Quasi-hyperbolic semigroups

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The class of *contraction* (power bounded) operators T (or operator semigroups) on X :

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is **comparatively** well-understood.

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Our aim is to try to understand an opposite class of *expansion* operators (operator semigroups) satisfying

$$\|T^n x\| \ge c\|x\|$$

at least in a certain sense to be made precise.

Hyperbolic operators

Definition A bounded linear operator T on a Banach space X is said to be *hyperbolic* if

$$X = X_s \oplus X_u,$$

where X_s and X_u are closed T-inv. subspaces of X, $T \restriction_{X_u}$ is invertible, and

$$\|(T \restriction_{X_s})^n\| \leq \frac{1}{2}, \qquad \|(T \restriction_{X_u})^{-n}\| \leq \frac{1}{2} \quad \text{for some } n \in \mathbb{N}.$$

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In other words, for some $\alpha < 1$ and $\beta > 1$,

$$\|T^n x\| \le C\alpha^n \|x\| \quad (x \in X_s, n \in \mathbb{N}), \quad \|T^n x\| \ge c\beta^n \|x\| \quad (x \in X_u, n \in \mathbb{N})$$

Note:

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T is hyperbolic if and only if $\sigma(T) \cap \Gamma = \emptyset$ (Γ is the unit circle).

One of motivations

Theorem [Krein] A difference equation

$$x_{n+1} = Tx_n + b_n, \qquad n \in \mathbb{Z},$$

admits a unique solution in $l^\infty(\mathbb{Z}, X)$ for every $(b_n)_{n\in\mathbb{Z}} \in l^\infty(\mathbb{Z}, X)$ if and only if T is hyperbolic.

Here $l^\infty(\mathbb{Z}, X)$ can be replaced by a variety of other spaces.

Observation If T is hyperbolic and Y is a closed T-invariant subspace of X, then $T\restriction_Y$ may not be hyperbolic, but each non-zero orbit contracts exponentially or expands exponentially, with uniform exponent.

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Definition (Eisenberg, Hedlund (1970)): Assume T is invertible.

- T is *expansive* if for each x there exists $n_x \in \mathbb{Z}$ such that

$$\|T^{n_x}x\| \geq 2\|x\|;$$

- T is *uniformly expansive* if there exists $n \in \mathbb{N}$ (independent of x) such that

$$\max(\|T^n x\|, \|T^{-n}x\|) \geq 2\|x\|$$

for all x.

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Hedlund (1971):

$$\text{T is uniformly expansive} \iff \sigma_{\mathrm{ap}}(T) \cap \Gamma = \emptyset$$
$$\iff \|(T-\lambda)x\| \geq c\|x\| \quad (x \in X, \lambda \in \Gamma)$$

Quasi-hyperbolic operators

T is not necessarily invertible

Definition T is *quasi-hyperbolic* if there exists $n \in \mathbb{N}$ (independent of x) such that

$$\max\left(\|T^{2n}x\|, \|x\|\right) \geq 2\|T^n x\|$$

for all $x \in X$.

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Elementary properties

- T hyperbolic $\Longrightarrow$ T quasi-hyperbolic
- T is uniformly expansive $\Longleftrightarrow$ T is quasi-hyperbolic and invertible
- T quasi-hyperbolic $\Longrightarrow$ $T\restriction_Y$ quasi-hyperbolic
- T is quasi-hyperbolic $\Longrightarrow$ $\sigma_{\mathrm{ap}}(T) \cap \Gamma = \emptyset$

Theorem (Read 1986,88; Müller 1988) Let T be a bounded linear operator on X. There is a Banach space Y and a bounded operator S on Y such that X is isometrically embedded in Y, $S\restriction_X = T$, $\|S\| = \|T\|$ and $\sigma(S) = \sigma_{\mathrm{ap}}(T)$.

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Corollary
T is the restriction of a hyperbolic operator to a closed invariant subspace.
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T is the restriction of a hyperbolic operator to a closed invariant subspace.
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T is quasi-hyperbolic $\Longleftrightarrow \sigma_{\mathrm{ap}}(T) \cap \Gamma = \emptyset$

Examples

1. Weighted shifts (Ridge, 1970)

$$X = l^2(\mathbb{Z}), \ \ S_w(x) = (w_n x_{n-1})_{n\in\mathbb{Z}}, \quad x = (x_n)_{n\in\mathbb{Z}} \in X$$

The *spectrum* of S_w is an annulus centered at 0.

The *approximate point spectrum* is either equal to the spectrum, or it is the union of two annuli.

