Positive Stabilization of Infinite-Dimensional Linear Systems

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Outline of the Talk

- Preliminary Concepts and Results: **Positive Semigroups**
- Algebraic Conditions of Positivity: **Positive Off-Diagonal Property**
- Positive Stabilization: **Spectral Decomposition**
- Example: **Heat Diffusion**
- Concluding Remarks and **Perspectives**

A real vector space X is called an **ordered vector space** if a partial order "$\leq$" is defined in X such that

$$x \leq y \text{ in } X \Rightarrow x + z \leq y + z \text{ for all } z \in X, \text{ and}$$
$$\lambda x \leq \lambda y \text{ for all } 0 \leq \lambda \in \mathbb{R}.$$

Given such a partial order, the positive cone of X is defined by

$$X^+ = \{x \in X \mid x \geq 0\}$$

[X^+ is a cone: $\alpha x + \beta y \in X^+$ whenever $x, y \in X^+$ and $0 \leq \alpha, \beta \in \mathbb{R}$. $X^+ \cap (-X^+) = \{0\}$, so X^+ is proper]

Conversely, **given a proper cone K in X, a partial order in X is defined** by setting $x \leq y$ whenever $y - x \in K$, and then

$(X, \leq)$ is **an ordered vector space with positive cone $X^+ = K$**.

A real Banach space $(X, \|\cdot\|)$ is called an **ordered Banach space** if

X is an ordered vector space such that
X^+ is norm closed, i.e. closed in the strong topology.

From now on we assume that

X is an ordered Banach space with **positive cone** X^+.

A family $(T(t))_{t\geq 0}$ in $\mathcal{L}(X)$ is called a **C_0-semigroup** if

$$T(0) = I \quad , \quad T(t+s) = T(t)T(s), \quad \forall t,s \geq 0$$

$$\lim_{t \to 0^+} \| T(t)x - x \| = 0, \quad \forall x \in X$$

The **infinitesimal generator** A of a C_0-semigroup $(T(t))_{t\geq 0}$ is defined by

$$Ax = \lim_{t \longrightarrow 0^+} \frac{T(t)x - x}{t}$$

on

$$D(A) = \{x \in X \mid \lim_{t \longrightarrow 0^+} \frac{T(t)x - x}{t} \text{ exists in } X\}$$

Definition

$(T(t))_{t\geq 0}$ is said to be **positive** if all the operators $T(t)$, $t \geq 0$, are positive, i.e.

$$T(t)X^+ \subset X^+ \text{ for all } t \geq 0$$

Proposition

A C_0-semigroup $(T(t))_{t\geq 0}$ is positive if and only if

$$\textit{its resolvent } R(\lambda, A) := (\lambda I - A)^{-1} \textit{ is positive for all } \lambda > \omega_0,$$

where

$$\omega_0 := \inf_{t>0} \frac{\log \parallel T(t) \parallel}{t} = \lim_{t \longrightarrow \infty} \frac{\log \parallel T(t) \parallel}{t}$$

is the **growth constant** *of $(T(t))_{t\geq 0}$.*

Characterization of the positivity of a C_0-semigroup in terms of its generator A :

Definition

A linear operator $A : D(A) \longrightarrow X$ is said to have the **Positive Off-Diagonal (POD) property** if

$$\langle Au, \phi \rangle \geq 0$$

whenever

$$0 \leq u \in D(A) \ \text{ and } \ \phi \in (X^*)^+ \text{ with } \langle u, \phi \rangle = 0$$

where

$$(X^*)^+ = \{ \phi \in X^* \ | \ \langle x, \phi \rangle \geq 0, \ \forall \ x \in X^+ \ \}$$

Theorem

Let A be the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$ in an ordered Banach space X with $int(X^+) \neq \varnothing$. The following assertions are equivalent:

- *(i) $(T(t))_{t\geq 0}$ is a positive C_0-semigroup.*
- *(ii) A satisfies the POD property.*

Moreover, if one of the two assertions above hold, then

$$s(A) = \inf\{\lambda \in \mathbb{R} \mid Au \leq \lambda u \text{ for some } u \in D(A) \cap int(X^+)\}$$

where

$$s(A) = \sup\{Re(\lambda) \mid \lambda \in \sigma(A)\}$$

denotes the **spectral bound of** A.

