Time optimal and switching controls for some distributed parameter systems

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Plan of the talk

- Bang-bang property of time optimal controls for parabolic equations
- Stability of dissipative systems with intermittent damping
- Conclusions and open questions

Bang-bang property of time optimal controls for parabolic equations

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Time optimal control

Consider the system $\dot{z} = Az + Bu, \quad z(0) = \psi$, where X, U be Hilbert spaces $A : \mathcal{D}(A) \to X$, let . Assume that $A < 0$, so that A generates $\mathbb{T}$. Let $B \in \mathcal{L}(U, X_{-1/2+\varepsilon})$ be a control operator (so that B is admissible) and assume that (A, B) is null controllable.
Let $M > 0$ and let

$$\mathcal{U}_{ad} = \{u \in L^\infty([0,\infty), U) \ \mid \ \|u(t)\| \leqslant M a.e.\}.$$

For given $\psi \in X$, the time optimal control problem consists in determining

$$\tau(\psi) = \min_{u \in \mathcal{U}_{ad}} \{t \ \mid \ z(\cdot, t) = 0\},$$

and the corresponding input u^*.

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The bang-bang property

For finite dimensional systems u^* has the bang-bang property:

$$\|u^*(t)\|_U = M \quad \text{a.e. in} \quad [0, \tau(\psi)].$$

No general result is available. See [4] for the 1D heat equation with boundary control and [5] for the nD heat equation with internal control.

References

[1] P. Borwein and T. Erdélyi, J. Amer. Math. Soc., 10 (1997), pp. 327–349.

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[4] V. J. Mizel and T. I. Seidman, SICON, 35 (1997), pp. 1204–1216.

[5] G. Wang, SICON, 47 (2008), pp. 1701–1720.

Sufficient conditions (I)

Proposition 1. (Fattorini, 1964). *Assume that for every $\tau > 0$ and for every set of positive measure $E \subset [0, \tau]$ there exists a control the exists a control $u \in L^\infty([0,\infty), U)$, $\text{supp} u \subset E$ such that $z(\tau) = 0$ (we say in this case that the pair (A, B) is E, τ null-controllable). Then the time optimal control u^* is bang-bang and it is unique.*

Proposition(Wang, 2008). For γ, $\mu > 0$ we denote

$$V_{\mu,\gamma} = \text{span}\{\varphi_k \mid \lambda_k^\gamma \leqslant \mu\}.$$

Assume that $\beta > -1/2$. Moreover, assume that there exist positive constants d_0, d_1 and $\gamma \in (0, 1)$ such that :

$$\|\psi\| \leqslant d_0 e^{d_1 \mu} \|B^* \psi\| \quad (\psi \in V_{\mu,\gamma}). \tag{1}$$

Then (A, B) is (E, T) null-controllable for every $T > 0$ and every set of positive measure $E \subset [0, T]$.

Sufficient conditions(II)

Remark.(Micu, Roventa,Tucsnak (2011)). For $\gamma,\ \mu > 0$ we denote

$$V_{\mu,\gamma} = \operatorname{span}\{\varphi_k \ \mid \ \lambda_k^\gamma \leqslant \mu\}.$$

Assume that $\beta > -1/2$. Moreover, assume that there exist positive constants d_0, d_1 and $\gamma \in (0,1)$ such that for every $s, \tau > 0$, with $0 < a \leqslant s < \tau \leqslant T$ and $e \subset [s,\tau]$ of positive measure we have

$$m(e)\|\mathbb{T}_\tau \psi\| \leqslant d_0 e^{d_1 \mu} \int_e \|B^* \mathbb{T}_t \psi\| \, \mathrm{d}t \quad (\psi \in V_{\mu,\gamma}). \tag{2}$$

Then the (A,B) is (E,T) null-controllable for every $T > 0$ and every set of positive measure $E \subset [0,T]$.

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A heat equation with Dirichlet boundary control

Theorem.(Micu, Roventa,Tucsnak (2011)) Let $\Omega \subset \mathbb{R}^N$ be a rectangular domain, let $T > 0$ and let $E \subset [0, T]$ be a set of positive measure.Then the system

$$\frac{\partial z}{\partial t} = \Delta z \text{ in } \Omega \times (0, \infty),$$

$$z = u \quad \text{on } \Gamma \times (0, \infty),$$

$$z = 0 \quad \text{on} \quad (\partial\Omega \setminus \Gamma) \times (0, \infty),$$

$$z(x, 0) = \psi(x) \text{ for } x \in \Omega,$$

is (E, T) null controllable in the state space $X = H^{-1}(\Omega)$.

Corollary. The time optimal controls for the above system are bang-bang.