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If

$$w_n = \begin{cases} 2, & n \geq 0 \\ \frac{1}{2}, & n < 0 \end{cases}$$

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If

$$w_n = \begin{cases} \frac{1}{2} & , n \geq 0 \\ 2 & , n < 0 \end{cases}$$

then $\sigma_{\mathrm{ap}}(S_w^r) = \{\lambda : \frac{1}{2} \leq |\lambda| \leq 2\}$.

2. Wave equations (Cooper, Koch 1995)

The problem

$$\begin{aligned} \Omega &= \left\{(x,t) \in \mathbb{R}^2_+ : 0 < x < 1 + \frac{\sin(\pi t)}{2\pi}\right\} \\ u_{tt} - u_{xx} &= 0 \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \\ u(\cdot, 0) &= f \in W_0^{1,2}(0,1) \\ u_t(\cdot, 0) &= g \in L^2(0,1) \end{aligned}$$

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For the monodromy operator $U(2,0) : (f,g) \mapsto (u(\cdot,2), u_t(\cdot,2))$ on $X = W_0^{1,2} \times L^2$:

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\sigma\left(U(2,0)\right) = \{\lambda : \frac{1}{\sqrt{3}} \le |\lambda| \le \sqrt{3}\}, \quad \sigma_{\mathrm{ap}}\left(U(2,0)\right) \cap \Gamma = \emptyset;
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$U(2,0)$ is quasi-hyperbolic and the energy $\|U(t,0)x\|^2, x \in X \setminus \{0\}$, grows exponentially in either forward or backward time.

3. Hyperbolic and quasi-hyperbolic operators appear naturally in the **smooth dynamics on manifolds** (operator-theoretical characterization of Anosov and quasi-Anosov maps; Mather-Manẽ theory)

Skip.

A question:

Is there a nice condition which characterises those operators T on X such that

- there exists a hyperbolic operator S on a Banach space Y
- X is continuously embedded in Y and
- $T = S \restriction_X$?

Hyperbolic Semigroups

Definition A C_0-semigroup $\mathcal{T} = \{T(t) : t \geq 0\}$ (with generator A) is hyperbolic if there is a splitting

$$X = X_s \oplus X_u,$$

where X_s and X_u are closed $\mathcal{T}$-invariant subspaces of X, $T(t) \restriction_{X_u}$ is invertible for some (or all) $t > 0$, and

$$\|T(t) \restriction_{X_s}\| < \frac{1}{2}, \quad \|(T(t) \restriction X_u)^{-1}\| < \frac{1}{2}$$

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For Hilbert spaces, $\mathcal{T}$ is hyperbolic $\Leftrightarrow$ $A - is$ is invertible for each $s \in \mathbb{R}$ and $\sup_{s\in\mathbb{R}} \|(A - is)^{-1}\| < \infty$ (Gearhart-Prüss).

A motivation for hyperbolicity

Theorem[Krein, Daletskii, Latushkin, Pruess, Schnaubelt, Zhikov, ...]
If A is the generator of a C_0-semigroup $\mathcal{T}$ then

$$x'(t) = Ax(t) + f(t), \qquad t \in \mathbb{R},$$

admits the unique bounded (mild) continuous solution on $\mathbb{R}$ for every bounded continuous f if and only if $\mathcal{T}$ is hyperbolic.

Quasi-hyperbolic Semigroups

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Properties:

$$\begin{aligned}
\mathcal{T} \text{ is quasi-hyperbolic} &\iff T(1) \text{ is quasi-hyperbolic} \\
&\iff \mathcal{T} \text{ is a restriction of} \\
&\ \text{a hyperbolic semigroup} \\
&\iff \sigma_{\mathrm{ap}}(T(1)) \cap \Gamma = \varnothing \\
&\implies \|(A - is)x\| \geq c\|x\|, s \in \mathbb{R}, \\
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Basic examples: weighted shift semigroups on $L^p(\mathbb{R})$

Remark *There exist a semigroup $\mathcal{T} = (T(t))_{t\geq 0}$ such that $\mathcal{T}$ is not quasi-hyperbolic, but A satisfies lower bounds*

$$\|(A - is)x\| \geq c\|x\|, \quad (s \in \mathbb{R}, x \in D(A)) :$$

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Let $a > 2q/p, \quad 1 < p < 2 < q < \infty$

$$X := L_p(\mathbb{R}, e^{2x} dx) \cap L_q(\mathbb{R}, w(x) dx), \quad w(x) := \begin{cases} e^{ax} & (x \leq 0), \\ 1 & (x > 0), \end{cases}$$

$$\|f\|_X = \left\{ \int_{\mathbb{R}} |f(x)|^p e^{2x}\, dx \right\}^{1/p} + \left\{ \int_{\mathbb{R}} |f(x)|^q w(x)\, dx \right\}^{1/q}.$$

