A NEW ν-METRIC IN CONTROL THEORY

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Outline

(1) What is the metric on ?

$$d_\nu : X \times X \longrightarrow [0, \infty)$$

X = set of "unstable control systems"

(2) Why is it needed ?

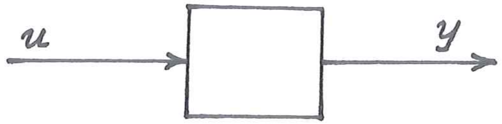
Robust stabilization problem

(3) The "classical" ν - metric

G. Vinnicombe ; 1993

(4) Our extension.

Control theory



$$\hat{y}(s) = g(s)\, \hat{u}(s)$$

transfer function

Stable system : "nice" inputs $\mapsto$ nice outputs

Classes of stable transfer functions

(1) $RH^\infty := H^\infty \cap \mathbb{C}(s)$

$u \in L^2(0,\infty) \Rightarrow y \in L^2(0,\infty)$

(2) $\mathcal{A}^+ = \left\{ \hat{\mu} : \begin{array}{l} \mu \text{ is a complex Borel measure on } \mathbb{R} \text{ s.t.} \\ \operatorname{supp} \mu \subset [0,\infty) \text{, without a singular nonatomic part} \end{array} \right\}$

$u \in L^p(0,\infty) \Rightarrow y \in L^p(0,\infty)$
$(1 \leq p \leq \infty)$

(3) $A(\mathbb{D}^n)$

Abstract approach

R = ring of stable transfer functions.

Unstable systems:

$$p \in \mathbb{F}(R) = \left\{ \frac{n}{d} : n, d \in R,\ d \neq 0 \right\}$$

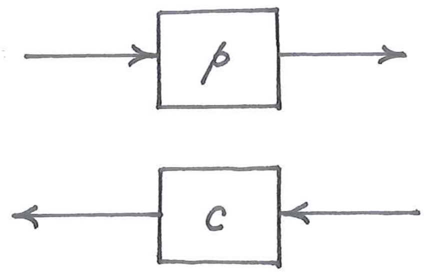
Example:
$$g(s) = \frac{1}{s-1} \notin RH^{\infty}$$

$$= \frac{\frac{1}{s+1}}{\frac{s-1}{s+1}} \in \mathbb{F}(RH^{\infty}).$$

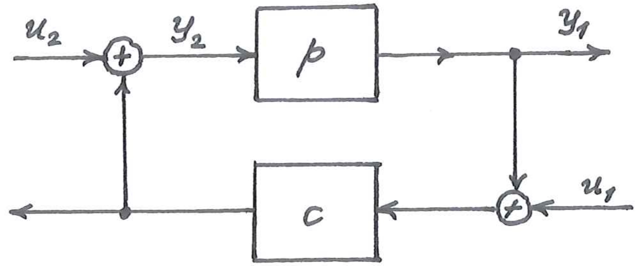
Stabilization problem

Given a $p \in \mathbb{F}(R)$,

find a $c \in \mathbb{F}(R)$,



such that their interconnection is stable.



That is, $H(p,c) := \begin{bmatrix} p \\ 1 \end{bmatrix} (1-cp)^{-1} \begin{bmatrix} -c & 1 \end{bmatrix} \in R^{2\times 2}$

closed loop transfer function $\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \mapsto \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right)$.

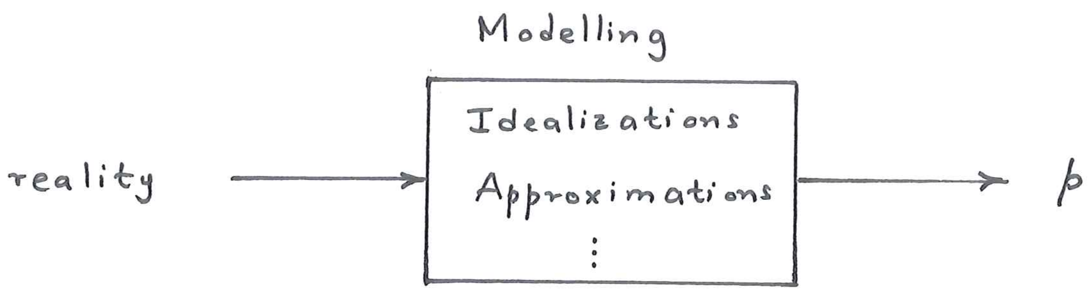
Solution

$p \in \mathbb{F}(R)$ has a coprime factorization if $p = \frac{n}{d}$, $n, d \in R$, $d \neq 0$

and $\exists \; x, y \in R$ s.t. $n x + d y = 1$.