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Main ingredients of the proof

First inequality.(Turàn, 1946) Let $0 < \alpha < \beta < \pi$. Then there exist $c_0,\ c_1 > 0$ such that for every $N \in \mathbb{R}^N$ we have

$$c_0 e^{c_1 N} \int_\alpha^\beta \left| \sum_{k=1}^N a_k \sin(kx) \right|^2 \mathrm{d}x \geqslant \sum_{k=1}^N |a_k|^2 \quad (a_k) \in \mathbb{R}^N.$$

Second inequality.(Borwein and Erdelyi, 1997) Let $T > 0$, $E \subset [0, T]$ of positive measure. Then there exists $K = K(E, T)$ such that

$$K(E,T) \int_E \left| \sum_{n \geqslant 1} a_n e^{-n^2 t} \right| \mathrm{d}t \geqslant \left(\sum_{n \geqslant 1} |a_n|^2 e^{-2n^2 T} \right)^{\frac{1}{2}} \qquad (a_n) \in l^2.$$

Stability of dissipative systems with intermittent damping

Problem statement

Consider a system of the form $\dot{z} = Az + Bu$, B bounded, and assume that there exists a stabilizing feedback law $u = u^* = Kz$. Consider now the system

$$\dot{z} = Az + \alpha(t)Bu, \tag{1}$$

where the *signal* α takes values in $[0, 1]$ and $\alpha(t) = 0$ for certain times t (i.e., the control may be switched off over possibly non-negligible subsets of time). Under which conditions imposed on α is the closed-loop system (1) with the same control u^* stable (in an appropriate sense)?

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Persistent signals in the skew-adjoint case

Assume that $A = -A^*$, $B \in \mathcal{L}(U, X)$, and take the stabilizing feedback law $u^* = -B^* z$. Consider now the system

$$\dot{z}(t) = Az(t) - \alpha(t) BB^* z(t), \tag{2}$$

where the *signal* α takes values in $[0, 1]$ and it is supposed to be self-persistent, i.e., there exist T, $\mu > 0$ such that

$$\int_t^{t+T} \alpha(s) \, \mathrm{d}s \geq \mu.$$

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Main result

For $T > 0$ and α as above we define $\Psi_{T,\alpha} \in \mathcal{L}(X, L^2([0,T],U))$ by

$$[\Psi_{T,\alpha} z](t) = B^* \mathbb{T}_t z \qquad (t \in [0,T], z \in X),$$

where $\mathbb{T}$ is the semigroup generated by A.

Theorem. (Hante, Sigalotti, Tucsnak (2011))

1. Assume that there exists $\vartheta > 0$ such that for all T-μ PE-signals $\alpha(\cdot)$ $\Psi_{\vartheta,\alpha}$ is one to one. Then the system (2) is weakly stable.

2. Assume that there exists $\vartheta > 0$ $\Psi_{\vartheta,\alpha}$ is bounded from below, uniformly with respect all T-μ PE-signals $\alpha(\cdot)$. Then the system (2) is exponentially stable.

If $\alpha(t) \in \{0,1\}$, the above conditions can be seen as (E,ϑ) observability properties (which are dual to (E,ϑ) controllability).

A Schrödinger equation with distributed damping

Let $\Omega \subset \mathbb{R}^N$ be smooth and consider the internally damped Schrödinger equation

$$
\begin{aligned}
i\dot{z}(t,x) + \Delta z(t,x) + i\alpha(t)d(x)^2 z(t,x) &= 0, & (t,x) \in (0,\infty)\times\Omega, \quad (1)\\
z(t,x) &= 0, & t \in (0,\infty)\times\partial\Omega, \quad (2)\\
z(0,x) &= z_0(x), & t \in \Omega, \quad (3)
\end{aligned}
$$

where $\alpha(\cdot)$ is a T-μ PE-signal. Assume that there exist $d_0 > 0$ and an open nonempty $\omega \subset \Omega$ such that

$$|d(x)| \geq d_0 \text{ for a. e. } x \text{ in } \omega.$$

The above system is weakly stable. Nothing is known on stronger stability properties.

A wave equation with distributed damping

The everywhere intermitently damped N-dimensional wave equation

$$\begin{aligned}
\ddot{v}(t,x) &= \Delta v(t,x) - \alpha(t) d(x)^2 \dot{v}(t,x), && (t,x) \in (0,\infty) \times \Omega, \\
v(0,x) &= v_0(x), && x \in \Omega, \\
\dot{v}(0,x) &= y_1(x), && x \in \Omega, \\
v(t,x) &= 0, && (t,x) \in (0,\infty) \times \partial\Omega,
\end{aligned}$$

where Ω is a bounded domain in $\mathbb{R}^N$ and $d \in L^\infty(\Omega)$ satisfies

$$|d(x)| \geq d_0 > 0 \quad \text{for almost all } x \in \Omega,$$

is exponentially stable.

If the damping is not acting in the entire Ω we do not have weak stabilitity.

Conclusions and open questions

- The controllability of infinite dimensional systems by means of controls which do not necessarily act on intervals is important in the study of time optimal control and of some switched systems.

- This is a difficult mathematical problem, essentially open for Schrödinger type equations (even in one space dimension) and for the heat equation controlled from the boundary.

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