Hamiltonians and Riccati equations for unbounded control and observation operators

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joint work with Birgit Jacob, Hans Zwart

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Riccati equation and Hamiltonian

Consider control algebraic Riccati equation

$$A^*X + XA - XBB^*X + C^*C = 0 \qquad (*)$$

on Hilbert space H. Associated Hamiltonian operator matrix:

$$T = \begin{pmatrix} A & -BB^* \\ -C^*C & -A^* \end{pmatrix} \quad \text{on} \quad H \times H.$$

Correspondence

X solution if and only if graph

$$G(X) = \left\{ \begin{pmatrix} x \\ Xx \end{pmatrix} \,\middle|\, x \in \mathcal{D}(X) \right\} \quad \text{is } T\text{-invariant.}$$

$$T \begin{pmatrix} x \\ Xx \end{pmatrix} = \begin{pmatrix} y \\ Xy \end{pmatrix} \Leftrightarrow \begin{cases} Ax - BB^*Xx = y \\ -C^*Cx - A^*Xx = Xy \end{cases} \Leftrightarrow (*)$$

History

Connection between Riccati equation and Hamiltonian:

- Matrix case: extensive theory.
- Kuiper, Zwart 1995:
  $BB^*, C^*C \in L(H)$, T Riesz-spectral.
  $\rightsquigarrow$ Existence and characterisation of bounded solutions.
- Langer, Ran, v.d. Rotten 2001:
  $BB^*, C^*C \in L(H)$, T exponentially dichotomous.
  $\rightsquigarrow$ Existence of nonnegative and nonpositive solution.
- W. 2008, 2010:
  $BB^*, C^*C : \mathcal{D} \subset H \to H$ unbounded, T has Riesz basis of fin.-dim. spectral subspaces.
  $\rightsquigarrow$ Existence of unbounded, characterisation of bounded solutions.

Setting

Consider

- A normal with compact resolvent.

$$\Rightarrow \quad \mathcal{D}(|A|^s) = H_s \subset H \subset H_{-s} \cong (H_s)^*, \\ \text{extensions } A, A^* : H_s \to H_{s-1}.$$

- $B \in L(U, H_{-s}),\ C \in L(H_s, Y)$ for some $s \in [0,1]$.

$$\Rightarrow \quad B^* \in L(H_s, U), \quad C^* \in L(Y, H_{-s}), \\ BB^*, C^*C \in L(H_s, H_{-s}).$$

Hamiltonian $T = \begin{pmatrix} A & -BB^* \\ -C^*C & -A^* \end{pmatrix}$ as operator?

For $v \in H_s \times H_s,\ Tv \in H_{-1} \times H_{-1}$ well defined. Consider

$$T : \mathcal{D}(T) \subset H \times H \to H \times H, \\ \mathcal{D}(T) = \{v \in H_s \times H_s \mid Tv \in H \times H\}.$$

Riesz basis properties for T

Suppose T has compact resolvent.

R1: T has Riesz basis of generalised eigenvectors $(\varphi_n)_{n\in\mathbb{N}}$, i.e., exists isomorphism $\Phi \in L(H \times H)$ s.t. $(\Phi\varphi_n)_n$ ON basis of $H \times H$.

R2: T has Riesz basis of fin.-dim. spectral subspaces $(V_n)_{n\in\mathbb{N}}$, i.e., exists iso $\Phi \in L(H \times H)$ s.t. $H \times H = \bigoplus_n \Phi(V_n)$ orthog., V_n fin.-dim., T-invariant, $\sigma(T|_{V_n})$ disjoint.

Then

- R1 $\Rightarrow$ R2
- R2 $\Rightarrow$ $V_n = \operatorname{span}\{\varphi_{n1}, \ldots, \varphi_{nd_n}\}$, φ_{nj} gen. eigenvectors
- R2 $\Rightarrow$ For $\sigma \subset \sigma(T)$,

$$W_\sigma = \overline{\{\varphi \,|\, \varphi \text{ gen. eigenvec. corresp. to } \sigma\}} \quad \text{is } T\text{-invariant.}$$

Existence of Riesz basis for T

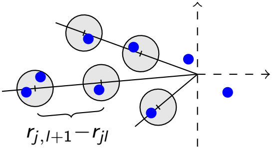
Theorem

Let

- $B \in L(U, H_{-s})$ *with* $s < 1/2$, $C \in L(H, Y)$,
- *almost all eigenvalues* λ_k *of* A *lie on discs* $D_\delta(e^{i\theta_j} r_{jl})$ *along finitely many rays in* $\mathbb{C}_-$,
- $\sum_{k=0}^{\infty} |\lambda_k|^{-2(1-2s)} < \infty, \quad \lim_{l\to\infty} r_{j,l+1} - r_{jl} = \infty.$

Then T *has compact resolvent and a Riesz basis of fin.-dim. spectral subspaces.*
If almost all discs contain only one simple λ_k*, then* T *has a Riesz basis of generalised eigenvectors.*

$\sigma(A)$



Krein space symmetry

General setting now:

- $B \in L(U, H_{-s})$, $C \in L(H_s, Y)$ with $0 \leq s \leq 1$,
- T has compact resolvent and Riesz basis of fin.-dim. spectral subspaces.

Indefinite inner product on $H \times H$:

$$\langle v|w\rangle = (Jv|w), \quad J = \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix}, \quad (\cdot|\cdot) \text{ usual inner product}$$

$\rightsquigarrow (H \times H, \langle\cdot|\cdot\rangle)$ Krein space.