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Let $\qquad (T(t)f)(s) = f(s+t)\ (s, t \in \mathbb{R}).$

Then $\qquad \sigma(A) \cap i\mathbb{R} = \emptyset, \quad \sup_{s\in\mathbb{R}} \|(is - A)^{-1}\| < \infty.$

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Then $\qquad \sigma(A) \cap i\mathbb{R} = \emptyset, \quad \sup_{s\in\mathbb{R}} \|(is - A)^{-1}\| < \infty$. However,

$$\forall t > 0 \quad \exists f \in X, \|f\| = 1 : \quad \|T(-t)f\|_X < 2\|f\|_X, \quad \|T(t)f\|_X < 2\|f\|_X.$$

Characterisations of quasi-hyperbolicity

Theorem a) Let $\mathcal{T}$ be a C_0-semigroup on a Hilbert space X with generator A. Then $\mathcal{T}$ is quasi-hyperbolic if and only if A satisfies lower bounds

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b)If X is a Banach space then $\mathcal{T}$ is quasi-hyperbolic if and only if the multiplication operator

$$(M_{A-i\cdot}f)(s) = (A - is)f(s)$$

is a *lower Fourier multiplier* on $L^p(\mathbb{R}, X), 1 \leq p < \infty$, i.e.

$$\|(\mathcal{F}^{-1}M_{A-i\cdot}\mathcal{F})f(s)\|_{L^p} \geq c\|f\|_{L^p}$$

for all Schwartz functions $f : \mathbb{R} \mapsto D(A)$, where $\mathcal{F}$ is the Fourier transform on $L^1(\mathbb{R}, X)$.

What do lower bounds for A imply?

For simplicity of statement, assume that $(T(t))_{t \in \mathbb{R}}$ is a C_0-group, i.e., each $T(t)$ is invertible.

Theorem Let A be the generator of a C_0-group $(T(t))_{t \in \mathbb{R}}$ on a Banach space X, and assume that

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Then for each non-zero x,

(i) $\|T(t)x\|$ grows faster than polynomially either as $t \to \infty$ or as $t \to -\infty$, and

(ii) There exists $\epsilon_x > 0$ such that

$$\int_{-\infty}^{\infty} \|T(t)x\| e^{-\epsilon_x |t|}\, dt = \infty.$$

Continuous embedding ?

If A satisfies

$$\|(A - is)x\| \geq c\|x\| \qquad (s \in \mathbb{R}, x \in D(A))$$

can X be continuously embedded in a space Y in such a way that there is a hyperbolic C_0-semigroup $\{S(t) : t \geq 0\}$ on Y such that $T(t) = S(t) \restriction_X$?

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If $(T(t))_{t \geq 0}$ has growth bound 0 (the spectral radius of $T(t)$ is 1) and A satisfies the condition above, then T is not quasi-hyperbolic, but such growth does occur (in negative time).

THANK YOU FOR YOUR ATTENTION !

M compact Riemann manifold, with tangent bundle TM, φ a diffeomorphism of M.

Definition φ is Anosov if $TM = TM_s \oplus TM_u$ where $D\varphi$ contracts TM_s exponentially in positive time and contracts TM_u exponentially in negative time.

$C(TM)$ Banach space of continuous sections of TM (with sup norm)

Define push-forward operator on $C(TM)$:

$$(E_\varphi f)(\theta) = D\varphi(\varphi^{-1}\theta)f(\varphi^{-1}\theta) \qquad (\theta \in M)$$

Mather (1968): φ is Anosov if and only if E_φ is hyperbolic.

Definition φ is quasi-Anosov if, for all $\theta \in M$ and all non-zero $x \in TM_\theta$,

$$\{(D_\varphi)^n(\theta)x : n \in \mathbb{Z}\}$$

is unbounded.

Manẽ (1977): φ is quasi-Anosov if and only if M can be embedded in a manifold N on which φ can be extended to an Anosov diffeomorphism.

Moreover, φ is quasi-Anosov if and only if $\sigma_{\mathrm{ap}}(E_\varphi) \cap \Gamma = \emptyset$, i.e.,

E_φ is quasi-hyperbolic