y_1, \ldots, \lambda x_m, (1-\lambda)y_{m+1}, \ 0, \ldots, 0)^T$

Now let $\hat{\underline{z}} = \mu \underline{z}' + (1-\mu)\underline{z}''; \mu \in (0,1); \underline{z}', \ \underline{z}'' \in P$ (see Definition 3).

To show:
$\underline{z}', \underline{z}'' \in [\underline{x}, \underline{y}]$
Proof by contradiction:
Assume that $\underline{z}' \notin [\underline{x}, \underline{y}]$ (wlog assume that $\underline{z}' \in P$ is not an extreme point and that $\underline{z}'$ does not ly on another edge of P).
$\Rightarrow \underline{z}'$ has at least one positive component in $\{z'_{m+2}, \ldots, z'_n\}$. Let wlog $z'_n > 0$.
Consider the last component of $\hat{z}$:

$$\hat{z}_n = \underbrace{\mu}_{>0} \cdot \underbrace{z'_n}_{>0} + \underbrace{(1-\mu)}_{>0} \cdot \underbrace{z''_n}_{>0} > 0$$

On the other hand, $\hat{\underline{z}} \in [\underline{x}, \underline{y}]$ implies that

$$\hat{z}_n = \underbrace{\lambda x_n}_{=0} + (1-\lambda) \underbrace{y_n}_{=0} = 0$$

which is a contradiction to $\underline{z}' \notin [\underline{x}, \underline{y}]$
$\Rightarrow \underline{z}' \in [\underline{x}, \underline{y}]$ and similarly $\underline{z}'' \in [\underline{x}, \underline{y}]$
$\Rightarrow [\underline{x}, \underline{y}]$ is an edge of $P \Rightarrow \underline{x}, \underline{y}$ are adjacent extreme points of P.