Heat kernel estimates for Laplacians on Lie groups

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18-7-2011

Outline

Control functions

Let $X_1, \ldots, X_N$ be C^∞-vector fields on a d-dimensional connected manifold M. Consider the system of ODE

$$\dot{x}(t) = \sum_{j=1}^{N} u_j(t)\, X_j(x(t))$$

with control functions $u_j \in L_1$.
A controlled path is a solution of this system. If $x(a) = p$ and $x(b) = q$ then the control steers the system from p to q with length

$$\int_a^b \Big(u_1(t)^2 + \ldots + u_N(t)^2 \Big)^{1/2} dt.$$

If $p \in M$ then $q \in M$ is called accessible from p if there exists a control which steers the system from p to q.
Then the distance $d(p,q)$ from p to q is the infimum of the appropriate lengths.

Existence distance

Theorem. (Chow, Carathéodory) The following are equivalent.

- For all $p, q \in M$ there exits a control which steers the system from p to q.
- The vector fields $X_1, \ldots, X_N$, together with their multi-commutators $[X_k, X_l]$, $[X_k, [X_l, X_m]]$, $\ldots$ span the tangent space $T_p M$ at every point $p \in M$. (Chow's condition, Hörmander condition)

If these conditions are satisfied then the map

$$M \times M \to [0, \infty)$$
$$(p, q) \mapsto d(p, q)$$

is a distance on M.

Lie groups

Suppose that the manifold is a connected Lie group G.
Let $X_1, \ldots, X_N$ be right invariant vector fields on G satisfying Chow's or Hörmander's condition. Define the balls

$$B(p, \rho) = \{q \in G : d(p, q) < \rho\}.$$

Let $|U|$ denote the (left) Haar measure of a measurable set $U \subset G$.

Theorem. There exist $D \in \mathbb{N}$ and $c \geq 1$ such that

$$c^{-1} \, \rho^D \leq |B(p, \rho)| \leq c \, \rho^D$$

uniformly for all $p \in G$ and $\rho \in (0, 1]$.

Local dimension

Define

$$\begin{aligned}
V_0 &= \{0\} \\
V_1 &= \mathrm{span}\{X_k : k \in \{1, \ldots, N\}\}, \\
V_2 &= V_1 + \mathrm{span}\{[X_k, X_l] : k, l \in \{1, \ldots, N\}\}, \\
V_3 &= V_2 + \mathrm{span}\{[X_k, [X_l, X_m]] : k, l, m \in \{1, \ldots, N\}\}, \\
&\ldots
\end{aligned}$$

Then

$$D = \sum_{j=1}^{\infty} j \Big(\dim V_j - \dim V_{j-1} \Big).$$

Sums of squares

Let $X_1, \ldots, X_N$ be C^∞-vector fields on a d-dimensional connected open subset $\Omega \subset \mathbb{R}^d$.
The sums of squares operator

$$H = -\sum_{j=1}^{N} (X_j)^2$$

acts on C_c^∞-functions. Hence it acts on distributions on Ω.
Theorem. (Hörmander) The following are equivalent.

- The operator H is hypoellitic, i.e. for every distribution F on Ω such that $HF \in C^\infty(\Omega)$ it follows that $F \in C^\infty(\Omega)$
- The vector fields $X_1, \ldots, X_N$, together with their multi-commutators $[X_k, X_l]$, $[X_k, [X_l, X_m]]$, $\ldots$ span the tangent space T_pM at every point $p \in M$. (Hörmander condition)

Lie groups

Suppose that the manifold is a connected Lie group G.
Let $X_1, \ldots, X_N$ be right invariant vector fields on G satisfying Chow's or Hörmander's condition. Let

$$H = - \sum_{j=1}^{N} (X_j)^2$$

with domain $D(H) = C_c^\infty(G)$. Let $p \in [1, \infty)$.

Theorem.

- The operator H is closable in $L_p(G)$ and $-\overline{H}$ generates a continuous semigroup S which is holomorphic in the right half-plane.
- The semigroup S has a smooth rapidly decreasing kernel K, i.e.,

  $$(S_t u)(g) = (K_t * u)(g) = \int_G K_t(h) \, u(h^{-1} g) \, dh$$

  for all $t > 0$, $u \in L_p(G)$ and $g \in G$.

