Local Exponential Stabilization of a 2 x 2 Quasilinear Hyperbolic System using Backstepping

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CDPS, Wuppertal, 2011

Outline

- 2x2 hyperbolic quasi-linear PDEs
- Backstepping control of the linearized system
- Well-posedness of kernel PDEs
- Stability of the nonlinear system
- Conclusions

2x2 Hyperbolic Quasi-Linear PDEs

$$z_t + \Lambda(z,x) z_x + f(z,x) = 0,$$

$x \in [0,1]$, where $z : [0,1] \times [0,\infty) \to \mathbb{R}^2$, $\Lambda : \mathbb{R}^2 \times [0,1] \to \mathcal{M}_{2,2}(\mathbb{R})$, $f : \mathbb{R}^2 \times [0,1] \to \mathbb{R}^2$.

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Consider w.l.o.

$$\Lambda(0,x) = \begin{bmatrix} \Lambda_1(x) & 0 \\ 0 & \Lambda_2(x) \end{bmatrix}$$

$\Lambda_1(x)$ and $\Lambda_2(x)$ are the speeds of propagation of $z = [z_1 \; z_2]^T$. According to their signs:

homodirectional	heterodirectional
$\forall x \in [0,1], \;\; \Lambda_1(x)\Lambda_2(x) > 0$	$\forall x \in [0,1], \;\; \Lambda_1(x)\Lambda_2(x) < 0$

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Heterodirectional $\longrightarrow$ one boundary condition on each side.

Homodirectional $\longrightarrow$ two boundary conditions on the same side.

Examples of Homodirectional Systems

- road traffic: Aw-Rascle model
- heat exchanger
- plug-flow chemical reactor
- population dynamics (Lotka-Volterra) in laser chambers

Examples of Heterodirectional Systems

- Saint-Venant model of water waves in a channel
- gas flow in pipes
- cardiovascular flow in flexible blood vessels

The control problem (hetero case)

$$z_t + \Lambda(z,x)z_x + f(z,x) = 0$$

with boundary conditions

$$\begin{aligned} z_1(0,t) &= q z_2(0,t), \qquad q \neq 0 \\ z_2(1,t) &= U(t) = \text{actuation} \end{aligned}$$

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Task: find fbk law for $U(t)$ to make $z \equiv 0$ locally exponentially stable

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Approach: (1) Stabilize linearized system using backstepping
(2) Prove local stability for nonlinear system

The linear case

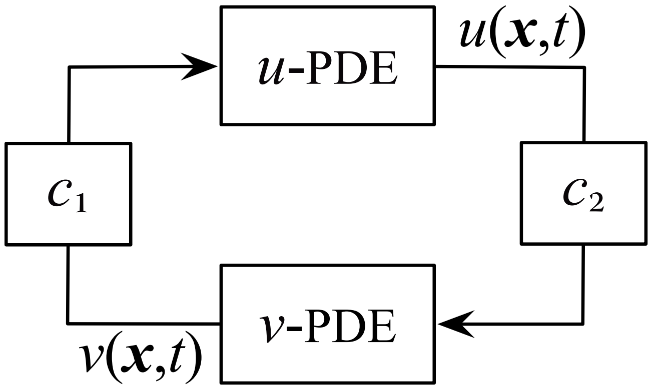
$$
\begin{aligned}
u_t &= -\varepsilon_1(x)u_x + c_1(x)v \\
v_t &= \varepsilon_2(x)v_x + c_2(x)u
\end{aligned}
$$

$x \in [0,1], \quad \varepsilon_1(x), \varepsilon_2(x) > 0$

with boundary conditions

$$
\begin{aligned}
u(t,0) &= qv(t,0) \\
v(t,1) &= U(t)
\end{aligned}
$$

Key Issue



A continuum of 1st-order (in time) subsystems with (potentially) positive feedback coupling and *small gain condition violated*.

Target system

$$\begin{aligned} \alpha_t &= -\varepsilon_1(x)\alpha_x \\ \beta_t &= \varepsilon_2(x)\beta_x \end{aligned}$$

with boundary conditions

$$\begin{aligned} \alpha(t,0) &= q\beta(t,0) \\ \beta(t,1) &= 0 \end{aligned}$$

Feedback connection severed throughout the domain, using control only at one boundary.

