Obtaining the boundary control formulas for Maxwell's equations

George WEISS
Tel Aviv University, Israel

Olof Staffans
Abo Akademi University, Finland

CDPS, Wuppertal, July 2011

Assumed background:

For lack of time, we assume that the following concepts are known:

- the spaces X_1 and X_{-1}.
- system nodes
- well-posed system (nodes)
- boundary control systems

Classical solutions of abstract linear equations

Let U, X, Y be Hilbert spaces. Let $A : \mathcal{D}(A) \to X$ be the generator of a strongly continuous semigroup $\mathbb{T}$ on X. Let $B \in \mathcal{L}(U, X_{-1})$, $C \in \mathcal{L}(X_1, Y)$, $D \in \mathcal{L}(U, Y)$ and define

$$Z = X_1 + (\beta I - A)^{-1} BU .$$

Assume that C has a continuous extension top Z. Then the operator

$$S = \begin{bmatrix} A & B \\ \hline C & D \end{bmatrix}$$

defined for those $\left[\begin{smallmatrix} x \\ v \end{smallmatrix}\right]$ for which $Ax + Bv \in X$, is a *compatible system node* on U, X and Y. A is called the *semigroup generator* of S, B is the *control operator* of S and C is the *observation operator* of S.

The *transfer function* of S is the $\mathcal{L}(U, Y)$-valued analytic function

$$\mathbf{G}(s) = \overline{C}(sI - A)^{-1}B + D \qquad \forall\, s \in \rho(A).$$

The system node S is usually associated with the equation

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \qquad \forall\, t \geq 0, \qquad (1)$$

or equivalently, for all $t \geq 0$,

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad y(t) = Cx(t) + Du(t).$$

A triple (x, u, y) is called a *classical solution* of (1) on $[0, \infty)$ if

(a) $x \in C^1([0, \infty); X)$,

(b) $u \in C([0, \infty); U)$, $y \in C([0, \infty); Y)$,

(c) $\begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \mathcal{D}(S)$ for all $t \geq 0$,

(d) (1) holds for all $t \geq 0$.

The following proposition guarantees that for a system node, we have plenty of classical solutions of the system equation (1).

Proposition. Let S be a system node on (U, X, Y). If $u \in C^2([0, \infty); U)$ and $\begin{bmatrix} x_0 \\ u(0) \end{bmatrix} \in \mathcal{D}(S)$, then the equation (1) has a unique classical solution (x, u, y) satisfying $x(0) = x_0$. Moreover, this classical solution satisfies

$$x \in C^2([0, \infty); X_{-1}).$$

Scattering passive systems

The system node S is called *scattering passive* if all the classical solutions of (1) satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|^2 \leq \|u(t)\|^2 - \|y(t)\|^2 \qquad \forall\ t \geq 0\,.$$

An equivalent condition is that

$$\|x(\tau)\|^2 + \int_0^\tau \|y(t)\|^2 \mathrm{d}t \leq \|x(0)\|^2 + \int_0^\tau \|u(t)\|^2 \mathrm{d}t \qquad \forall\ t \geq 0\,.$$

The system node S is called *scattering energy preserving* if the power balance equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|^2 = \|u(t)\|^2 - \|y(t)\|^2 \qquad \forall\ t \geq 0$$

holds for all classical solutions of (1).

The *dual* of a system node S on (U, X, Y) is its adjoint S^*, which is a system node on (Y, X, U). The semigroup generator A^d, the control operators B^d, the observation operator C^d and transfer functions $\mathbf{G}^d$ of the dual system node S^* are given by

$$A^d = A^*, \qquad B^d = C^*, \qquad C^d = B^*, \qquad \mathbf{G}^d(s) = \mathbf{G}(\overline{s})^*.$$

The system node S is called *scattering conservative* if both S and S^* are scattering energy preserving.
Any scattering passive system is *well-posed*, meaning that for some (hence, for every) $\tau > 0$ there exists $K_\tau \geq 0$ such that for any classical solution of (1),

$$\|x(\tau)\|^2 + \|y\|^2_{L^2[0,\tau]} \leq K_\tau \left(\|x(0)\|^2 + \|u\|^2_{L^2[0,\tau]} \right).$$

Indeed, for scattering passive systems we have $K_\tau = 1$.
Any well-posed system is a compatible system node.

Our aim

It is not easy to establish that a system is scattering passive (or conservative). (Algebraic conditions for conservative systems were derived by J. Malinen, O. Staffans and G. Weiss in 2006.) It is of interest to identify large classes of systems where the operators A, B, C, D have a special structure observed in models of mathematical physics, which implies that the system is scattering passive or conservative. Such a special class of conservative systems ("from thin air") has been introduced in two papers by M. Tucsnak and G. Weiss in 2003. A special class of systems described by several first order PDEs in one space dimension has been studied by H. Zwart, Y. Le Gorrec, B. Maschke and J. Villegas, about 2006-2009.

In a paper in preparation we (GW and OS) give a larger special class, that includes "thin air" systems. We were led to introduce this class by our failure to fit Maxwell's equations into the "thin air" framework. The new class of systems is also more flexible for allowing time-varying and nonlinear extensions.

