

On the Past Time Final State and Initial State Future Time Optimal Control Problems

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The Continuous Time Linear System

This talk is about optimal control of a linear time-invariant i/s/o (input/state/output) systems whose dynamics is described by an equation of the type

$$\Sigma : \begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad t \in \mathbb{R}^+, \quad x(0) = x_0. \quad (1)$$

It has a

input space \mathcal{U} (a Hilbert space),
state space \mathcal{X} (a Hilbert space),
output space \mathcal{Y} (a Hilbert space).

The operator S is supposed to be a **system node** (or more generally, an **operator node**). It need not be well-posed.

A system node S has a **main operator** A , a **control operator** B , an **observation operator** C , and a **transfer function** $\widehat{\mathfrak{D}}$ defined on $\rho(A)$.

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Two Cost Minimization Problems

- In the **initial state future time** cost minimization problem we fix an initial state $x_0 \in \mathcal{X}$ and minimize the future cost

$$J_{\text{fut}}(x_0, u, y) = \int_0^{\infty} (\|u(t)\|_{\mathcal{U}}^2 + \|y(t)\|_{\mathcal{Y}}^2) dt, \quad (2)$$

over a suitable set of generalized stable future trajectories $\begin{bmatrix} x \\ u \\ y \end{bmatrix}$ of Σ with the given initial state $x(0) = x_0$. (This cost may be zero, or finite and nonzero, or $+\infty$.)

- In the **past time final state** cost minimization problem we fix a final state $x_0 \in \mathcal{X}$ and minimize the past cost

$$J_{\text{past}}(x_0, u, y) = \int_{-\infty}^0 (\|u(t)\|_{\mathcal{U}}^2 + \|y(t)\|_{\mathcal{Y}}^2) dt, \quad (3)$$

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Classical Stable Trajectories of (1)

Recall the original equation:

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad t \in \mathbb{R}^+. \quad (1)$$

Definition

Let $S: [\mathcal{X}] \supset \text{Dom}(S) \rightarrow [\mathcal{Y}]$ be a closed operator.

- 1 A triple (x, u, y) is called a **classical solution** of (1) on the interval I (where $I = \mathbb{R}^+$ or $I = \mathbb{R}^-$) if $x \in C^1(I; \mathcal{X})$, $u \in C(I; \mathcal{U})$, $y \in C(I; \mathcal{Y})$, and (1) holds.
 - 2 This trajectory is (externally) **stable** if, in addition, $u \in L^2(I; \mathcal{U})$ and $y \in L^2(I; \mathcal{Y})$.
- Note that we do not require the **state** $x(t)$ to be bounded (because this is irrelevant at the moment).

Generalized Future Trajectories (Motivation)

- By taking Laplace transforms in the equation

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad t \in \mathbb{R}^+, \quad x(0) = x_0, \quad (1)$$

we find that the Laplace transform of a classical stable future trajectory (x, u, y) satisfies

$$\begin{bmatrix} \lambda \hat{x}(\lambda) - x_0 \\ \hat{y}(\lambda) \end{bmatrix} = S \begin{bmatrix} \hat{x}(\lambda) \\ \hat{u}(\lambda) \end{bmatrix}, \quad \Re \lambda > 0.$$

- For each $\lambda \in \rho(A) \cap \mathbb{C}^+$ we can solve for $\hat{x}(\lambda)$ and $\hat{y}(\lambda)$ in terms of x_0 and $\hat{u}(\lambda)$ to get

$$\hat{x}(\lambda) = (\lambda - A)^{-1}x_0 + (\lambda - A|_X)^{-1}B\hat{u}(\lambda), \quad (4)$$

$$\hat{y}(\lambda) = C(\lambda - A)^{-1}x_0 + \hat{\mathcal{D}}(\lambda)\hat{u}(\lambda). \quad (5)$$

- In the sequel we ignore (4) but use (5) as a definition of a generalized stable future trajectory of Σ . (The cost $J_{\text{fut}}(x_0, u, y)$ depends only on x_0 , u , and y .)