Cascade of two exp. stable transport PDEs ($\beta \to \alpha$).

Backstepping transformation

$$
\begin{aligned}
\alpha(t,x) &= u(t,x) - \int_0^x K^{uu}(x,\xi)u(t,\xi)d\xi - \int_0^x K^{uv}(x,\xi)v(t,\xi)d\xi \\
\beta(t,x) &= v(t,x) - \int_0^x K^{vu}(x,\xi)u(t,\xi)d\xi - \int_0^x K^{vv}(x,\xi)v(t,\xi)d\xi
\end{aligned}
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$$\beta(t,x) = v(t,x) - \int_0^x K^{vu}(x,\xi)u(t,\xi)d\xi - \int_0^x K^{vv}(x,\xi)v(t,\xi)d\xi$$

Control law (set $\beta(t,1) = 0$)

$$U(t) = \int_0^1 K^{vu}(1,\xi)u(t,\xi)d\xi + \int_0^1 K^{vv}(1,\xi)v(t,\xi)d\xi$$

Kernel PDEs

First, for K^{uu} and K^{uv}:

$$\begin{aligned}\left(\varepsilon_1(x)\partial_x + \varepsilon_1(\xi)\partial_\xi\right) K^{uu} &= -\varepsilon_1'(\xi)K^{uu} - c_2(\xi)K^{uv}\\ \left(\varepsilon_1(x)\partial_x - \varepsilon_2(\xi)\partial_\xi\right) K^{uv} &= \varepsilon_2'(\xi)K^{uv} - c_1(\xi)K^{uu},\end{aligned}$$

with boundary conditions

$$\begin{aligned}K^{uu}(x,0) &= \frac{\varepsilon_2(0)}{q\varepsilon_1(0)}K^{uv}(x,0)\\ K^{uv}(x,x) &= \frac{c_1(x)}{\varepsilon_1(x)+\varepsilon_2(x)}.\end{aligned}$$

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A 2×2 system of first-order linear hyperbolic PDE that evolves in the triangular domain $\mathcal{T} = \{(x,\xi) : 0 \le \xi \le x \le 1\}$.

Second, for K^{vu} and K^{vv}:

$$\begin{aligned}
\left(\varepsilon_2(x)\partial_x + \varepsilon_2(\xi)\partial_\xi\right) K^{vv} &= -\varepsilon_2'(\xi) K^{vv} + c_1(\xi) K^{vu}, \\
\left(\varepsilon_2(x)\partial_x - \varepsilon_1(\xi)\partial_\xi\right) K^{vu} &= \varepsilon_1'(\xi) K^{vu} + c_2(\xi) K^{vv},
\end{aligned}$$

with boundary conditions

$$\begin{aligned}
K^{vv}(x,0) &= \frac{q\varepsilon_1(0)}{\varepsilon_2(0)} K^{vu}(x,0) \\
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\end{aligned}
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Uncoupled with the previous PDE.

Linear example: constant coefficients

Benchmark system

$$\begin{aligned} u_t + u_x &= \omega v \\ v_t - v_x &= \omega u \end{aligned}$$

with boundary conditions $u(t,0) = qv(t,0)$, $v(t,1) = U(t)$

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Open-loop unstable for large ω.