Algebraic Conditions of Positivity: POD Property

Algebraic conditions of positivity for systems defined on a space whose positive cone has an empty interior ?

Fact

a) The positive cone l^2_+ of l^2 has an empty interior.
b) The positive cone of any infinite-dimensional separable Hilbert space (e.g. L^2) has an empty interior.

Indeed:

$$\text{every } x = (x_n) \in l^2_+ \longrightarrow 0$$

$\Longrightarrow$
for any ball $B = B(x, \epsilon)$, there exists a sequence $y = (y_n)$ which belongs to B but not to l^2_+.

In this case, the **POD property of the generator is still necessary but not sufficient** for the positivity of the semigroup.

Algebraic Conditions of Positivity: POD Property

Let Z be an ordered Banach space such that $int(Z^+) = \varnothing$.
Let $\{e_n\}_{n\geq 1}$ be a **positive Schauder basis** of Z, i.e. each element z of Z has a unique representation of the form

$$z = \sum_{n=1}^{\infty} \alpha_n e_n$$

such that the linear functional

$$z \longmapsto \alpha_n =: \langle z, e_n \rangle \text{ is bounded}$$

where $\alpha_n :=$ the *nth* coordinate of z with respect to the basis $\{e_n\}_{n\geq 1}$ and the **positive cone** is given by

$$Z^+ = \left\{ z = \sum_{n=1}^{\infty} \alpha_n e_n \;\middle|\; \alpha_n \geq 0, \;\forall n \right\}.$$

Algebraic Conditions of Positivity: POD Property

Consider a closed linear operator $A : D(A) \subset Z \longrightarrow Z$.
Assume that: $\{e_n\}_{n\geq 1} \subset D(A)$,
A is the infinitesimal generator of a C_0-semigroup $(T_A(t))_{t\geq 0}$.

Definition

1) The operator A is said to be **Metzler** if $a_{nk} = \langle Ae_k, e_n\rangle \geq 0, \ \forall n \neq k$.
2) The system $\dot{z}(t) = Az(t)$ is said to be **positive** if
Z^+ is $T_A(t)$-invariant, i.e.

$$T_A(t)Z^+ \subset Z^+, \ \forall t \geq 0$$

Proposition

If the system $\dot{z}(t) = Az(t)$ is positive on Z, then

A satisfies the POD property.

Algebraic Conditions of Positivity: POD Property

The POD property of the generator of a C_0-semigroup guarantees the positivity of the latter on invariant finite-dimensional subspaces.

Theorem

Assume that

$$Z_N := span\{e_1, e_2, ..., e_N\} \text{ (where } N < \infty)$$

is $T_A(t)$-invariant for all $t \geq 0$.

If A is Metzler and $a_{nk} > 0$ for all $n \neq k$ such that $1 \leq n, k \leq N$, then the system $\dot{z}(t) = Az(t)$ is positive on Z_N, i.e.

$$T_A(t)Z_N^+ \subset Z_N^+, \forall t \geq 0$$

where

$$Z_N^+ = Z_N \cap Z^+ := \text{ the positive cone of } Z_N$$

Algebraic Conditions of Positivity: POD Property

Theorem

Assume that

$$A \text{ is Metzler}$$

and

$$Z_N \text{ is } T_A(t)\text{-invariant for all } t \geq 0.$$

Then

$$\text{the system } \dot{z}(t) = Az(t) \text{ is positive on } Z_N.$$

Hint: Consider $A_\epsilon := A + B_\epsilon$ where $B_\epsilon z := \sum_{k=1}^{\infty} \langle z, e_k \rangle B_\epsilon e_k$ and
$\langle B_\epsilon e_k, e_n \rangle = \epsilon > 0$ for all $n, k \leq N$ and
$\langle B_\epsilon e_k, e_n \rangle = 0$ for all n, k such that n or $k > N$.

