Weighted admissibility of linear systems on Bergman spaces

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Discrete linear system

Discrete Time System

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Definition:

For $\alpha \in (-1,1)$, $C \in X^*$ is α-**admissible** for T if

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α-admissibility $\sim$ measurement depends continuously on initial value

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Discrete Weighted Weiss Conjecture

- Hence: **If C is α-admissible for T** then

$$\textbf{(RC)}_\alpha : \qquad \|C(I-\omega T)^{-1}\|_{X^*} \leq \frac{k}{(1-|\omega|^2)^{\frac{1-\alpha}{2}}}, \qquad \omega \in \mathbb{D}.$$

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- Exists a (more famous!) conjecture in **continuous time**.
- Related results to those presented in this talk (work by Haak, Le Merdy, Partington, Jacob, Weiss and more!)

When is the Weiss conjecture true/false?

TRUE in the following:

(i) [Harper '06] If $\alpha = 0$ and T a **contraction** ($\|T\|_{\mathcal{L}(X)} \leq 1$).

(ii) [W '08] If $\alpha \in (0,1)$ and T is a **normal contraction**.

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FALSE in the cases [W '09]:

(i) If $\alpha \in (-1,0)$ conjecture fails for a **normal contraction** T.

(ii) If $\alpha \in (0,1)$ fails for the **unilateral shift** on $H^2(\mathbb{D})$.

Dirichlet Spaces; Carleson measures

Definition:

For $\beta > -1$, **weighted Dirichlet space** $\mathcal{D}_\beta(\mathbb{D})$ contains analytic $f : \mathbb{D} \to \mathbb{C}$ such that

$$\|f\|^2_{\mathcal{D}_\beta(\mathbb{D})} := |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 (1 - |z|^2)^\beta \, dA(z) < \infty.$$

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Definition:

A measure μ satisfying $\mathcal{D}_\beta(\mathbb{D}) \hookrightarrow L^2(\mathbb{D}, \mu)$ is called a $\mathcal{D}_\beta(\mathbb{D})$-**Carleson measure**.

Normal operators; measure connection

Lemma

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(ii) *Resolvent condition $\textbf{(RC)}_\alpha$ holds iff*

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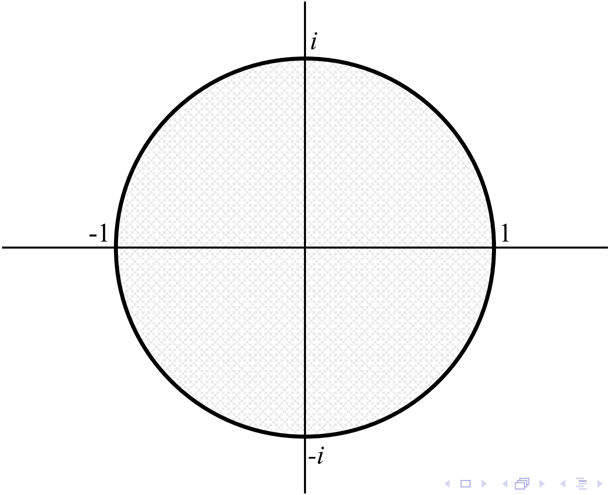
The Boxes $S(I)$

- $S(I) := \{re^{i\theta} : 1 - \frac{|I|}{2\pi} \leq r < 1, e^{i\theta} \in I\} \subset \mathbb{D}$



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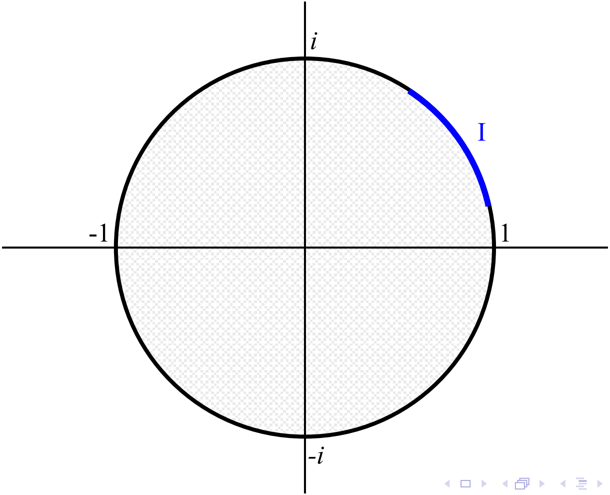