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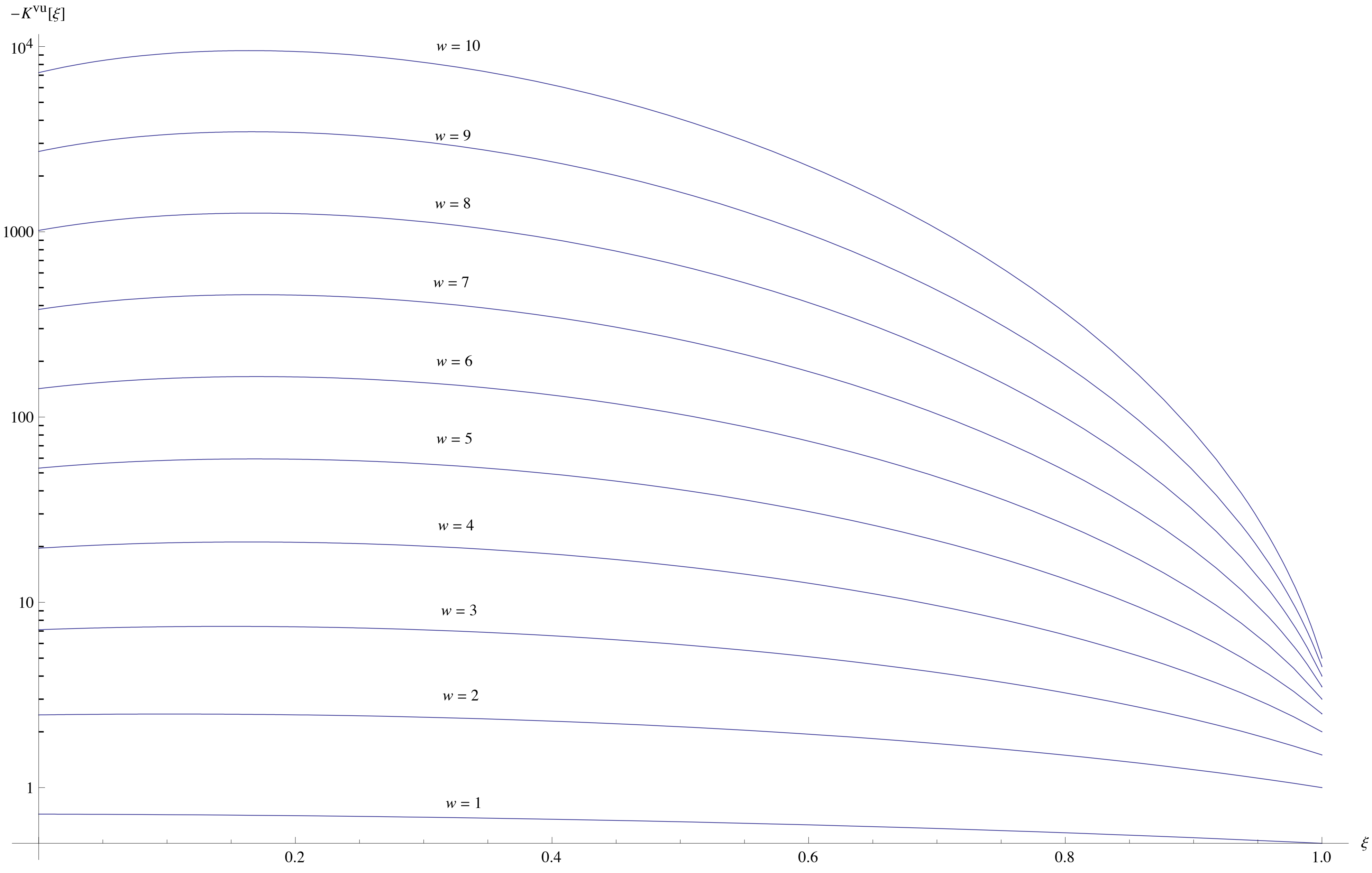
For large enough ω no choices of k_1, k_2 in static output fbk law $U = k_1 u(t,0) + k_2 u(t,1)$ can achieve stability.