Algebraic Conditions of Positivity: POD Property

Corollary

Assume that

$$A \text{ has the POD property}$$

and

$$Z_N \text{ is } T_A(t)\text{-invariant for all } t \geq 0.$$

Then

$$\text{the system } \dot{z}(t) = Az(t) \text{ is positive on } Z_N.$$

Indeed: for all n, $z \longmapsto \langle z, e_n \rangle$ is a positive bounded linear functional such that, for all $k \neq n$, $\langle e_k, e_n \rangle = 0$ (where $0 \leq e_k \in D(A)$).

It follows by the POD property that A is a Metzler operator.

Consider the **infinite dimensional linear system (Σ)** described by the following abstract differential equation

$$\begin{cases} \dot{z}(t) &= Az(t) + Bu(t), \\ z(0) &= z_0 \in D(A), \end{cases}$$

where

A is the infinitesimal generator of a C_0-semigroup $(T_A(t))_{t\geq 0}$ on an ordered Banach space Z with positive cone Z^+,
B is a bounded linear operator from $\mathcal{U}$ to Z,
$\mathcal{U} = \{u : \mathbb{R}^+ \longrightarrow U, \text{ continuous}\}$ and
U is a control ordered Banach space with a positive cone U^+.

Definition

The system (Σ), i.e. the pair (A, B), is said to be **positive** if for every $z_0 \in Z^+$ and all inputs $u \in \mathcal{U}^+$, *i.e.* $\forall u \in \mathcal{U}$ such that $u(t) \in U^+$, $\forall t \geq 0$, the state trajectories $z(t)$ remain in Z^+ for all $t \geq 0$.

Definition

The system (Σ), i.e. the pair (A, B), is **positively stabilizable** if there exists a state feedback control law $K \in \mathcal{L}(Z, \mathcal{U})$ such that the C_0-semigroup generated by $A - BK$ is an exponentially stable positive semigroup.

Conditions of existence of a state feedback such that the corresponding closed loop system is exponentially stable and positive ?

Positive Stabilization: Spectral Decomposition

Theorem

The system (Σ) is positive $\Longleftrightarrow$ A is the infinitesimal generator of a positive C_0-semigroup and B is a positive operator.

Consider $U = \mathbb{R}^m$ and B the bounded linear operator given by

$$Bu = \sum_{i=1}^{m} b_i u_i,$$

where $u = \begin{bmatrix} u_1 & \cdots & u_m \end{bmatrix}^t$ and $b_i \in Z_N \;\; \text{for} \;\; i = 1, ..., m$.

Corollary

Assume that A is Metzler, Z_N is $T_A(t)$-invariant for all $t \geq 0$ and B is a positive operator.
Then for every $z_0 \in Z_N^+$ and for every u such that $Im(u) \subset \mathbb{R}_+^m$, the corresponding state trajectory $z(\cdot)$ of the controlled system (Σ) remains in Z_N^+.

Positive Stabilization: Spectral Decomposition

Assume that the state space Z is an ordered Hilbert space and that:
(H1) $\exists\, \delta > 0$ such that the set $\sigma(A) \cap \{s \in \mathbb{C} \mid Re(s) > -\delta\}$ contains only a finite number of elements of the spectrum $\sigma(A)$, and
(H2) A satisfies the spectrum decomposition assumption at δ.
Then the spectrum of A can be decomposed as follows:

$$\sigma_\delta^+(A) = \sigma(A) \cap \{\lambda \in \mathbb{C} \mid Re(\lambda) > -\delta\},$$
$$\sigma_\delta^-(A) = \sigma(A) \cap \{\lambda \in \mathbb{C} \mid Re(\lambda) \leq -\delta\}.$$

The **spectral projection**

$$P_\delta z = \frac{1}{2\pi j} \int_{\Gamma_\delta} (\lambda I - A)^{-1} z d\lambda$$

induces a decomposition of the state space

$$Z = Z_u \oplus Z_s, \quad Z_u := P_\delta Z, \quad Z_s := (I - P_\delta) Z,$$

where $Z_u := P_\delta Z$ is finite-dimensional.

