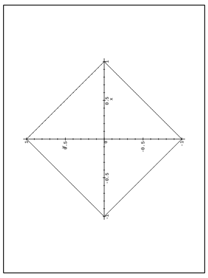
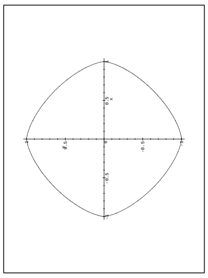
Location Analysis, Handout 2:
Single Facility Weber Problems with l_p Distances,
$1/P/\bullet/l_p/\sum$

l_p Distances:

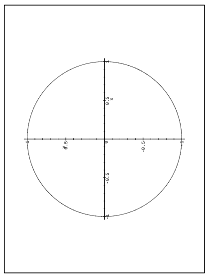
$$
\begin{array}{rcll}
l_p(X,Y) & = & (|x_1-y_1|^p+|x_2-y_2|^p)^{\frac{1}{p}}, & X,Y\in I\!R^2, \quad ,1\le p<\infty \\
l_\infty(X,Y) & = & \max\{|x_1-y_1|,|x_2-y_2|\}, & X,Y\in I\!R^2
\end{array}
$$



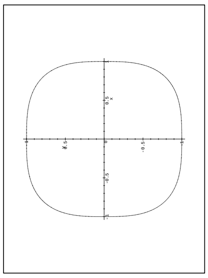
$l_1(X,\underline{0})=1$



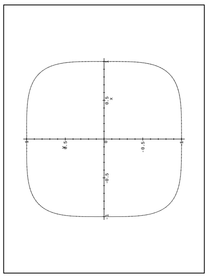
$l_{1.5}(X,\underline{0})=1$



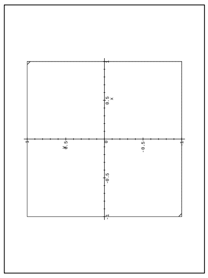
$l_2(X,\underline{0})=1$



$l_3(X,\underline{0})=1$



$l_4(X,\underline{0})=1$



$l_\infty(X,\underline{0})=1$

Property 1: $l_p(X,Y)$ decreases as p increases.

Porperty 2: As $p\to\infty$, $l_p(X,Y)$ becomes the larger of $|x_1-y_1|$ and $|x_2-y_2|$.

Weiszfeld Algorithm for $1/P/\bullet/l_2/\sum$

Input:	Existing facilities $a_1, \ldots, a_n \in I\!R^2$; weights $w_1, \ldots, w_n > 0$.
Step 1:	If $CR_r \le w_r$ for $r \in \{1, \ldots, n\}$, set $X^* := a_r$, STOP.
Step 2:	Select a starting solution $X^{(0)} = (x_1^{(0)}, x_2^{(0)})$; set $l := 0$. ($X^{(0)}$ could be chosen, for example, as the optimal solution of $1/P/\bullet/l_2^2/\sum$.)
Step 3:	For $k = 1, 2$ do $$x_k^{(l+1)} := \frac{\sum_{j=1}^n \frac{w_j a_{jk}}{l_2(X^{(l)}, a_j)}}{\sum_{j=1}^n \frac{w_j}{l_2(X^{(l)}, a_j)}}$$
Step 4:	If $X^{(l+1)}$ satisfies a stopping criterion, set $X^* := X^{(l+1)}$, STOP. Otherwise, set $l := l + 1$ and goto Step 3.
Output:	Approximation X^* of an optimal solution for $1/P/\bullet/l_2/\sum$.

Hyperbolic Approximation Algorithm for $1/P/\bullet/l_p/\sum$, $1 < p < \infty$

Input:	Existing facilities $a_1, \ldots, a_n \in I\!R^2$; weights $w_1, \ldots, w_n > 0$; $p \in I\!R$ with $1 < p < \infty$.
Step 1:	If $CRP_r \le w_r$ for $r \in \{1, \ldots, n\}$, set $X^* := a_r$, STOP.
Step 2:	Select a starting solution $X^{(0)} = (x_1^{(0)}, x_2^{(0)})$; set $l := 0$.
Step 3:	For $k = 1, 2$ do $$x_k^{(l+1)} := \frac{\sum_{j=1}^n \frac{w_j a_{jk}}{d'(X^{(l)}, a_j) \cdot d''(x_k^{(l)}, a_{jk})}}{\sum_{j=1}^n \frac{w_j}{d'(X^{(l)}, a_j) \cdot d''(x_k^{(l)}, a_{jk})}}$$
Step 4:	If $X^{(l+1)}$ satisfies a stopping criterion, set $X^* := X^{(l+1)}$, STOP. Otherwise, set $l := l + 1$ and goto Step 3.
Output:	Approximation X^* of an optimal solution for $1/P/\bullet/l_p/\sum$.

$$CR_r = \left[\left(\sum_{\substack{j=1 \\ j \neq r}}^n \frac{w_j (a_{r1} - a_{j1})}{l_2(a_r, a_j)} \right)^2 + \left(\sum_{\substack{j=1 \\ j \neq r}}^n \frac{w_j (a_{r2} - a_{j2})}{l_2(a_r, a_j)} \right)^2 \right]^{\frac{1}{2}}$$

$$CRP_r = \left[\left| \sum_{\substack{j=1 \\ j \neq r}}^n \frac{w_j \operatorname{sign}(a_{r1} - a_{j1}) |a_{r1} - a_{j1}|^{p-1}}{(l_p(a_r, a_j))^{p-1}} \right|^{\frac{p}{p-1}} + \left| \sum_{\substack{j=1 \\ j \neq r}}^n \frac{w_j \operatorname{sign}(a_{r2} - a_{j2}) |a_{r2} - a_{j2}|^{p-1}}{(l_p(a_r, a_j))^{p-1}} \right|^{\frac{p}{p-1}} \right]^{\frac{p-1}{p}}$$

$$d'(X, a_j) = \left(((x_1 - a_{j1})^2 + \epsilon)^{\frac{p}{2}} + ((x_2 - a_{j2})^2 + \epsilon)^{\frac{p}{2}} \right)^{1 - \frac{1}{p}}$$

$$d''(x_k, a_{jk}) = \left((x_k - a_{jk})^2 + \epsilon \right)^{1 - \frac{p}{2}}, \qquad k = 1, 2.$$