Byrnes-Isidori form for infinite dimensional systems

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joint work with A. Ilchmann (Ilmenau) and B. Jacob (Wuppertal)

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Assumptions

$$\begin{aligned}
\dot{z}(t) &= Az(t) + u(t)b, \quad z(0) = z_0, \\
y(t) &= (z(t), c).
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4. Relative degree r:

$$b \in \text{dom } A^r \quad \text{and} \quad c \in \text{dom } {A^*}^r$$

and

$(A^{r-1}b, c) \neq 0$ and for $j = 0, 1, \ldots, r-2$ we have $(A^j b, c) = 0$.

Decomposition of H

Lemma

Assume we have relative degree r. Then

$$H = ls\{c\} \dot{+} ls\{A^*c\} \dot{+} \cdots \dot{+} ls\{{A^*}^{r-1}c\} \dot{+} H_0, \qquad (1)$$

where

$$H_0 := \{b\}^\perp \cap \{Ab\}^\perp \cap \cdots \cap \{A^{r-1}b\}^\perp. \qquad (2)$$

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Theorem

With respect to $H = ls\{c\} \dot{+} ls\{A^*c\} \dot{+} \cdots \dot{+} ls\{{A^*}^{r-1}c\} \dot{+} H_0$*:*

$$x = (P^0 x)c + (P^1 x)A^*c + \cdots + (P^{r-1}x){A^*}^{r-1}c + P_{H_0}x,$$

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$$P_j^m x := \left(\frac{\left({A^*}^{j-1}c, A^{r-(m+1)}b\right)}{(c, A^{r-1}b)} - \sum_{k=m+2}^{j-1} P_k^m {A^*}^{j-1}c \right) \frac{(x, A^{r-j}b)}{(c, A^{r-1}b)},$$

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$$P_{H_0}x := x - \sum_{j=0}^{r-1} (P^j x){A^*}^j c.$$

Byrnes-Isidori form

Set dom $\widehat{A} := \mathbb{R}^r \times (H_0 \cap \text{dom } A)$ and define $\widehat{A}$ in $\mathbb{R}^r \times H_0$

$$\widehat{A} = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & & 1 & 0 \\ P^0 A^{*^r} c & P^1 A^{*^r} c & \cdots & P^{r-1} A^{*^r} c & S \\ R & 0 & \cdots & 0 & Q \end{bmatrix}$$

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Theorem

$$U : H \to \mathbb{R}^r \times H_0, \qquad x \mapsto Ux := \begin{pmatrix} P^0 x \\ \vdots \\ P^{r-1} x \\ P_{H_0} x \end{pmatrix}.$$

Then we have

$$AU^* = U^* \widehat{A}.$$

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Rewrite the system in Byrnes-Isidori form

Theorem

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Funnel

Let φ smooth, $\varphi(0) = 0$, $\varphi(t) > 0$, $\liminf_{t\to\infty} \varphi(t) > 0$.

$$\mathcal{F}_\varphi := \mathrm{graph}\ \Big(t \mapsto \big\{z \in \mathbb{R} \big| \ \ \varphi(t)|z| < 1\big\}\Big).$$

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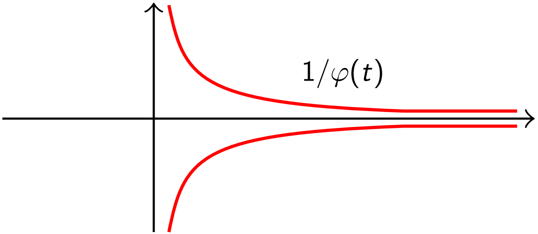

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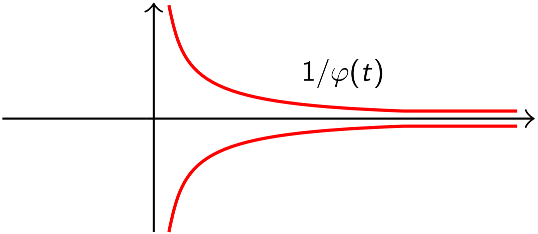

For a cont. differentiable bounded reference y_{ref} let

$$u(t) = -k(t)\,\operatorname{sgn}(b, c)\, e(t), \ k(t) = \frac{1}{1-\varphi(t)|e(t)|}, \ e(t) = y(t) - y_{\text{ref}}(t).$$

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Assume A_4 is the generator of an exponentially stable semigroup

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$$e(t) \in \mathcal{F}_\varphi.$$

Example: Heat equation

$$\dot{z}(t) = \frac{d^2 z(t)}{dx^2} + u(t)b, \quad z(0) = z_0,$$

$$\frac{dz(t)}{dx}(0) = \frac{dz(t)}{dx}(0) = 0,$$

$$y(t) = (z(t), c)_{L^2(0,1)}.$$

$c(x) \equiv 1$, b smooth with compact support in $(0,1)$. $(b, c) \neq 0$, i.e. relative degree 1.

Thank you.