Positive Stabilization: Spectral Decomposition

Using the subscript notations "u" for **unstable** and "s" for **stable**, one can write the operators A and B as:

$$A = \begin{bmatrix} A_u & 0 \\ 0 & A_s \end{bmatrix} \text{ where } A_u := A_{|Z_u}, \quad A_s := A_{|Z_s},$$

with $\sigma(A_u) := \sigma_\delta^+(A), \;\; \sigma(A_s) := \sigma_\delta^-(A)$,

$$B = \begin{bmatrix} B_u \\ B_s \end{bmatrix} \text{ where } B_u := P_\delta B \;\text{ and }\; B_s := (I - P_\delta)B.$$

Positive Stabilization: Spectral Decomposition

The **spectrum decomposition assumption** is valid for a wide class of infinite-dimensional systems:
e.g. systems whose generator is a Riesz-spectral operator, parabolic systems and systems described by delay differential equations.

A_u may have some stable eigenvalues.

A_s is the infinitesimal generator of an exponentially stable C_0-semigroup.

Proposition

$$(A, B) \text{ is exponentially stabilizable}$$

$\Longleftrightarrow$

$$(A_u, B_u) \text{ is exponentially stabilizable}$$

Positive Stabilization: Spectral Decomposition

Let

$$Z_u^+ = Z_u \cap Z^+ \text{ and } Z_s^+ = Z_s \cap Z^+$$

Z_u^+ and Z_s^+ are proper cones and therefore define an order on Z_u and Z_s. Clearly:

$$Z_u^+ \oplus Z_s^+ \subset Z^+$$

Lemma

If A is the infinitesimal generator of a positive C_0-semigroup, then A_u and A_s are infinitesimal generators of positive C_0-semigroups.

If, in addition,

$$Z_u^+ \oplus Z_s^+ = Z^+$$

the converse holds, i.e.

$$T_A(t)Z^+ \subset Z^+, \forall t \geq 0 \iff \begin{cases} T_{A_u}(t)Z_u^+ \subset Z_u^+, \forall t \geq 0 \\ T_{A_s}(t)Z_s^+ \subset Z_s^+, \forall t \geq 0. \end{cases}$$

Positive Stabilization: Spectral Decomposition

Theorem

Assume that

$$A \text{ is the infinitesimal generator of a positive } C_0\text{-semigroup}$$

and (A_u, B_u) *is positively stabilizable such that there exists a state feedback* $K_u \in \mathcal{L}(Z_u, \mathcal{U})$ *such that the operator*

$$-B_s K_u \in \mathcal{L}(Z_u, Z_s) \text{ is positive.}$$

Then

$$(A, B) \text{ is positively stabilizable,}$$

i.e. there exists a state feedback $K \in \mathcal{L}(Z, \mathcal{U})$ *such that* $A - BK$ *is the infinitesimal generator of an exponentially stable and positive* C_0*-semigroup with respect to the cone* $Z_u^+ \oplus Z_s^+$.