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Fixing the Transfer Function

Throughout the rest of this talk I **fix some (connected) component Ω of $\rho(A) \cap \mathbb{C}^+$** . (If $\rho(A) \cap \mathbb{C}^+$ is connected, then $\Omega = \rho(A) \cap \mathbb{C}^+$.)

For example, $\Omega = \Omega_\infty =$ the component of $\rho(A)$ which contains some right half-plane.

Generalized Future Trajectories (Definition)

Definition

By the set of **generalized stable future trajectories** of Σ we mean

the set of all triples $\begin{bmatrix} x_0 \\ u \\ y \end{bmatrix} \in \begin{bmatrix} \mathcal{X} \\ L^2(\mathbb{R}^+; \mathcal{U}) \\ L^2(\mathbb{R}^+; \mathcal{Y}) \end{bmatrix}$ which satisfy

$$\hat{y}(\lambda) = \hat{\mathcal{D}}(\lambda)\hat{u}(\lambda) + C(\lambda - A)^{-1}x_0, \quad \lambda \in \Omega, \quad (6)$$

where \hat{u} and \hat{y} are the Laplace transforms of u and y , respectively.

- We call x_0 the **initial state**, u the **input component**, and y the **output component**.
- Note that we here do not actually define the state component $x(t)$ of the trajectory for $t > 0$, but only for $t = 0$.
- However, the input u and output y are almost everywhere defined L^2 -functions.

Solution of Initial State Future Time Cost Minimization Problem

Because of the way in which I have defined the notion of a “generalized stable future trajectory of Σ ” (and the assumption that S is a system node), the following result is true:

- The minimum of the future cost function

$$J_{\text{fut}}(x_0, u, y) = \int_0^{\infty} (\|u(t)\|_{\mathcal{U}}^2 + \|y(t)\|_{\mathcal{Y}}^2) dt, \quad (2)$$

is always achieved at some generalized stable future trajectory (x_0, u, y) of Σ .

- Thus, I can compute the future cost $\|x_0\|_{\text{fut}}^2$ of every possible initial state x_0 .
- This cost may be $+\infty$, or it may be zero, or it may be finite and nonzero.

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Generalized Past Trajectories

We define the notion of a **general stable past trajectory** in a slightly different way, by taking the closure in $\left[\begin{array}{c} L^2(\mathbb{R}^+; \mathcal{U}) \\ L^2(\mathbb{R}^+; \mathcal{Y}) \end{array} \right]^{\mathcal{X}}$ of the span \mathfrak{V}_- of all classical exponential past trajectories:

$$\mathfrak{V}_- := \text{span} \left\{ \left[\begin{array}{c} x_0 \\ u \\ y \end{array} \right] = \left[\begin{array}{c} (\lambda - A|_{\mathcal{X}})^{-1} B u_0 \\ e_\lambda u_0 \\ e_\lambda \widehat{\mathcal{D}}(\lambda) u_0 \end{array} \right] \mid \begin{array}{l} \lambda \in \Omega, \\ u_0 \in \mathcal{U} \end{array} \right\}. \quad (7)$$

Here e_λ is the function

$$e_\lambda(t) = e^{\lambda t}, \quad t \in \mathbb{R}^-.$$

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States with Finite or Nonzero Costs

We now encounter the following two crucial questions:

- Which initial states $x_0 \in \mathcal{X}$ have a finite future cost $\|x_0\|_{\text{fut}}^2$?
- Which final states $x_0 \in \mathcal{X}$ have a nonzero past cost $\|x_0\|_{\text{past}}^2$?