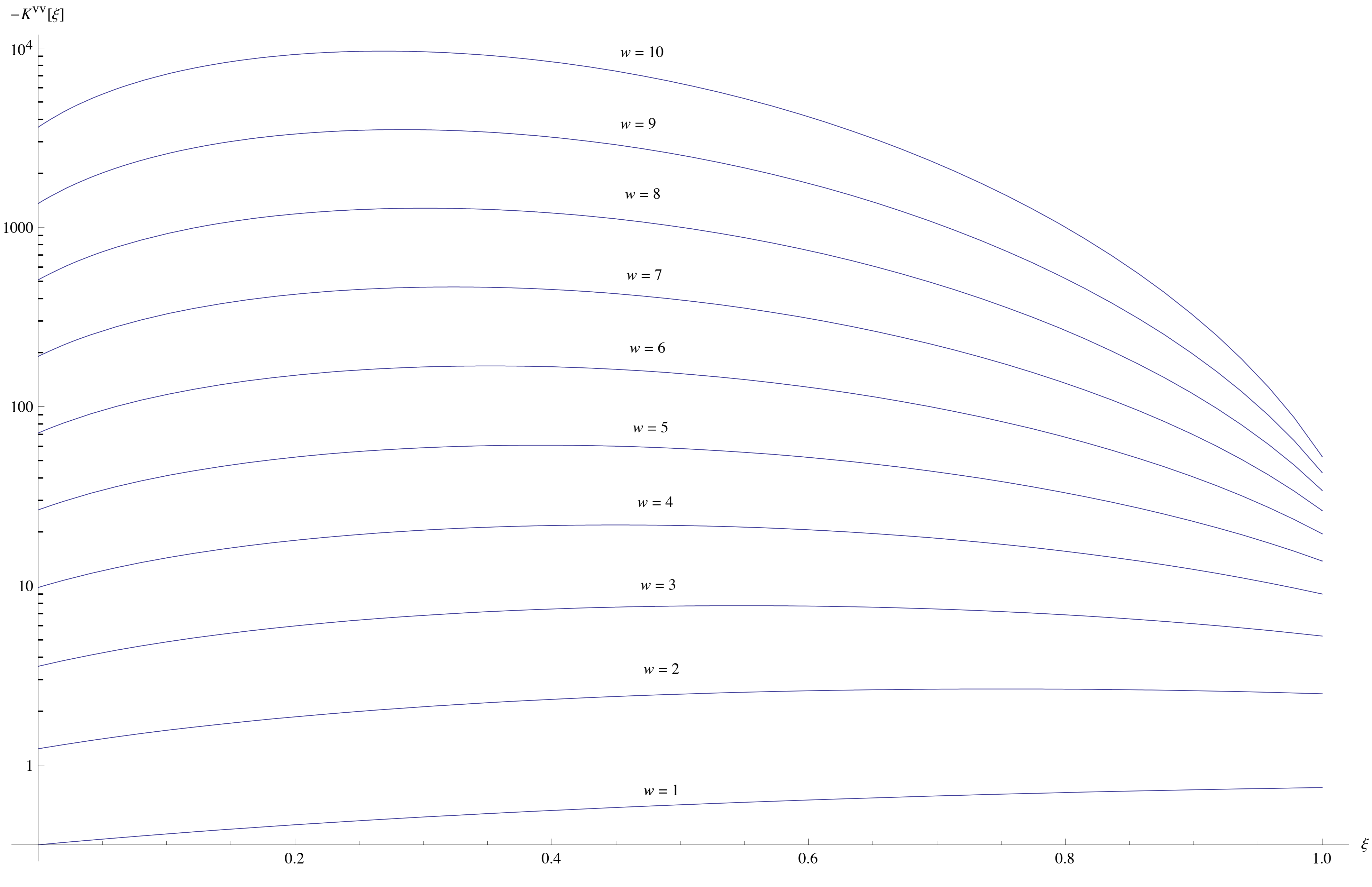
Recall the backstepping controller

$$U(t) = \int_0^1 K^{vu}(1,\xi)u(t,\xi)d\xi + \int_0^1 K^{vv}(1,\xi)v(t,\xi)d\xi$$

Control gain kernel $K^{vu}(1,\xi)$ for $q = 0.5$ and $\omega = 1$–10



Control gain kernel $K^{vv}(1, \xi)$ for $q = 0.5$ and $\omega = 1$–10



Note the log scale. The growth in ω seems exponential.