Hamiltonian J-skew-selfadjoint, $T = -T^{\langle * \rangle}$.

$\Rightarrow \sigma(T)$ symmetric w.r.t. $i\mathbb{R}$.

Existence of solutions X

Theorem

Let

- (A, B) *approximately controllable,*
- *no non-observable eigenvalues of* A *on* $i\mathbb{R}$.

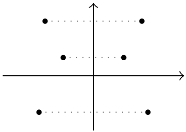
Then $\sigma(T) \cap i\mathbb{R} = \varnothing$, *and for* $\sigma \subset \sigma(T)$ *skew-conjugate we have* $W_\sigma = G(X)$ *with* X *selfadjoint solution of*

$$A^*X + X(A - BB^*X) + C^*C = 0$$

on dense subspace $\mathcal{D}_X \subset H$.

$X_\pm$ *corresp. to* $\sigma = \sigma(T) \cap \mathbb{C}_\mp$ *is nonnegative/nonpositive.*

$\sigma \subset \sigma(T)$ skew-conjugate if $\sigma(T) = \sigma \uplus -\sigma^*$.



Existence of solutions X

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Let

- (A, B) *approximately controllable,*
- *no non-observable eigenvalues of* A *on* $i\mathbb{R}$*.*

Then $\sigma(T) \cap i\mathbb{R} = \varnothing$*, and for* $\sigma \subset \sigma(T)$ *skew-conjugate we have* $W_\sigma = G(X)$ *with* X *selfadjoint solution of*

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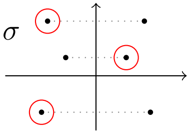

Idea of the proof

Existence of X:

- $T = -T^{\langle * \rangle}$, σ skew-conj. $\Rightarrow W_\sigma = W_\sigma^{\langle \perp \rangle}$
- (A, B) approx. contr. $\Leftrightarrow \ker(A-\lambda) \cap \ker B^* = \{0\}\ \forall \lambda \in \mathbb{C}$
- $\Rightarrow W_\sigma = G(X)$
- $G(X) = G(X)^{\langle \perp \rangle} \Rightarrow X$ selfadjoint

X_+ nonnegative:

- Consider $[v|w] = (J_2 v|w)$, $J_2 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$
- $\operatorname{Re}[Tv|v] \leq 0$
- $\Rightarrow G(X_+)$ is J_2-nonnegative, i.e. $[v|v] \geq 0$ for $v \in G(X_+)$
- $\Rightarrow X_+$ nonnegative

Bounded solutions

Theorem

Let

- (A, B) *approximately controllable,*
- *no non-observable eigenvalues of* A *on* $i\mathbb{R}$*,*
- T *has Riesz basis of gen. eigenvectors, whose part corresp. to* $\mathbb{C}_-$ *is quadratically close to an ON system of* $H \times \{0\}$*.*

If $\sigma \subset \sigma(T)$ *skew-conj. and* $\sigma \cap \mathbb{C}_+$ *finite, then* $W_\sigma = G(X)$ *with* X *bounded selfadjoint solution of*

$$A^*X + XA - XBB^*X + C^*C = 0 \quad \text{on} \quad \mathcal{D}_X.$$

For $A_X = A - BB^*X$, $\mathcal{D}(A_X) = \mathcal{D}_X$, *we get* $\sigma(A_X) = \sigma$*.*

Existence of Riesz basis 2

Theorem

Let

- $B \in L(U, H_{-s})$ *with* $s < 1/2$, $C \in L(H, Y)$,
- *almost all eigenvalues* λ_k *of* A *are simple, lie on discs* $D_\delta(e^{i\theta_j} r_{jl})$ *along finitely many rays in* $\mathbb{C}_-$, *each disc contains only one* λ_k,
- $\sum_{l=0}^{\infty} r_{jl}^{-2q} < \infty$, $r_{j,l+1}^{1-q} - r_{jl}^{1-q} \geq \beta > 0$ *with* $0 < q \leq 1 - 2s$.

Then T *admits a Riesz basis of gen. eigenvectors, whose part corresp. to* $\mathbb{C}_-$ *is quadratically close to an ON system of* $H \times \{0\}$.

Example: heat equation with boundary control

Consider

$$
\begin{aligned}
&H = L^2([0,1]), \\
&Ax = x'', \quad \mathcal{D}(A) = \{x \in H^2([0,1]) \,|\, x'(0) = x(1) = 0\}, \\
&B^* x = x(0), \\
&\text{any } C \in L(H, Y).
\end{aligned}
$$

Then

- $B \in L(\mathbb{C}, H_{-s})$ for all $s > 1/4$,
- Previous theorem applies with $1/4 < s < 3/8$, $q = 1 - 2s$,
- (A, B) approx. contr., A has no imag. eigenvalues,
- Existence of (bounded) solutions.

Open questions

- Existence of bounded solutions under weaker assumption of Riesz basis of fin.-dim. spectral subspaces?
- Characterisation of solutions? E.g. if X solution, then $G(X) = \overline{\operatorname{span}\{\text{certain gen. eigenvectors}\}}$?
- Non-selfadjoint solutions?
- A not normal?

References

- C. Wyss, B. Jacob, H. Zwart. *Hamiltonians and Riccati equations for linear systems with unbounded control and observation operators*. Submitted. Preprint 2011.
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