The new special class of passive systems

We consider a linear system Σ with state space $X = H \oplus E$, where H and E are Hilbert spaces. The Hilbert space U is both the input space and the output space of Σ. We identify H, E and U with their duals. The Hilbert space E_0 is a dense subspace of E and the embedding $E_0 \hookrightarrow E$ is continuous. We denote by E_0' the dual of E_0 with respect to the pivot space E, so that

$$E_0 \subset E \subset E_0',$$

densely and with continuous embeddings. We denote $X_0 = H \oplus E_0$, so that $X_0' = H \oplus E_0'$. We assume that

$$L \in \mathcal{L}(E_0, H), \qquad K \in \mathcal{L}(E_0, U), \qquad G \in \mathcal{L}(E_0, E_0'),$$

$$\mathrm{Re}\,\langle G w_0, w_0 \rangle_{E_0', E_0} \leq 0 \qquad \forall\, w_0 \in E_0,$$

and we define $\overline{A} \in \mathcal{L}(X_0, X_0')$, $B \in \mathcal{L}(U, X_0')$ and $\overline{C} \in \mathcal{L}(X_0, U)$ by

$$\overline{A} = \begin{bmatrix} 0 & -L \\ L^* & G - \frac{1}{2}K^*K \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ K^* \end{bmatrix}, \qquad \overline{C} = \begin{bmatrix} 0 & -K \end{bmatrix}.$$

The equations of the system are (as at (1))

$$\dot{x}(t) = \overline{A}x(t) + Bu(t), \qquad y(t) = \overline{C}x(t) + u(t), \tag{2}$$

where x is the state trajectory, u is the input function and y is the output function. Note that the differential equation above is an equation in X_0'. We define the domain $\mathcal{D}(A)$ by

$$\mathcal{D}(A) = \{x_0 \in X_0 \mid \overline{A}x_0 \in X\}$$

and we denote by A and C the restrictions of $\overline{A}$ and $\overline{C}$ to $\mathcal{D}(A)$. More explicitly,

$$\mathcal{D}(A) = \left\{ \begin{bmatrix} z_0 \\ w_0 \end{bmatrix} \in X_0 \;\middle|\; L^* z_0 + \left(G - \tfrac{1}{2}K^*K\right) w_0 \in E \right\}.$$

We assume that $\begin{bmatrix} L \\ K \end{bmatrix}$ (with domain E_0) is closed as an unbounded operator $E \to H \oplus U$. This implies that A is maximal dissipative and hence it generates a semigroup of contractions.

Main abstract result

Under the above assumptions, the equations (2) determine a scattering passive system node with state space $X = H \oplus E$ and input and output space U. This system node is scattering conservative if and only if

$$\mathrm{Re}\,\langle G w_0, w_0 \rangle_{E_0', E_0} = 0 \qquad \forall\, w_0 \in E_0 .$$

This syste node is

$$S_{\mathrm{sca}} = \begin{bmatrix} A & B \\ \hline C & I \end{bmatrix}$$

with the domain $\mathcal{D}(S_{\mathrm{sca}})$ given by

$$\left\{ \begin{bmatrix} z_0 \\ w_0 \\ u_0 \end{bmatrix} \in H \times E_0 \times U \;\middle|\; L^* z_0 + \left(G - \tfrac{1}{2} K^* K \right) w_0 + K^* u_0 \in E \right\} .$$

Proposition about classical solutions

We use the notation and the assumptions of the main result. If the input function u and the initial state $\begin{bmatrix} z(0) \\ w(0) \end{bmatrix}$ of $S_{\rm sca}$ satisfy

$$u \in \mathcal{H}^1_{\rm loc}((0,\infty); U), \qquad \begin{bmatrix} z(0) \\ w(0) \\ u(0) \end{bmatrix} \in \mathcal{D}(S_{\rm sca}), \tag{3}$$

then the corresponding state trajectory $\begin{bmatrix} z \\ w \end{bmatrix}$ and output function y of $S_{\rm sca}$ satisfy $y \in \mathcal{H}^1_{\rm loc}((0,\infty); Y)$,

$$\begin{bmatrix} z \\ w \end{bmatrix} \in C^1([0,\infty); H \oplus E), \quad \begin{bmatrix} z \\ w \\ u \end{bmatrix} \in C([0,\infty); \mathcal{D}(S_{\rm sca})),$$

and

$$\begin{bmatrix} \dot z(t) \\ \dot w(t) \\ y(t) \end{bmatrix} = S_{\rm sca} \begin{bmatrix} z(t) \\ w(t) \\ u(t) \end{bmatrix} \qquad \forall\, t > 0.$$

Power balance formula

We use the notation and the assumptions of the main result. If the functions $u, x = \left[\begin{smallmatrix} z \\ w \end{smallmatrix}\right]$ and y are a classical solution of

$$\begin{bmatrix} \dot{z}(t) \\ \dot{w}(t) \\ y(t) \end{bmatrix} = S_{\mathrm{sca}} \begin{bmatrix} z(t) \\ w(t) \\ u(t) \end{bmatrix} \qquad \forall\, t > 0,$$

then the following power balance equation holds for $t \geq 0$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|^2 = \|u(t)\|^2 - \|y(t)\|^2 + 2\mathrm{Re}\,\langle Gw(t), w(t)\rangle\,.$$

The dual system node S^*_{sca} has the same structure, but with L, K and G replaced with $-L, -K$ and G^*. Therefore, its classical solutions satisfy the same power balance equation. (Hence, as already mentioned, S_{sca} is scattering conservative if and only if $\mathrm{Re}\,\langle Gw_0, w_0\rangle = 0$ for all $w_0 \in E_0$.)

The transfer function of $S_{\rm sca}$ is

$$\mathbf{G}(s) = I - K\left[sI + \frac{1}{2}K^*K - G + \frac{1}{s}L^*L\right]^{-1}K^*,$$

for all s in the open right half-plane.