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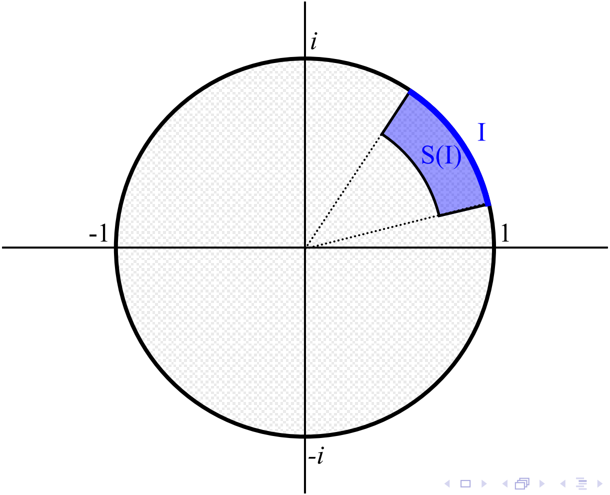

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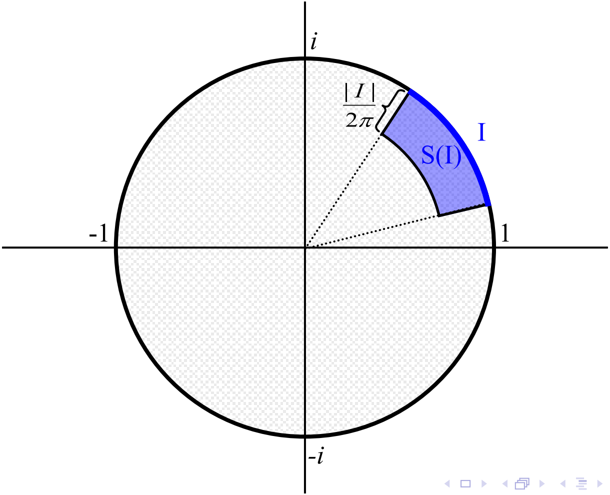

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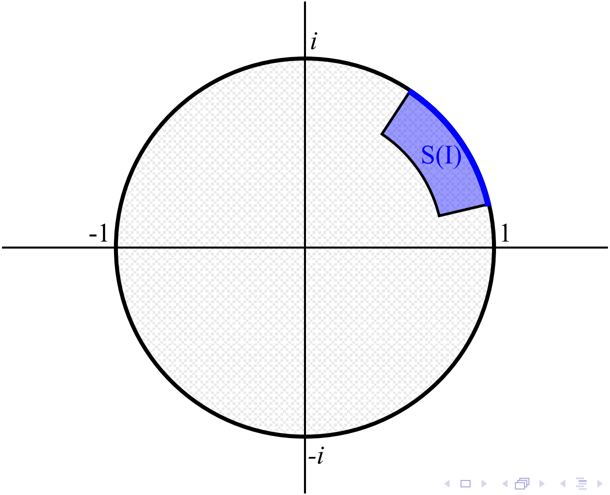

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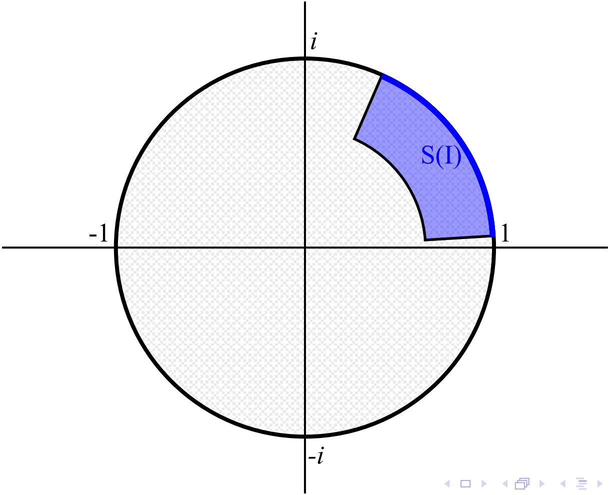