Example: Heat Diffusion

Heat diffusion model with Neumann boundary conditions:

$$\begin{cases} \dfrac{\partial z}{\partial t}(x,t) &= \dfrac{\partial^2 z}{\partial x^2}(x,t) + b_1 u(t) \\ \dfrac{\partial z}{\partial x}(0,t) &= 0 = \dfrac{\partial z}{\partial x}(1,t). \end{cases}$$

Described on $Z = L^2(0,1)$ by:

$$\dot{z}(t) = Az(t) + Bu(t) \ , \ z(0) = z_0 \in D(A),$$

where $Az = \dfrac{d^2 z}{dx^2}$ is defined on its domain

$$D(A) = \{z \in L_2(0,1) \mid z, \frac{dz}{dx} \text{are absolutely continuous,}$$

$$\frac{d^2 z}{dx^2} \in L_2(0,1) \text{ and } \frac{dz}{dx}(0) = \frac{dz}{dx}(1) = 0\},$$

and $B \in \mathcal{L}(\mathbb{R}, L_2(0,1))$ is given by

$$Bu = b_1 u, \quad \text{where} \quad b_1 \in L_2(0,1)$$

Example: Heat Diffusion

A has a pure point spectrum $\sigma(A)$ which consists of the simple eigenvalues $\lambda_n = -n^2\pi^2$, $n \geq 0$, and the corresponding eigenvectors $\varphi_0 = 1$ and $\varphi_n(x) = \sqrt{2}\cos(n\pi x)$, $n \geq 1$, form an orthonormal basis of $L_2(0,1)$.

So A is the Riesz spectral operator given by

$$Az = \sum_{n=0}^{\infty} -(n\pi)^2 \langle z, \varphi_n \rangle \varphi_n, \quad \text{for } z \in D(A)$$

and is the infinitesimal generator of the C_0-semigroup:

$$T_A(t)z_0 = \langle z_0, 1 \rangle + \sum_{n=1}^{\infty} 2e^{-(n\pi)^2 t} \langle z_0, \cos n\pi(\cdot) \rangle \cos n\pi(\cdot)$$

Example: Heat Diffusion

$(T_A(t))_{t\geq 0}$ is a positive C_0-semigroup, i.e.

$$T_A(t)(L_2(0,1))^+ \subset (L_2(0,1))^+, \ \ \forall t \geq 0$$

where

$$L_2(0,1))^+ = \{h \in L_2(0,1) \ | \ h \geq 0 \text{ almost everywhere}\}.$$

A satisfies the spectrum decomposition assumption, so w.l.g. :

$$A = \begin{bmatrix} A_u & 0 \\ 0 & A_s \end{bmatrix}, \ \text{ where } \ A_u = A|_{L_2^u(0,1)}, A_s = A|_{L_2^s(0,1)}$$

where

$$\begin{aligned} L_2^u(0,1) &= \mathrm{span}\{\varphi_0\} = \{\text{the constant functions}\} \\ L_2^s(0,1) &= \overline{\mathrm{span}}\{\varphi_n, \ n \geq 1\} \end{aligned}$$

Example: Heat Diffusion

$T_{A_u}(t) = 1,\ t \geq 0$, is a positive unstable C_0-semigroup on $L_2^u(0,1)$ and

$$T_{A_s}(t) \text{ is positive on } L_2^s(0,1).$$

Indeed: let $z_s \in (L_2^s(0,1))^+ = L_2^s(0,1) \cap (L_2(0,1))^+$. Then $\langle z_s, 1\rangle = 0$. It follows that

$$\begin{aligned} T_{A_s}(t)z_s(\cdot) &= \sum_{n=1}^{\infty} 2e^{-n\pi^2 t}\langle z_s(.), \cos n\pi(\cdot)\rangle \cos n\pi(\cdot) \\ &= T_A(t)z_s(\cdot) \\ &\in L_2^s(0,1) \cap (L_2(0,1))^+ \subset (L_2^s(0,1))^+. \end{aligned}$$

Hence, for all $t \geq 0$,

$$T_{A_s}(t)(L_2^s(0,1))^+ \subset (L_2^s(0,1))^+$$

Let $b_1 = \alpha$ be a strictly positive constant function. Then

$$B_u u = \alpha u$$

is a positive operator from $\mathbb{R}$ to $L_2^u(0,1)$ and $B_s = 0$. So, $\forall k_u \in \mathbb{R}_+^0$, $A_u - B_u k_u$ is the infinitesimal generator of the positive exponentially stable C_0-semigroup given by

$$T_{A_u - B_u k_u}(t) z_u = e^{-\alpha k_u t} z_u, \quad \forall z_u \in L_2^u(0,1)$$

Hence (A, B) is positively stabilizable.