Control $v(t,1)$ puts a strong emphasis on $u(t,0.2)$ and $v(t,0.3)$ — highly **non-collocated!**

Inverse backstepping transformation

$$\begin{aligned} u(t,x) &= \alpha(t,x) + \int_0^x L^{\alpha\alpha}(x,\xi)\alpha(t,\xi)d\xi + \int_0^x L^{\alpha\beta}(x,\xi)\beta(t,\xi)d\xi, \\ v(t,x) &= \beta(t,x) + \int_0^x L^{\beta\alpha}(x,\xi)\alpha(t,\xi)d\xi + \int_0^x L^{\beta\beta}(x,\xi)\beta(t,\xi)d\xi, \end{aligned}$$

One gets again four PDEs:

$$\begin{aligned} \left(\varepsilon_1(x)\partial_x + \varepsilon_1(\xi)\partial_\xi\right) L^{\alpha\alpha} &= -\varepsilon_1'(\xi)L^{\alpha\alpha} + c_1(x)L^{\beta\alpha}, \\ \left(\varepsilon_1(x)\partial_x - \varepsilon_2(\xi)\partial_\xi\right) L^{\alpha\beta} &= \varepsilon_2'(\xi)L^{\alpha\beta} + c_1(x)L^{\beta\beta}, \\ \left(\varepsilon_2(x)\partial_x - \varepsilon_1(\xi)\partial_\xi\right) L^{\beta\alpha} &= \varepsilon_1'(\xi)L^{\beta\alpha} - c_2(x)L^{\alpha\alpha} \\ \left(\varepsilon_2(x)\partial_x + \varepsilon_2(\xi)\partial_\xi\right) L^{\beta\beta} &= -\varepsilon_2'(\xi)L^{\beta\beta} - c_2(x)L^{\alpha\beta} \end{aligned}$$

with boundary conditions

$$\begin{aligned} L^{\alpha\alpha}(x,0) &= \frac{\varepsilon_2(0)}{q\varepsilon_1(0)}L^{\alpha\beta}(x,0), \quad L^{\alpha\beta}(x,x) = \frac{c_1(x)}{\varepsilon_1(x)+\varepsilon_2(x)} \\ L^{\beta\alpha}(x,x) &= -\frac{c_2(x)}{\varepsilon_1(x)+\varepsilon_2(x)}, \quad L^{\beta\beta}(x,0) = \frac{q\varepsilon_1(0)}{\varepsilon_2(0)}L^{\beta\alpha}(x,0) \end{aligned}$$

Kernel well-posedness

Consider the following “generalized Goursat problem” of which the direct and inverse kernel equations are a particular case:

$$
\begin{aligned}
\left(\varepsilon_1(x)\partial_x+\varepsilon_1(\xi)\partial_\xi\right)F^1 &= g_1(x,\xi)+\sum_{i=1}^{4}C_{1i}(x,\xi)F^i(x,\xi),\\
\left(\varepsilon_1(x)\partial_x-\varepsilon_2(\xi)\partial_\xi\right)F^2 &= g_2(x,\xi)+\sum_{i=1}^{4}C_{2i}(x,\xi)F^i(x,\xi),\\
\left(\varepsilon_2(x)\partial_x-\varepsilon_1(\xi)\partial_\xi\right)F^3 &= g_3(x,\xi)+\sum_{i=1}^{4}C_{3i}(x,\xi)F^i(x,\xi),\\
\left(\varepsilon_2(x)\partial_x+\varepsilon_2(\xi)\partial_\xi\right)F^4 &= g_4(x,\xi)+\sum_{i=1}^{4}C_{4i}(x,\xi)F^i(x,\xi),
\end{aligned}
$$

with boundary conditions

$$
\begin{aligned}
F^1(x,0) &= h_1(x)+q_1(x)F^2(x,0)+q_2(x)F^3(x,0),\\
F^2(x,x) &= h_2(x), \quad F^3(x,x)=h_3(x),\\
F^4(x,0) &= h_4(x)+q_3(x)F^2(x,0)+q_4(x)F^3(x,0).
\end{aligned}
$$

evolving in the domain $\mathcal{T}=\{(x,\xi):0\leq\xi\leq x\leq 1\}$.