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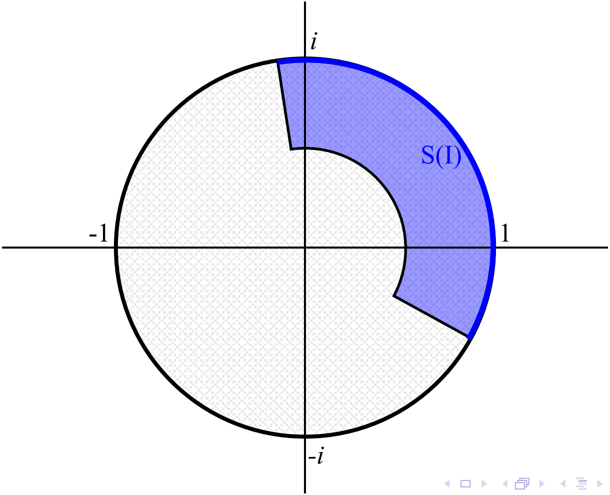

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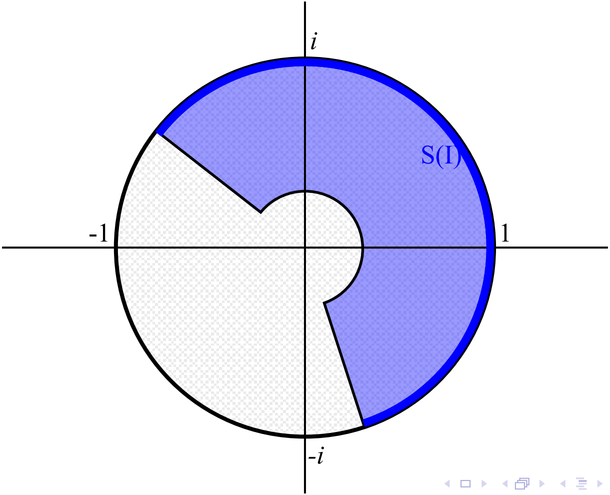

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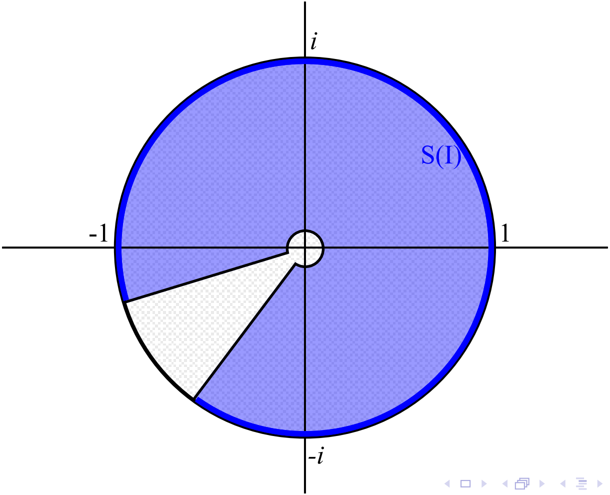

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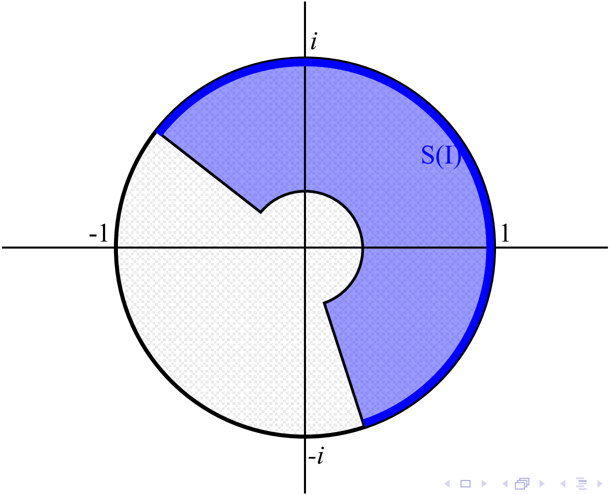