It turns out that the answer to these questions are related to the following three questions:

- Does the transfer function $\hat{\mathcal{D}}$ have a right H^∞ factorization over \mathbb{C}^+ ?
- Does the transfer function $\hat{\mathcal{D}}$ have a left H^∞ factorization over \mathbb{C}^+ ?
- Does the transfer function $\hat{\mathcal{D}}$ have a doubly coprime H^∞ factorization over \mathbb{C}^+ ?

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Definition

- 1 $\widehat{\mathcal{D}}$ has a **right $H^\infty(\mathbb{C}^+)$ factorization** valid in Ω if there exist two functions $M \in H^\infty(\mathbb{C}^+; \mathcal{L}(\mathcal{U}))$ and $N \in H^\infty(\mathbb{C}^+; \mathcal{L}(\mathcal{U}; \mathcal{Y}))$ such that $M(\lambda)$ has a bounded inverse for all $\lambda \in \Omega$ and $\widehat{\mathcal{D}}(\lambda) = N(\lambda)M(\lambda)^{-1}$ for all $\lambda \in \Omega$.
- 2 The factorization in (i) is **normalized** if the function $\begin{bmatrix} M \\ N \end{bmatrix}$ is inner.
- 3 The factorization in (i) is **weakly coprime** if N and M have no common right H^∞ factors.

Definition

- 1 $\widehat{\mathcal{D}}$ has a **left $H^\infty(\mathbb{C}^+)$ factorization** valid in Ω if there exist two functions $\widetilde{M} \in H^\infty(\mathbb{C}^+; \mathcal{L}(\mathcal{Y}))$ and $\widetilde{N} \in H^\infty(\mathbb{C}^+; \mathcal{L}(\mathcal{U}; \mathcal{Y}))$ such that $\widetilde{M}(\lambda)$ has a bounded inverse for all $\lambda \in \Omega$ and $\widehat{\mathcal{D}}(\lambda) = \widetilde{M}(\lambda)^{-1}\widetilde{N}(\lambda)$ for all $\lambda \in \Omega$.
- 2 The factorization in (i) is **normalized** if the operator function $\begin{bmatrix} \widetilde{N} & \widetilde{M} \end{bmatrix}$ is co-inner.
- 3 The factorization in (i) is **weakly coprime** if \widetilde{N} and \widetilde{M} have no common left H^∞ factors.

Doubly Coprime (Bezout) Factorizations

Definition

$\widehat{\mathcal{D}}$ has a **doubly coprime (Bezout) $H^\infty(\mathbb{C}^+)$ factorization** valid in Ω if there exist functions $M \in H^\infty(\mathbb{C}^+; \mathcal{L}(\mathcal{U}))$, $N \in H^\infty(\mathbb{C}^+; \mathcal{L}(\mathcal{U}; \mathcal{Y}))$, $\tilde{X} \in H^\infty(\mathbb{C}^+; \mathcal{L}(\mathcal{U}))$, $\tilde{Y} \in H^\infty(\mathbb{C}^+; \mathcal{L}(\mathcal{Y}; \mathcal{U}))$, $\tilde{M} \in H^\infty(\mathbb{C}^+; \mathcal{L}(\mathcal{Y}))$, $\tilde{N} \in H^\infty(\mathbb{C}^+; \mathcal{L}(\mathcal{U}; \mathcal{Y}))$, $X \in H^\infty(\mathbb{C}^+; \mathcal{L}(\mathcal{Y}))$ and $Y \in H^\infty(\mathbb{C}^+; \mathcal{L}(\mathcal{U}; \mathcal{Y}))$ such that $\begin{bmatrix} M \\ N \end{bmatrix}$ is a right $H^\infty(\mathbb{C}^+)$ factorization valid in Ω , $[\tilde{M}, \tilde{N}]$ is a left $H^\infty(\mathbb{C}^+)$ factorization valid in Ω and

$$\begin{bmatrix} M & Y \\ N & X \end{bmatrix} \begin{bmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{bmatrix} = \begin{bmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & Y \\ N & X \end{bmatrix} = \begin{bmatrix} 1_{\mathcal{Y}} & 0 \\ 0 & 1_{\mathcal{U}} \end{bmatrix}$$

on \mathbb{C}^+ .

