Invertible solutions of the Lyapunov equation

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1 Introduction

In this talk we study the implications of the existence of a bounded and boundedly invertible solution, X, to the Sylvester equation

$$\langle A_1 z_1, X z_2\rangle + \langle z_1, X A_2 z_2\rangle = 0, \tag{1}$$

where $z_1 \in D(A_1)$, $z_2 \in D(A_2)$. A_1 and A_2 are closed, densely defined linear operators on Z with Z a Hilbert space.

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We show that this does not hold in general. We begin by deriving an equivalent condition to (1) when A_1 and A_2 are both infinitesimal generators.

Lemma

Let A_1, A_2 be the infinitesimal generators of the C_0-semigroups $(T_1(t))_{t\geq 0}$ and $(T_2(t))_{t\geq 0}$, respectively. Then $X \in \mathcal{L}(Z)$ satisfies the Sylvester equation

$$\langle A_1 z_1, X z_2\rangle + \langle z_1, X A_2 z_2\rangle = 0, \quad z_1 \in D(A_1), z_2 \in D(A_2)$$

if and only if

$$T_1^*(t) X T_2(t) = X, \qquad \text{for all } t \geq 0.$$

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if and only if

$$T_1^*(t) X T_2(t) = X, \qquad \text{for all } t \geq 0.$$

If X is (boundedly) invertible, then

$$X^{-1} T_1^*(t) X T_2(t) = I, \qquad \text{for all } t \geq 0.$$

Thus $\left(X^{-1} T_1^*(t) X\right)_{t\geq 0}$ is the left-inverse of $(T_2(t))_{t\geq 0}$. □

Proof: Easy by differentiating (one direction) or (other implication) substituting $z_1 = T_1(t)z_{10}$ and $z_2 = T_2(t)z_{20}$, with $z_{10} \in D(A_1)$, $z_{20} \in D(A_2)$. □

2 Left-Invertibility

We begin with the definition of a left-invertible semigroup.

Definition

The C_0-semigroup $(T(t))_{t\geq 0}$ is left-invertible if there exists a function $t \mapsto m(t)$ such that $m(t) > 0$ and for all $z_0 \in Z$ there holds

$$m(t)\|z_0\| \leq \|T(t)z_0\|, \qquad t \geq 0. \tag{3}$$

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Louis and Wexler, 1983, showed the following equivalence.

Theorem

Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on the Hilbert space Z. Then the following are equivalent:

1. **$(T(t))_{t\geq 0}$ is left-invertible;**
2. **There exists a C_0-semigroup $(S(t))_{t\geq 0}$ such that $S(t)T(t) = I$ for all $t \geq 0$.**

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Hence if left-invertible, then the left-inverse can be chosen as a semigroup. The proof of Louis and Wexler uses optimal control. We present a new/adjusted proof using an invertible solution of a Lyapunov equation.

Proof:

$2. \Rightarrow 1.$ is trivial. So we concentrate on the other implication.

Let A be the infinitesimal generator of $(T(t))_{t\geq 0}$. Choose $\omega \in \mathbb{R}$ such that $A - \omega I$ is exponentially stable.

Now for $z \in Z$

$$m_0\|z\|^2 = \int_0^\infty m(t)^2 e^{-2\omega t}\|z\|^2 dt$$
$$\leq \int_0^\infty e^{-2\omega t}\|T(t)z\|^2 dt \leq M\|z\|^2$$

Define $\langle z, Xz\rangle = \int_0^\infty e^{-2\omega t}\|T(t)z\|^2$. Then

- $m_0 I \leq X \leq MI$
- X **is the solution to the Lyapunov equation**

$$\langle (A - \omega I)z_1, Xz_2 \rangle + \langle z_1, X(A - \omega I)z_2 \rangle = -\langle z_1, z_2 \rangle,$$

for $z_1, z_2 \in D(A)$**.**

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for $z_1, z_2 \in D(A)$.

We rewrite this Lyapunov equation to the Sylvester equation

$$\langle (A - \omega I + X^{-1}) z_1, X z_2 \rangle + \langle z_1, X(A - \omega I) z_2 \rangle = 0.$$

The Sylvester equation

$$\langle (A - \omega I + X^{-1}) z_1, X z_2 \rangle + \langle z_1, X(A - \omega I) z_2 \rangle = 0.$$

is a special case of our general Sylvester equation. Thus by the previous lemma we know that the semigroup generated by $A - \omega I$ is left invertible, and

$$X^{-1} T_1(t)^* X T(t) e^{-\omega t} = I$$

where $(T_1(t))_{t \geq 0}$ is the semigroup generated by $A - \omega I + X^{-1}$. Thus $S(t) := X^{-1} T_1(t)^* X e^{-\omega t}$ is the left-inverse of $T(t)$. □

Looking at the proof, the following is an easy consequence.

Corollary

If $C \in \mathcal{L}(Z, Y)$ is exactly observable, i.e. there exists an $m_0 > 0$ and $t_1 > 0$ such that

$$\int_0^{t_1} \|CT(t)z\|^2 dt \geq m_0 \|z\|^2$$

then $(T(t))_{t \geq 0}$ is left-invertible.