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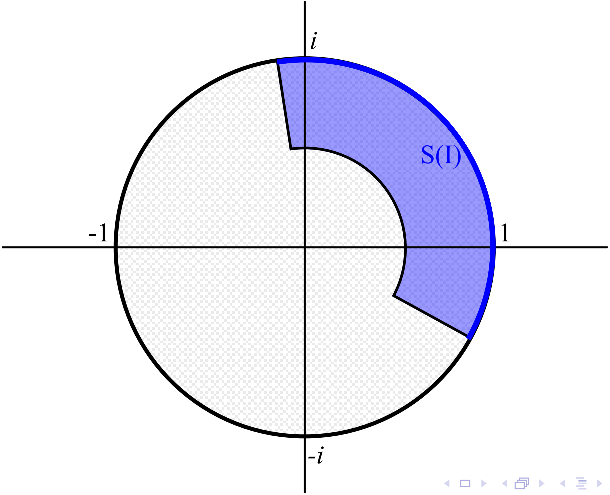

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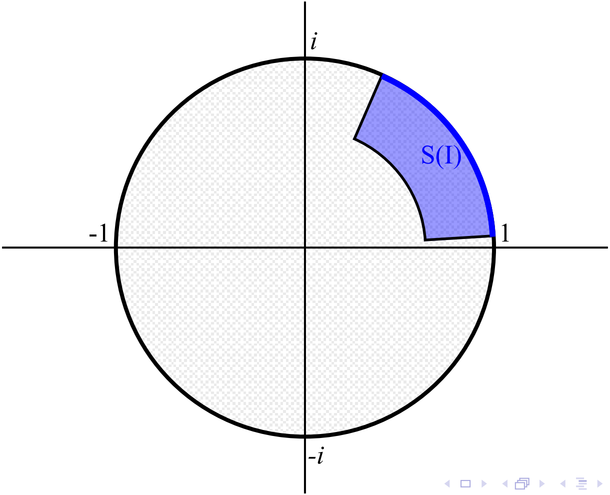

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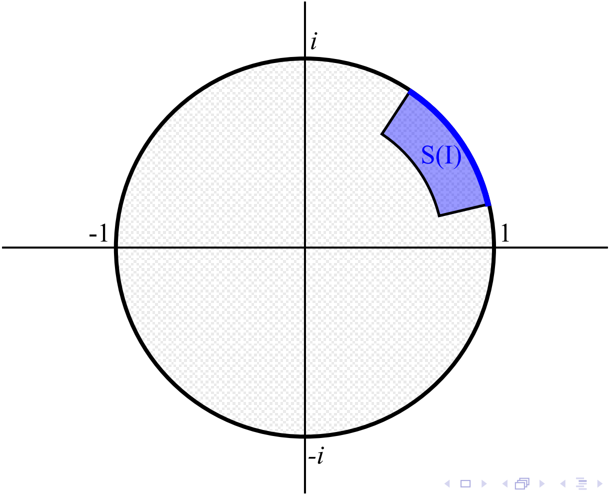

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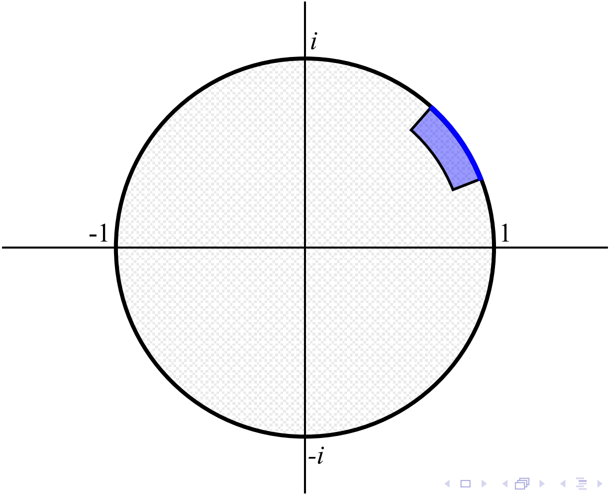