The Input Finite Future Cost Condition

Definition

- 1 The system Σ satisfies the **input finite future cost condition** at the point $\alpha \in \Omega$ if $(\alpha - A|_{\mathcal{X}})^{-1}Bu_0$ has a finite future cost for every $u_0 \in \mathcal{U}$.
- 2 The system Σ satisfies the **finite future cost condition** if every initial state in \mathcal{X} has a finite future cost.

Note that $(\alpha - A|_{\mathcal{X}})^{-1}Bu_0$ is the **state at time zero** of the stable classical **past exponential trajectory** corresponding to the input function $u(t) = e^{\alpha t}u_0$.

- Out of these the **finite future cost condition** (= “optimizability”) is the **standard assumption** in papers dealing with the initial state future time cost minimization problem.
- However, the **important condition** is not the finite future cost condition but the **input finite future cost condition**.

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Theorem

The following conditions are equivalent for the system Σ :

- ① Σ satisfies the **input finite future cost** condition at some point (or equivalently, at every point) $\alpha \in \Omega$.
- ② The **control Riccati equation** for Σ has an α -normalized nonnegative solution for some (or equivalently, for all) $\alpha \in \Omega$.
- ③ The transfer function $\widehat{\mathcal{D}}$ of Σ has a **right H^∞ -factorization** valid in some open subset of Ω .
- ④ The transfer function $\widehat{\mathcal{D}}$ of Σ has a **normalized weakly coprime right H^∞ -factorization** valid in Ω .

When these equivalent conditions hold, then the **optimal future cost** is equal to the **minimal** α -normalized nonnegative solution of the continuous time **control Riccati equation** for all $\alpha \in \Omega$.

Definition

Let $S = \begin{bmatrix} S_0 \\ S_1 \end{bmatrix}$ be a system node with main operator A , and control operator B , and let $\alpha \in \rho(A) \cap \mathbb{C}^+$. By an α -normalized solution of the (generalized) continuous time **control Riccati equation** induced by S we mean a **closed nonnegative sesquilinear symmetric form** q on \mathcal{X} with **domain** \mathcal{Z} satisfying the following conditions:

- 1 $(\alpha - A)^{-1}\mathcal{Z} \subset \mathcal{Z}$ and $(\alpha - A|_{\mathcal{X}})^{-1}BU \subset \mathcal{Z}$;
- 2 q satisfies the “natural” α -normalized control Riccati equation (see next two pages).

Interpretation of Control Riccati Equation

- The **classical interpretation** of the control Riccati equation is that we are looking for a **feedback pair** $S_2 \begin{bmatrix} x \\ u \end{bmatrix} = Kx - u$ and a nonnegative **Riccati operator** Q such that the solution of the equation

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} S_0 \\ S_1 \\ S_2 \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}$$

satisfies the **energy balance equation**

$$\frac{d}{dt} \left\| Q^{1/2} x(t) \right\|_x^2 + \|y(t)\|_y^2 + \|u(t)\|_u^2 = \|v(t)\|_u^2,$$

- The **optimal solution** to the forward cost minimization with initial state $x(0) = x_0$ is obtained for the input u for which $v(t) \equiv 0$, i.e., $u(t) = Kx(t)$.
- Thus, the minimizing input $u(t)$ is of feedback type $u(t) = Kx(t)$.