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then $(T(t))_{t \geq 0}$ is left-invertible.

The left-inverse semigroup is "generated" by $A - \omega I + X^{-1}C^*C$. $\square$

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Example

Consider the left-shift semigroup on $L^2(0, 1)$, i.e.

$$(T(t)f)(\eta) = \begin{cases} f(\eta + t) & \eta + t \in [0, 1] \\ 0 & \eta + t \geq 1 \end{cases}$$

with the observation at $\eta = 0$, i.e.,

$$Cf = f(0).$$

Then $(T(t))_{t\geq 0}$ not left-invertible, but it is exactly observable

$$\int_0^1 |CT(t)f|^2 dt = \|f\|^2.$$

□

Remark

One could understand the difficulty as follows:

Is the operator $A + X^{-1}C^*C$ an infinitesimal generator?

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Theorem

Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on the Hilbert space Z with generator A. Then the following are equivalent:

1. **$(T(t))_{t\geq 0}$ is left-invertible;**
2. **There exists a bounded operator Q and an equivalent inner product such that $A+Q$ generates an isometric semigroup in the new norm.**

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□

Markus Haase has proved a similar result for generators of groups.

Proof: $1. \Rightarrow 2.$

By our previous proof we have the existence of a $X \in \mathcal{L}(Z)$, boundedly invertible, such that

$$\langle (A - \omega I)z_1, Xz_2 \rangle + \langle z_1, X(A - \omega I)z_2 \rangle = -\langle z_1, z_2 \rangle,$$

for $z_1, z_2 \in D(A)$. Now we write it as

$$\langle (A - \omega I + \frac{1}{2}X^{-1})z_1, Xz_2 \rangle + \langle z_1, X(A - \omega I + \frac{1}{2}X^{-1})z_2 \rangle = 0.$$

By defining $Q = -\omega I + \frac{1}{2}X^{-1}$, and taking as new inner product $\langle z_1, z_2 \rangle_{\text{new}} = \langle z_1, Xz_2 \rangle$, we obtain the desired result. $\square$

3 When do we have that $A_1^* X = -X A_2$?

We return to our general Sylvester equation, see (1),

$$\langle A_1 z_1, X z_2 \rangle + \langle z_1, X A_2 z_2 \rangle = 0,$$

and wonder when $A_1^* X = -X A_2$.

We have the following result:

Theorem

Assume that $X \in \mathcal{L}(Z)$ is boundedly invertible and satisfies the Sylvester equation

$$\langle A_1 z_1, X z_2 \rangle + \langle z_1, X A_2 z_2 \rangle = 0.$$

If A_1, A_2 satisfy

- **The intersection of $\rho(A_2)$ with the complement of the point spectrum of $-A_1^*$ is non-empty, or**
- $X D(A_2) = D(A_1^*)$,

then

$$A_1^* = -X A_2 X^{-1}$$

Hence if A_1, A_2 generate a C_0-semigroup, then they generate a group and

$$T_1^*(t) = X T_2(-t) X^{-1}.$$

□

4 Riccati equations

It is clear that if X is an invertible, self-adjoint solution to the Lyapunov equation

$$\langle Az_1, Xz_2\rangle + \langle z_1, XAz_2\rangle = -\langle z_1, z_2\rangle,$$

then X^{-1} satisfies the Riccati equation

$$\langle AX^{-1}z_1, z_2\rangle + \langle z_1, AX^{-1}z_2\rangle + \langle X^{-1}z_1, X^{-1}z_2\rangle = 0.$$

However, there are other relations.

The ARE

$$A^*X + XA - XBB^*X + C^*C = 0$$

can be written as (weak form)

$$\begin{aligned} -\langle Cz_1, Cz_2 \rangle &= \langle Az_1, Xz_2 \rangle + \langle z_1, X(A - BB^*X)z_2 \rangle \\ &= \end{aligned}$$

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This last (Lyapunov) equation also holds when B is unbounded.

Theorem

Assume $C \in \mathcal{L}(Z, Y)$ and let X be a self-adjoint, invertible solution of

$$\langle Az_1, Xz_2\rangle + \langle z_1, XA_{\mathrm{opt}}z_2\rangle = -\langle Cz_1, Cz_2\rangle,$$

and assume further that $\rho(-A^* - X^{-1}C^*C) \cap \rho(A_{\mathrm{opt}}) \neq \emptyset$. Then

1. $D(A_{\mathrm{opt}}) = X^{-1}D(A^*)$
2. A_{opt} **and** $A + X^{-1}C^*C$ **generate a** C_0**-group,** $(T_{\mathrm{opt}}(t))_{t\geq 0}$**, and** $(T_{X^{-1}C^*C}(t))_{t\geq 0}$**, respectively, and**

$$T_{\mathrm{opt}}(t) = X^{-1}T_{X^{-1}C^*C}(-t)^*X.$$