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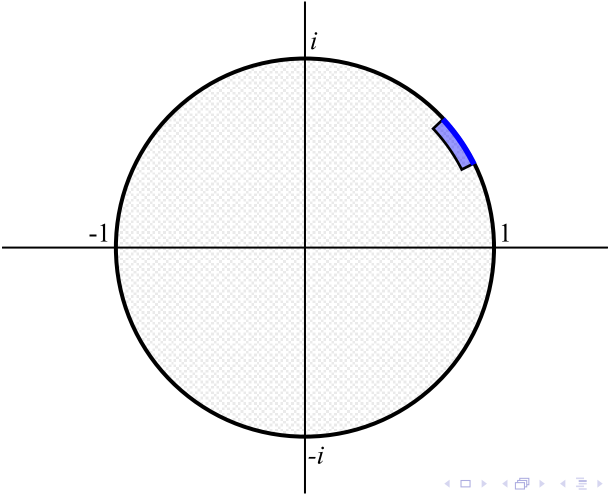

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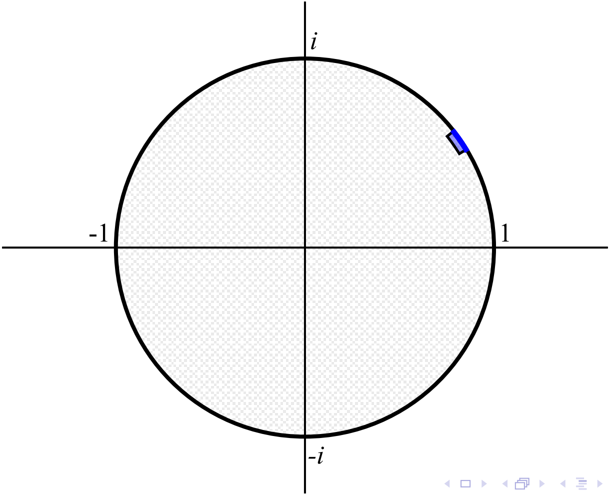

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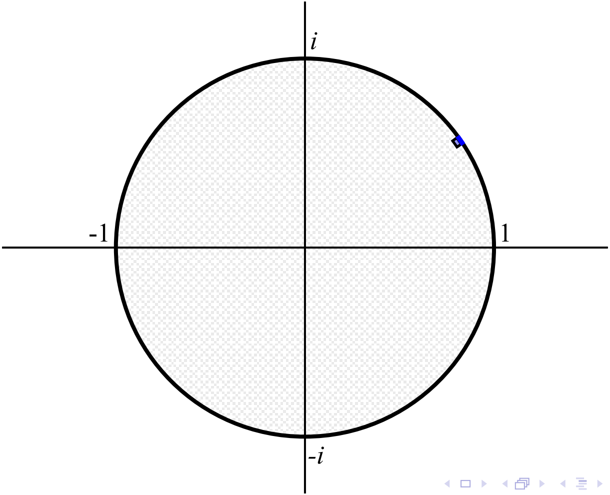

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Carleson measures vs Geometric Characterisation

Theorem (Carleson, Luecking)

Let $\alpha \in [0,1)$. A positive measure μ on $\mathbb{D}$ is $\mathcal{D}_{1+\alpha}(\mathbb{D})$-Carleson iff

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False for Shift on $H^2(\mathbb{D})$

- The **Unilateral Shift** $S : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ is given by

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- If Weighted Weiss conjecture true for S, very likely true for **all contraction operators**.

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Let $\alpha \in (0,1)$. Suppose that $C \in H^2(\mathbb{D})^*$ is given by $Cf := \langle f, c\rangle_{H^2}$, for some $c \in H^2(\mathbb{D})$. Then

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$$|c'(z)|^2(1 - |z|^2)dA(z) \tag{1}$$

is a $\mathcal{A}_{\alpha-1}(\mathbb{D})$*-Carleson measure;*

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is a $H^2(\mathbb{D})$-Carleson measure.
3. **Exists $c \in H^2(\mathbb{D})$ satisfying (1) but not (2).**

When is weighted conjecture true for contractions?