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The Control Riccati Equation (continues)

Formally: the statement that **the “natural” α -normalized control Riccati equation holds**” means that

- 2 There exists an operator $S_2: \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \rightarrow \mathcal{U}$ with

$$\text{Dom}(S_2) = \left\{ \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} \in \text{Dom}(S_0) \mid x_0 \in \mathcal{Z} \text{ and } S_0 \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} \in \mathcal{Z} \right\} \quad (8)$$

such that for all $\begin{bmatrix} x_0 \\ u_0 \end{bmatrix} \in \text{Dom}(S_2)$,

$$2\Re q \left[S_0 \begin{bmatrix} x_0 \\ u_0 \end{bmatrix}, x_0 \right]_{\mathcal{X}} + \left\| S_1 \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} \right\|_{\mathcal{Y}}^2 + \|u_0\|_{\mathcal{U}}^2 = \left\| S_2 \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} \right\|_{\mathcal{U}}^2, \quad (9)$$

and

$$S_2 \begin{bmatrix} (\alpha - A|_{\mathcal{X}})^{-1} B \\ 1_{\mathcal{U}} \end{bmatrix} \text{ has a bounded inverse in } \mathcal{L}(\mathcal{U}). \quad (10)$$

The Output Coercive Past Cost Condition

Definition

The system Σ satisfies the **output coercive past cost condition** at the point $\alpha \in \Omega$ if there exists a constant $M > 0$ such that

$$\|C(\alpha - A)^{-1}x_0\|_y^2 \leq M(\|u\|_{L^2(\mathbb{R}^-; \mathcal{U})}^2 + \|y\|_{L^2(\mathbb{R}^-; \mathcal{Y})}^2) \quad (11)$$

for every generalized stable past trajectory $\begin{bmatrix} x_0 \\ u \\ y \end{bmatrix}$ of Σ .

This is the **dual** of the **input finite future cost condition**:

- the system Σ satisfies the output coercive past cost condition at some point $\alpha \in \Omega$ if and only if the dual system Σ_* (with system node S^*) satisfies the input finite future cost condition at the point $\bar{\alpha} \in \bar{\Omega}$.

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Theorem

The following conditions are equivalent for the system Σ :

- 1 Σ satisfies the **output coercive past cost** condition at some point (or equivalently, at every point) $\alpha \in \Omega$.
- 2 The **filter Riccati equation** for Σ has an α -normalized nonnegative solution for some $\alpha \in \Omega$.
- 3 The transfer function $\widehat{\mathcal{D}}$ of Σ has a **left H^∞ -factorization** valid in some open subset of Ω .
- 4 The transfer function $\widehat{\mathcal{D}}$ of Σ has a **weakly coprime left H^∞ -factorization** valid in Ω .

When these equivalent conditions hold, then the optimal past cost is the **inverse** of the **minimal** α -normalized nonnegative solution of the continuous time **filter Riccati equation** for all $\alpha \in \Omega$.

Filter Riccati equation for $S =$ **control** Riccati equation for S^*

The Past Cost Dominance Condition

Definition

The system Σ satisfies the **past cost dominance condition** (with respect to Ω) if the optimal future cost $\|\cdot\|_{\text{fut}}^2$ is dominated by the optimal past cost $\|\cdot\|_{\text{past}}^2$, i.e., there is a finite constant M such that $\|x\|_{\text{fut}}^2 \leq M\|x\|_{\text{past}}^2$ for every $x \in \mathcal{X}$.

- The **past cost dominance** condition implies both the **input finite future cost** condition and the **output coercive past cost** condition.
- In particular, the past cost dominance condition implies that **both** the control Riccati equation and the filter **Riccati equation** for Σ **have nonnegative solutions** p and q .
- The **converse is not true**.

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Past Cost Dominance \iff Doubly Coprime H^∞ Factorization

Theorem

The following conditions are equivalent for the system Σ :

- 1 Σ satisfies the *past cost dominance condition* (with respect to Ω).
- 2 For some (or equivalently, for all) $\alpha \in \Omega$ the *control Riccati equation* for Σ has an α -normalized nonnegative solution q and the *filter Riccati equation* for Σ has an α -normalized nonnegative solution p , and q is dominated by the inverse of p .
- 3 The transfer function $\hat{\mathcal{D}}$ of Σ has a *doubly coprime H^∞ -factorization* valid in Ω (or equivalently, in some open subset of Ω).