Theorem (continued)

Let $|g| = d(g, e)$, where e is the identity element of G.

Theorem.

- There exist $b, b', c, c' > 0$ and $\omega, \omega' \geq 0$ such that

$$c' \, t^{-D/2} \, e^{-b'|g|^2 \, t^{-1}} \, e^{-\omega' t} \leq K_t(g) \leq c \, t^{-D/2} \, e^{-b|g|^2 \, t^{-1}} \, e^{\omega t}$$

for all $t > 0$ and $g \in G$.

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  for all $t > 0$ and $g \in G$.

- For every multi-index α there are $b, c, \omega > 0$ such that

$$|(X^\alpha K_t)(g)| \leq c\, t^{-D/2}\, t^{-|\alpha|/2}\, e^{-b|g|^2\, t^{-1}}\, e^{\omega t}$$

  for all $t > 0$ and $g \in G$.

Extensions

Similar theorem (except the derivatives bounds) is valid for operators of the form

$$H = -\sum_{k,l=1}^{N} X_k \, c_{kl} \, X_l$$

with $c_{kl} \in L_\infty(G, \mathbb{R})$ and satisfying the ellipticity condition

$$\mathrm{Re} \sum_{k,l=1}^{N} \xi_k \, c_{kl}(g) \, \overline{\xi_l} \geq \mu \, |\xi|^2$$

for some $\mu > 0$, uniformly for all $\xi \in \mathbb{C}^N$ and $g \in G$.

Also complex uniformly continuous coefficients are possible, but then one also looses the lower bounds for the kernel.

Extensions

Let $m \in 2\mathbb{N}$ and let

$$H = (-1)^{m/2} \sum_{k=1}^{N} X_k^m$$

with domain $D(H) = C_c^\infty(G)$. Fix $p \in [1, \infty)$.
Then H is closable in $L_p(G)$, the operator $-\overline{H}$ generates a continuous semigroup which is holomorphic in the right half-plane and has a convolution kernel K.
There exist $b, c > 0$ and $\omega \in \mathbb{R}$ such that

$$|K_t(g)| \leq c\, t^{-D/m}\, e^{-b(|g|^m t^{-1})^{1/(m-1)}}\, e^{\omega t}$$

for all $t > 0$ and $g \in G$.
For every multi-index α there exist $b, c > 0$ and $\omega \in \mathbb{R}$ such that

$$|(X^\alpha K_t)(g)| \leq c\, t^{-D/m}\, t^{-|\alpha|/m}\, e^{-b(|g|^m t^{-1})^{1/(m-1)}}\, e^{\omega t}$$

for all $t > 0$ and $g \in G$.

Large time

Theorem. Let $c_{kl} \in \mathbb{C}$ and constant. Suppose

$$\mathrm{Re} \sum_{k,l=1}^{N} \xi_k \, c_{kl} \, \overline{\xi_l} \geq \mu \, |\xi|^2$$

for some $\mu > 0$, uniformly for all $\xi \in \mathbb{C}^N$. Let

$$H = - \sum_{k,l=1}^{N} c_{kl} \, X_k \, X_l$$

with domain $D(H) = C_c^\infty(G)$. Fix $p \in [1, \infty)$.

- Then H is closable, the operator $-\overline{H}$ generates a continuous semigroup in $L_p(G)$, which is holomorphic in a sector and has a convolution kernel K.

Large time (continued)

- There exist $b, c > 0$ such that

$$|K_t(g)| \leq c\,|B(e,t)|^{-1/2}\, e^{-b|g|^2 t^{-1}}$$

for all $t > 0$ and $g \in G$.

- For all $p \in (1, \infty)$ one has $D(\overline{H}^{1/2}) = \bigcap_{k=1}^{N} D(X_k)$ and there exist $c, c' > 0$ such that

$$c' \sum_{k=1}^{N} \|X_k u\|_p \leq \|\overline{H}^{1/2} u\|_p \leq c \sum_{k=1}^{N} \|X_k u\|_p$$

for all $u \in D(\overline{H}^{1/2})$.