- Weighted conjecture **false** for the shift S on $X = H^2(\mathbb{D})$ for any $\alpha \in (0,1)$.

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Question:

Is weighted Weiss Conjecture true for S on any of the spaces $\mathcal{A}_\beta(\mathbb{D})$?

Shift on weighted Bergman space

- For $\beta > -1$, the **weighted Bergman space** $\mathcal{A}_\beta(\mathbb{D})$ contains analytic functions $f = \sum f_n z^n$ with

$$\begin{aligned} \|f\|^2_{\mathcal{A}_\beta(\mathbb{D})} &:= \int_{\mathbb{D}} |f(z)|^2 (1-|z|^2)^\beta \, dA(z) \\ &\sim \sum_{n=0}^{\infty} (1+n)^{-(1+\beta)} |f_n|^2 < \infty. \end{aligned}$$

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- Consider α-admissibility of the shift $(Sf)(z) := zf(z)$ on $\mathcal{A}_\beta(\mathbb{D})$.
- Link the problem to **Little Hankel Operators**.

The Little Hankel Operator

- Anti-analytic Bergman space

$$\overline{\mathcal{A}_\beta(\mathbb{D})} := \left\{ \overline{f} : f \in \mathcal{A}_\beta(\mathbb{D}) \right\}.$$

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The **Little Hankel Operator** h_f on $\mathcal{A}_\beta(\mathbb{D})$ with **symbol** f is defined by

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- α-admissibility on $\mathcal{A}_\beta(\mathbb{D})$ is closely linked to boundedness of little Hankel operators.

Admissibility and Little Hankel operators

- Suppose that $C \in \mathcal{A}_\beta(\mathbb{D})^*$ given by $Cf := \langle f, c \rangle_\beta$. Then,

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- Where $\overline{z}^n$ is a basis element in $\overline{\mathcal{A}_\beta(\mathbb{D})}$.

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- As a consequence,

$$\sup_{f \in \mathcal{A}_\beta(\mathbb{D})} \sum_{n=0}^{\infty} (1+n)^\alpha |CS^n f|^2 \sim \sup_{f \in \mathcal{A}_{\alpha-1}(\mathbb{D})} \| h_{\overline{c}} f \|^2_{\overline{\mathcal{A}_\beta(\mathbb{D})}}.$$

Admissibility and Little Hankel operators

- Hence, $C \in \mathcal{A}_\beta(\mathbb{D})^*$ is α-admissible for S if and only if

$$h_{\bar{c}} : \mathcal{A}_{\alpha-1}(\mathbb{D}) \to \overline{\mathcal{A}_\beta(\mathbb{D})}$$

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- Want to link the resolvent condition $(\mathbf{RC})_\alpha$ to Little Hankel operators.

- To do this, use Reproducing Kernels.

Reproducing Kernels

- Reproducing kernels for $\mathcal{A}_\beta(\mathbb{D})$ are

$$k_\omega^\beta(z) := \frac{(1-|\omega|^2)^{1+\frac{\beta}{2}}}{(1-\bar{\omega}z)^{2+\beta}}, \qquad \omega, z \in \mathbb{D}.$$

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- Can show

$$\sup_{\omega\in\mathbb{D}} \left\| h_{\bar{c}} k_\omega^{\alpha-1} \right\|_{\overline{\mathcal{A}_\beta(\mathbb{D})}} \sim \sup_{\omega\in\mathbb{D}} (1-|\omega|^2)^{\frac{1-\alpha}{2}} \|C(I-\bar{\omega}S)^{-1}\|_{\mathcal{A}_\beta(\mathbb{D})^*},$$

providing a link with the resolvent condition.