Technical Progress 0: Removal of Irrelevant Assumptions

(Almost) all of the **existing literature** makes at least the following two additional assumptions:

- The transfer function $\hat{\mathcal{D}}$ is defined in some right-half plane, and usually it is even **bounded** in this right-half plane.
- The main operator A generates a strongly continuous semigroup (i.e., S is a **system node**).

However, our removal of these two conditions is **not so significant**.

- The reason why we do not use either of the above assumptions is that **they would not simplify any of the proofs** (on the contrary, they just add irrelevant additional structure which obscures the basic simplicity of the solution).

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Conceptual Advance 1: Unbounded Riccati Operator

A much more significant fact is that we **allow the Riccati operator Q** (or the quadratic form q) to be **unbounded**. This makes it possible to prove simple **necessary and sufficient conditions** for the existence of a coprime factorization.

The literature says:

- The function \hat{D} has a right H^∞ factorization if and only if \hat{D} has a (minimal) realization which satisfies the finite future cost condition.
- Suppose that \hat{D} is the transfer function of some (maybe even well-posed) system Σ which does not satisfy the finite cost condition. Then the above result tells us absolutely nothing.
- However, our new result does apply also in this case, and it says that
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Conceptual Advance 2: Non-Densely Defined Riccati Operator with Nontrivial Kernel

Due to the fact that we allow the Riccati operators Q and P (or the quadratic forms q and p) to have a non-dense domain and a nontrivial kernel we can

- remove all controllability and observability assumptions on the underlying system Σ

Conceptual Advance 3: The Past Time Final State Cost Minimization Problem

- There seems to be **virtually nothing written about the infinite-dimensional past time final state cost minimization problem** in the literature.
- We show that the solution of the past time final state cost minimization problem is the **inverse of the initial state future time cost minimization problem for the dual system Σ_*** .

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Conceptual Advance 4: The Coupling Condition for Existence of Doubly Coprime Factorization

- We have shown: $\hat{\mathcal{D}}$ has a doubly coprime H factorization \iff the future cost of Σ is dominated by the past cost.
- This can be interpreted as (previously unknown) **coupling condition** between the solutions of the control and filter Riccati equations:
 \iff the product of the two Riccati Operators Q and P is bounded
(although Q and/or P may be separately unbounded).
- This is the natural condition that one obtains from the H^∞ minimization problem by letting the norm parameter $\gamma \rightarrow \infty$.
- This is in sharp contrast to the prevailing theory, which says that there is no coupling between the H^2 -optimal control and the H^2 -optimal filter. Indeed, there is formally no coupling, but as a matter of fact, P and Q are coupled in the above sense if and only if $\hat{\mathcal{D}}$ has a doubly coprime factorization.

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Conceptual Advance 5: Construction of Stabilizable and Detectable Realization

The following result is more or less true (work in progress):

How to construct a stabilizable and detectable realization

- A necessary condition for $\widehat{\mathcal{D}}$ to have a stabilizable and detectable realization is that $\widehat{\mathcal{D}}$ has a doubly coprime factorization.
- Suppose that $\widehat{\mathcal{D}}$ has a doubly coprime factorization, and that $\widehat{\mathcal{D}}$ is bounded in some right half-plane (i.e., “well-posed”).
 - Choose an arbitrary system (or operator) node realization Σ of $\widehat{\mathcal{D}}$.
 - Restrict Σ to the reachable subspace, and factor out the unobservable subspace.
 - Replace the original norm by the half way interpolation of $\|\cdot\|_{\text{fut}}$ and $\|\cdot\|_{\text{past}}$, and complete the space with respect to this norm (the resulting realization will be minimal and LQG balanced).
- Then the resulting system is well-posed, stabilizable and detectable (and unique up to unitary similarity)

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