Transforming the PDEs into integral equations along the characteristic lines, and using the method of successive approximations, we prove:

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Theorem Under the assumptions

$$q_i, h_i \in \mathcal{C}([0,1]), \quad g_i, C_{ji} \in \mathcal{C}(\mathcal{T}), \quad i, j = 1,2,3,4$$

and $\varepsilon_1, \varepsilon_2 \in \mathcal{C}([0,1])$ with $\varepsilon_1(x), \varepsilon_2(x) > 0$, there exists a unique $\mathcal{C}(\mathcal{T})$ solution F^i, $i = 1,2,3,4$.

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Theorem Under the additional assumptions

$$\varepsilon_i, q_i, h_i \in \mathcal{C}^N([0,1]), \quad g_i, C_{ji} \in \mathcal{C}^N(\mathcal{T}),$$

there exists a unique $\mathcal{C}^N(\mathcal{T})$ solution F^i, $i = 1,2,3,4$.

An observer-based controller with sensing of $u(1,t)$

$$
\begin{aligned}
\hat{u}_t &= -\varepsilon_1 \hat{u}_x + c_1(x)\hat{v} - \varepsilon_1 P^{uu}(x,1)\left(u(t,1) - \hat{u}(t,1)\right) \\
\hat{v}_t &= \varepsilon_2 \hat{v}_x + c_2(x)\hat{u} - \varepsilon_1 P^{vu}(x,1)\left(u(t,1) - \hat{u}(t,1)\right) \\
\hat{u}(t,0) &= q\hat{v}(t,0) \\
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$$

Observer gains obtained from

$$
\begin{aligned}
\left(\varepsilon_1(x)\partial_x + \varepsilon_1(\xi)\partial_\xi\right) P^{uu} &= -\varepsilon_1'(\xi)P^{uu} - c_1(x)P^{vu}\\
\left(\varepsilon_2(x)\partial_x - \varepsilon_1(\xi)\partial_\xi\right) P^{vu} &= \varepsilon_1'(\xi)P^{vu} + c_2(x)P^{uu}
\end{aligned}
$$

$$
P^{uu}(0,\xi) = qP^{vu}(0,\xi), \quad P^{vu}(x,x) = -\frac{c_2(x)}{\varepsilon_1(x)+\varepsilon_2(x)}
$$

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\left(\varepsilon_1(x)\partial_x - \varepsilon_2(\xi)\partial_\xi\right) P^{uv} &= \varepsilon_2'(\xi)P^{uv} - c_1(x)P^{vv}\\
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$$
P^{vv}(0,\xi) = \frac{1}{q}P^{uv}(0,\xi), \quad P^{uv}(x,x) = \frac{c_1(x)}{\varepsilon_1(x)+\varepsilon_2(x)}
$$

Back to the Nonlinear Case

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Consider only the state-fbk problem here, but output-fbk also possible.

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Compute

$$\frac{\partial f}{\partial z}(z,x)\bigg|_{z=0} = \begin{bmatrix} f_{11}(x) & f_{12}(x) \\ f_{21}(x) & f_{22}(x) \end{bmatrix}$$

Define

$$\begin{aligned} \varphi_1(x) &= \exp\left(\int_0^x \frac{f_{11}(s)}{\Lambda_1(s)}ds\right) \\ \varphi_2(x) &= \exp\left(-\int_0^x \frac{f_{22}(s)}{\Lambda_2(s)}ds\right) \end{aligned}$$

In re-scaled variables

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the system is

$$\underbrace{w_t - \Sigma(x) w_x - C(x) w}_{\text{design bks contr for this syst}} \quad + \underbrace{\Lambda_{NL}(w,x) w_x + f_{NL}(w,x)}_{\text{nonlinear perturbations}} = 0$$

where

$$\Sigma(x) = -\Lambda(0,x) = \begin{bmatrix} -\Lambda_1(x) & 0 \\ 0 & -\Lambda_2(x) \end{bmatrix}, \qquad C(x) = \begin{bmatrix} 0 & -f_{12} \\ -f_{21} & 0 \end{bmatrix}$$

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Control law in z variables:

$$U = \varphi_2(1) \int_0^1 \frac{K^{vu}(1,\xi)}{\varphi_1(\xi)} z_1(\xi,t) d\xi + \varphi_2(1) \int_0^1 \frac{K^{vv}(1,\xi)}{\varphi_2(\xi)} z_2(\xi,t) d\xi$$

Proof of stability

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Direct and inverse bkst transforms applied to obtain the system + perturbation term in target variables.