Reproducing Kernel thesis

- Hence, Conjecture is true for S on $\mathcal{A}_\beta(\mathbb{D})$ iff boundedness of

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- Link between Weiss Conjecture and RKT highlighted by Harper in '06.

Little Hankel Operators satisfy RKT

Theorem (Case $\alpha = \beta$, Janson et. al. '87)

Suppose that $-1 < \alpha \leq \beta$. Then h_f satisfies RKT for $f \in L^2(\mathbb{D}, dA_\beta)$.

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Corollary (Jacob, W.)

Let S be the shift on $\mathcal{A}_\beta(\mathbb{D})$ for $\beta > -1$. Suppose that $0 \leq \alpha \leq 1 + \beta$. Then $C \in \mathcal{A}_\beta(\mathbb{D})^*$ is α-admissible for S iff

$$\sup_{\omega \in \mathbb{D}} (1 - |\omega|^2)^{\frac{1-\alpha}{2}} \|C(I - \bar{\omega} S)^{-1}\|_{\mathcal{A}_\beta(\mathbb{D})^*} < \infty.$$

Continuous time systems

- Shift on $\mathcal{A}_\beta(\mathbb{D})$ is equivalent to the **right-shift semigroup**

$$(S(t)f)(\tau) := \begin{cases} f(\tau - t), & t \geq \tau; \\ 0, & t < \tau; \end{cases} \qquad t \geq 0,$$

on weighted L^2-space

$$L^2_\alpha(\mathbb{R}_+) := \{f : \|f\|^2_{L^2_\alpha(\mathbb{R}_+)} := \int_0^\infty t^{-\alpha}|f(t)|^2 dt < \infty\}.$$

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- For $\alpha > 0$, possible to translate admissibility results between continuous and discrete time.

Continuous Time Admissibility

Definition (Haak, Le Merdy '05)

For $\alpha > -1$, C is said to be α-admissible w.r.t $(S(t))_{t\geq 0} \subset \mathcal{L}(X)$ if

$$\int_0^\infty t^\alpha \|CS(t)x\|_X^2 dt \leq M^2 \|x\|_X^2, \qquad x \in D(A).$$

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- Equivalent weighted result for continuous time systems.

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- Equivalent weighted result for continuous time systems.

Theorem (Jacob, W.)

Suppose that $0 \leq \alpha \leq 1+\beta$. Let $(S(t))_{t\geq 0}$ be the right-shift semigroup on $L^2_\beta(\mathbb{R}_+)$ with generator A. Then $C \in \mathcal{L}(D(A), \mathbb{C})$ is α-admissible iff

$$\sup_{\lambda \in \mathbb{R}_+} (\mathrm{Re}\lambda)^{\frac{1-\alpha}{2}} \|CR(\lambda, A)\|_{L^2_\beta(\mathbb{R}_+)^*} < \infty.$$

Link to Integral Operators

- For $\alpha, \beta > 0$, define an Integral Operator with **symbol** h:

$$(\Gamma_h^{\alpha,\beta} u)(t) := \int_0^\infty \frac{s^{\frac{\alpha}{2}} t^{\frac{\beta}{2}} h(s+t)}{(s+t)^\beta} u(s) ds, \qquad t > 0.$$

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Corollary (Integral Operator RKT)

Suppose that $0 \leq \alpha \leq 1 + \beta$ *and* $h \in L^2_\beta(\mathbb{R}_+)$. *Then*

$$\Gamma_h^{\alpha,\beta} : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$$

is bounded iff

$$\sup_{\lambda \in \mathbb{C}_+} \|\Gamma_h^{\alpha,\beta} e_\lambda\|_{L^2_\beta(\mathbb{R}_+)} < \infty, \qquad \left(e_\lambda(t) := (\mathrm{Re}\lambda)^{-\frac{1}{2}} e^{-\lambda t} \right).$$

A Question

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Question:

For each $\alpha > 0$, is there an identifiable class of operators $\mathcal{O}_\alpha$,

$$\{\text{normal ops.}\} \subset \mathcal{O}_\alpha \subset \{\text{contraction ops.}\}$$

for which the weighted Weiss conjecture is true?

The End

Thank you!