Moreover, for all $K = \left[\begin{array}{cc} k & 0 \end{array}\right] \in \mathcal{L}(L_2(0,1),\mathbb{R})$, with $k \in \mathbb{R}^0_+$,
$A - BK = \left[\begin{array}{cc} A_u - B_u k & 0 \\ 0 & A_s \end{array}\right]$ is the infinitesimal generator of a positive exponentially stable C_0-semigroup, or equivalently, the closed loop system

$$\left\{\begin{array}{lcl} \dfrac{\partial z}{\partial t}(x,t)(t) & = & \dfrac{\partial^2 z}{\partial x^2}(x,t) - b_1(x)k\langle\varphi_0, z(\cdot,t)\rangle \\ \dfrac{\partial z}{\partial x}(0,t) & = & \dfrac{\partial z}{\partial x}(1,t) = 0 \end{array}\right. \quad (0.1)$$

is a positive exponentially stable system for all $k \in \mathbb{R}^0_+$ with respect to the cone $(L_2^u(0,1))^+ \oplus (L_2^s(0,1))^+$.

Example: Heat Diffusion

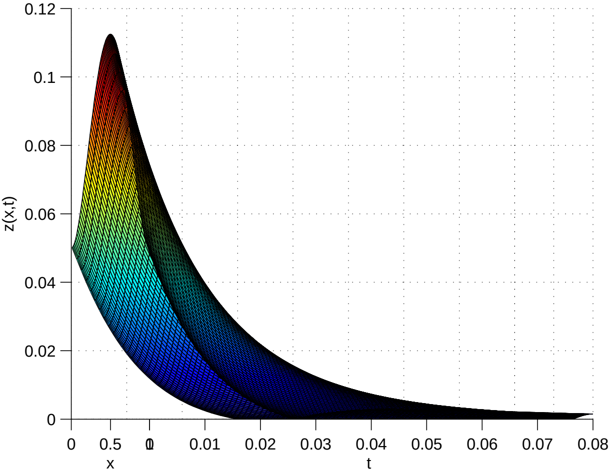

Figure: $z_0(x) = (x(x-1))^2 + 0.05$

Example: Heat Diffusion

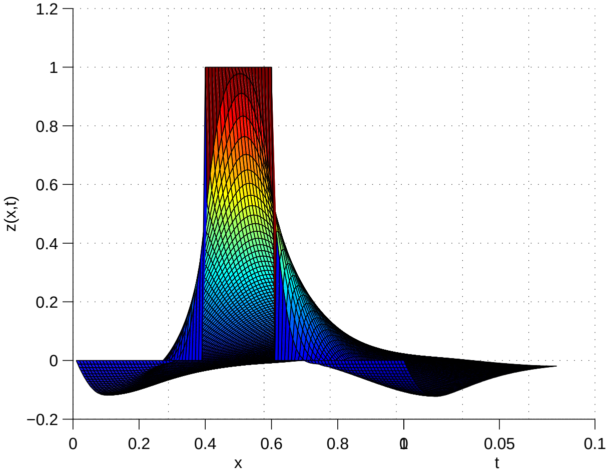

Figure: z_0 = characteristic function of [0.4,0.6]

Concluding Remarks and Perspectives

- The Metzler property guarantees the positivity whenever the positive initial condition is chosen in a specific finite-dimensional subspace.
- Necessary and sufficient conditions for the positivity of controlled systems.
- Sufficient conditions for the existence of a stabilizing state feedback such that the closed loop system remains positive.
- Positive stabilization without using spectral decomposition assumption is currently under investigation.