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To go to H_1 and H_2, take t-derivatives instead of x-derivatives to simplify the process.

L_2 analysis step

System written in target variables $\gamma = [\alpha\ \beta]^T$

$$\underbrace{\gamma_t - \Sigma(x)\gamma_x}_{\text{linear stable part}} \quad + \quad \underbrace{F_3[\gamma, \gamma_x] + F_4[\gamma]}_{\text{nonlinear perturbation}} \quad = 0,$$

F_3 and F_4 are functionals in terms of backstepping kernels. Boundary conditions:

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Lyapunov function:

$$U = \int_0^1 \gamma^T(x,t) D(x) \gamma(x,t) dx, \qquad D(x) = \begin{bmatrix} \lambda_1 \frac{e^{\mu(1-x)}}{\Lambda_1(x)} & 0 \\ 0 & \left(q^2 \lambda_1 e^{\mu} + \lambda_2\right) \frac{e^{\mu x}}{\Lambda_2(x)} \end{bmatrix}$$

with $\mu = \lambda_1 \max_{x \in [0,1]} \left\{ \frac{1}{\Lambda_1(x)}, \frac{1}{\Lambda_2(x)} \right\}$ and choosing $\lambda_1, \lambda_2 > 0$.

L_2 analysis step

System written in target variables $\gamma = [\alpha\ \beta]^T$

$$\underbrace{\gamma_t - \Sigma(x)\gamma_x}_{\text{linear stable part}} + \underbrace{F_3[\gamma, \gamma_x] + F_4[\gamma]}_{\text{nonlinear perturbation}} = 0,$$

F_3 and F_4 are functionals in terms of backstepping kernels. Boundary conditions:

$$\begin{aligned} \alpha(0,t) &= q\beta(0,t) \\ \beta(1,t) &= 0 \end{aligned}$$

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with $\mu = \lambda_1 \max_{x\in[0,1]} \left\{ \frac{1}{\Lambda_1(x)}, \frac{1}{\Lambda_2(x)} \right\}$ and choosing $\lambda_1, \lambda_2 > 0$. Then if $\|\gamma\|_\infty < \delta_1$

$$\dot{U} \leq -\lambda_1 U - \lambda_2 \left(\alpha^2(1,t) + \beta^2(0,t)\right) + C_1 U^{3/2} + C_2 \|\gamma_x\|_\infty U,$$

H_1 analysis step

$$\underbrace{\gamma_{tt} - \Sigma(x)\gamma_{tx}}_{\text{linear stable part}} + \underbrace{F_1[\gamma]\gamma_{tx} + F_5[\gamma, \gamma_x, \gamma_t] + F_6[\gamma, \gamma_t]}_{\text{nonlinear perturbation}} = 0,$$

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Lyapunov function:

$$V = \int_0^1 \gamma_t^T(x,t) R[\gamma](x)\gamma_t(x,t)dx, \qquad R[\gamma](x) = D(x) + \begin{bmatrix} 0 & \psi[\gamma] \\ \psi[\gamma] & 0 \end{bmatrix}$$

where $\psi[\gamma] = \frac{D_{11}(x)(F_1[\gamma])_{12} - D_{22}(x)(F_1[\gamma])_{21}}{\varepsilon_2(x) + \varepsilon_1(x) + (F_1[\gamma])_{11} - (F_1[\gamma])_{22}}$.

