Location Analysis, Handout 4:
Multi-Facility Center Problems,
$m/P/\bullet/l_p/\max$

LP-Based Algorithm 5.2. for $m/P/\bullet/l_\infty/\max$

Input: Existing facilities $a_1,\ldots,a_n \in \mathbb{R}^2$; number m of new facilities sought; positive weights $w_{1,ij}$, $i=1,\ldots,m$, $j=1,\ldots,n$ and $w_{2,ir}$, $i=1,\ldots,m-1$, $r=i+1,\ldots,m$.

Step 1: For $k=1,2$ do: Solve the LP

$$\begin{array}{lrrrrl}
\min & z_k & & & & \\
\text{s.t.} & & -x_{ik} & +\frac{1}{w_{1,ij}}z_k & \geq -a_{jk} & \left.\right\} \forall i \in \{1,\ldots,m\},\ j \in \{1,\ldots,n\} \\
 & & x_{ik} & +\frac{1}{w_{1,ij}}z_k & \geq a_{jk} & \\
 & -x_{ik} & +x_{rk} & +\frac{1}{w_{2,ir}}z_k & \geq 0 & \left.\right\} \forall i,r \in \{1,\ldots,m\},\ i<r \\
 & x_{ik} & -x_{rk} & +\frac{1}{w_{2,ir}}z_k & \geq 0 &
\end{array}$$

and determine the optimal solution $x^*_{1k},\ldots,x^*_{mk},z^*_k$.

Step 2: Set $X^*_i := (x^*_{i1},x^*_{i2})$, $i=1,\ldots,m$, $X^* := (X^*_1,\ldots,X^*_m)$, and $z^* := \max\{z^*_1,z^*_2\}$.

Output: Optimal solution X^* of $m/P/\bullet/l_\infty/\max$ with objective value z^*.

Algorithm 5.3. for $m/P/\bullet/l_1/\max$

Input: Existing facilities $a_1,\ldots,a_n \in \mathbb{R}^2$; number m of new facilities sought; positive weights $w_{1,ij}$, $i=1,\ldots,m$, $j=1,\ldots,n$ and $w_{2,ir}$, $i=1,\ldots,m-1$, $r=i+1,\ldots,m$.

Step 1: Find an optimal solution $\tilde{X}$ of the corresponding problem of type $m/P/\bullet/l_\infty/\max$ with existing facility locations at $T^{-1}(a_1),\ldots,T^{-1}(a_n)$.

Output: Optimal solution $X^* := T(\tilde{X})$ of $m/P/\bullet/l_1/\max$.

Single-Facility Center Problems, $1/P/\bullet/l_p/\max$

Algorithm 5.9. for $1/P/\bullet/l_\infty/\max$

Input: Existing facilities $a_1,\dots,a_n \in I\!R^2$; positive weights w_j, $j=1,\dots,n$.

Step 1: Determine $z^* := \max\{|z_{jl}^k| \,:\, j,l \in \{1,\dots,n\}, j<l; k\in\{1,2\}\}$, where

$$z_{jl}^k := \frac{w_j w_l}{w_j + w_l}(a_{jk} - a_{lk}), \quad j,l \in \{1,\dots,n\},\ j<l;\ k\in\{1,2\}.$$

Step 2: Set $\mathcal{X}^* := [A_1^-(z^*), A_1^+(z^*)] \times [A_2^-(z^*), A_2^+(z^*)]$, where for $k=1,2$

$$A_k^+(z) := \min_{j=1,\dots,n} A_{jk}^+(z), \qquad A_{jk}^+(z) := a_{jk} + \tfrac{1}{w_j}z, \quad j=1,\dots,n$$
$$A_k^-(z) := \max_{j=1,\dots,n} A_{jk}^-(z), \qquad A_{jk}^-(z) := a_{jk} - \tfrac{1}{w_j}z, \quad j=1,\dots,n$$

Output: Set of optimal solutions $\mathcal{X}^*$ of $1/P/\bullet/l_\infty/\max$, optimal objective value z^*.

Algorithm 5.10. for $1/P/w_j=1/l_\infty/\max$

Input: Existing facility locations $a_1,\dots,a_n \in I\!R^2$.

Step 1: Determine

$$a_1^+ := \min_{j=1,\dots,n} a_{j1} \qquad a_1^- := \max_{j=1,\dots,n} a_{j1}$$
$$a_2^+ := \min_{j=1,\dots,n} a_{j2} \qquad a_2^- := \max_{j=1,\dots,n} a_{j2}$$

Step 2: Determine

$$z_1 := \frac{a_1^- - a_1^+}{2}, \quad z_2 := \frac{a_2^- - a_2^+}{2}$$

and $z^* := \max\{z_1, z_2\}$.

Step 3: Set $\mathcal{X}^* := [a_1^- - z^*, a_1^+ + z^*] \times [a_2^- - z^*, a_2^+ + z^*]$.

Output: Set of optimal solutions $\mathcal{X}^*$ of $1/P/w_j=1/l_\infty/\max$, optimal objective value z^*.