H_1 analysis step

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where $\psi[\gamma] = \frac{D_{11}(x)(F_1[\gamma])_{12} - D_{22}(x)(F_1[\gamma])_{21}}{\varepsilon_2(x) + \varepsilon_1(x) + (F_1[\gamma])_{11} - (F_1[\gamma])_{22}}$. Then if $\|\gamma\|_\infty < \delta_2$

$$\dot{V} \leq -\lambda_3 V - \lambda_4 \left(\alpha_t^2(1,t) + \beta_t^2(0,t) \right) + C_3 V \|\gamma_t\|_\infty$$

H_2 analysis

$$\underbrace{\gamma_{ttt} - \Sigma(x)\gamma_{ttx}}_{\text{linear stable part}} + \underbrace{F_1[\gamma]\gamma_{ttx} + F_7[\gamma, \gamma_x, \gamma_t, \gamma_{tx}, \gamma_{tt}] + F_8[\gamma, \gamma_t, \gamma_{tt}]}_{\text{nonlinear perturbation}} = 0,$$

H_2 **analysis**

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H_2 analysis

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Lyapunov function:

$$W = \int_0^1 \gamma_{tt}^T(x,t) R[\gamma](x) \gamma_{tt}(x,t) dx$$

Then if $\|\gamma\|_\infty + \|\gamma_t\|_\infty < \delta_3$

$$\dot{W} \leq -\lambda_5 W - \lambda_6 \left(\alpha_{tt}^2(1,t) + \beta_{tt}^2(0,t) \right) + C_4 W V^{1/2} + C_5 V W^{1/2} + C_6 W^{3/2}$$

To relate γ_t, γ_{tt} and the H_1, H_2 norms of γ use the following lemmas:

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Lemma 1. If $\|\gamma\|_\infty < \delta_4$

$$
\begin{aligned}
\|\gamma_t\|_\infty &\leq c_1 (\|\gamma_x\|_\infty + \|\gamma\|_\infty) \\
\|\gamma_x\|_\infty &\leq c_3 (\|\gamma_t\|_\infty + \|\gamma\|_\infty) \\
\|\gamma_t\|_{L^2} &\leq c_2 \left(\|\gamma_x\|_{L^2} + \|\gamma\|_{L^2}\right) \\
\|\gamma_x\|_{L^2} &\leq c_4 \left(\|\gamma_t\|_{L^2} + \|\gamma\|_{L^2}\right)
\end{aligned}
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\|\gamma_t\|_{L^2} &\leq c_2\left(\|\gamma_x\|_{L^2} + \|\gamma\|_{L^2}\right)\\
\|\gamma_x\|_{L^2} &\leq c_4\left(\|\gamma_t\|_{L^2} + \|\gamma\|_{L^2}\right)
\end{aligned}$$

Lemma 2. If $\|\gamma\|_\infty + \|\gamma_t\|_\infty < \delta_5$

$$\begin{aligned}
\|\gamma_{tt}\|_\infty &\leq c_1\left(\|\gamma_{xx}\|_\infty + \|\gamma_x\|_\infty + \|\gamma\|_\infty\right)\\
\|\gamma_{xx}\|_\infty &\leq c_3\left(\|\gamma_{tt}\|_\infty + \|\gamma_t\|_\infty + \|\gamma\|_\infty\right)\\
\|\gamma_{tt}\|_{L^2} &\leq c_2\left(\|\gamma_{xx}\|_{L^2} + \|\gamma_x\|_{L^2} + \|\gamma\|_{L^2}\right)\\
\|\gamma_{xx}\|_{L^2} &\leq c_4\left(\|\gamma_{tt}\|_{L^2} + \|\gamma_t\|_{L^2} + \|\gamma\|_{L^2}\right)
\end{aligned}$$

The nonlinear result

Define Lyap. fcn.

$$S = U + V + W$$

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for $\lambda, C > 0$.

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for $\lambda, C > 0$.

Noting $\|\gamma\|_\infty + \|\gamma_t\|_\infty \leq C_7 S$ and that S is equivalent to the H^2 norm of γ we obtain

Theorem [Vazquez, Coron, Krstic, 2011 CDC]

With the linear backstepping controller, $\exists \delta_0, M_0, \gamma_0 > 0$ *such that*

$$\|w_0\|_{H_2} \leq \delta_0$$

$$\Downarrow$$

$$\|w(\cdot,t)\|_{H_2} \leq M_0 \mathrm{e}^{-\gamma_0 t} \|w_0\|_{H_2}.$$

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Interesting open problem: $N \times N$ systems (slugging flows in offshore oil rig risers)