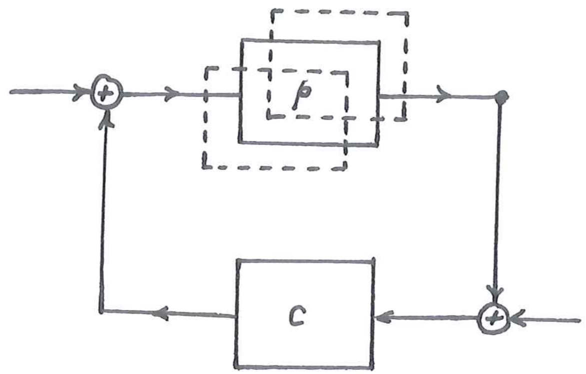
(Then $c := -\frac{x}{y}$ stabilizes p.)

But in reality, p is not known exactly:



Result: p is all wrong!

Robust stabilization



Want c to stabilize not only p, but all $\tilde{p}$:s "near" p.

What is an appropriate notion of closeness between unstable plants?

Want: d which (1) is a metric on {stabilizable plants}

(2) is easy to compute

(3) has good properties in robust stabilization.

Classical ν-metric d_ν (Glenn Vinnicombe; 1993)

$R = RH^\infty \subset C(\mathbb{T})$

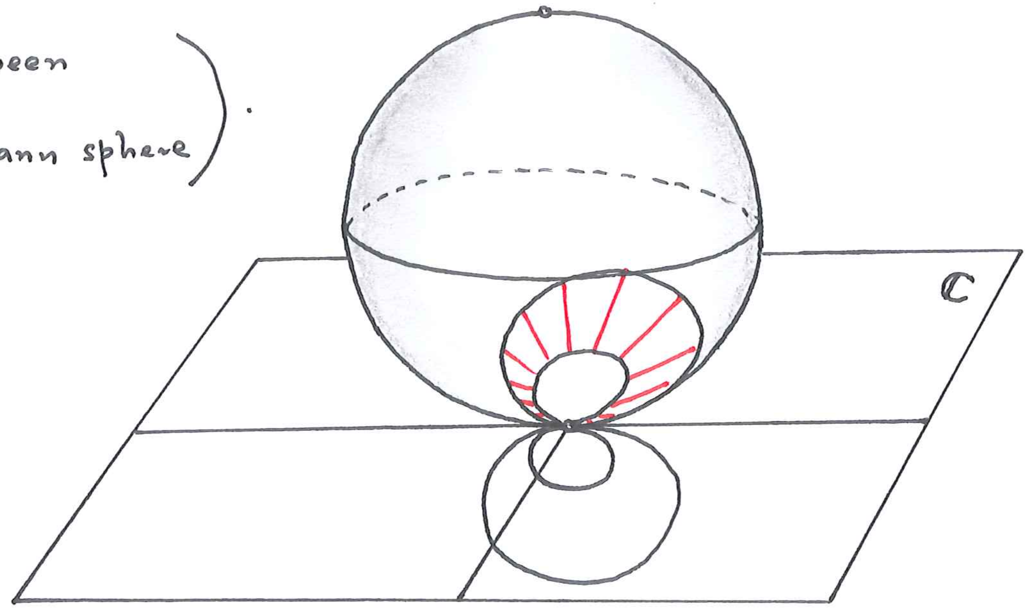
$X = \{ p \in F(RH^\infty) : p$ has a normalized coprime factorization$\}$

$p = \frac{n}{d}$; $n, d \in R$, $d \neq 0$; $\exists\ x, y \in R$ s.t. $nx + dy = 1$ and $|n|^2 + |d|^2 = 1$ on $i\mathbb{R}$.

For $p_1, p_2 \in X$,
$$d_\nu(p_1, p_2) := \begin{cases} \|n_2 d_1 - n_1 d_2\|_\infty & \text{if } \omega(n_1 \bar{n}_2 + d_1 \bar{d}_2) = 0 \\ 1 & \text{otherwise} \end{cases}$$

If $d_\nu(p_1, p_2) < 1$, then

$$d_\nu(p_1, p_2) = \sup_{y \in \mathbb{R}} \left(\begin{array}{c} \text{chordal distance between} \\ p_1(iy),\, p_2(iy) \text{ on Riemann sphere} \end{array} \right).$$



Why winding number constraint?

$p \in RH^\infty$ stable ; stabilized by $c = 0$.

But every neighbourhood of p in the chordal metric has unstable plants.

So stabilizability is not a robust property of the plant.

d_ν's good property w.r.t. robust stabilization.

Stability margin $\quad \mu_{p,c} := \dfrac{1}{\| H(p,c) \|_\infty}$ $\quad$ large $\mu_{p,c} \Rightarrow$ more stable ; better performance

Measures "How stable is the closed loop system?"

Theorem $\quad \mu_{\tilde{p},c} \geq \mu_{p,c} - d_\nu(p, \tilde{p})$

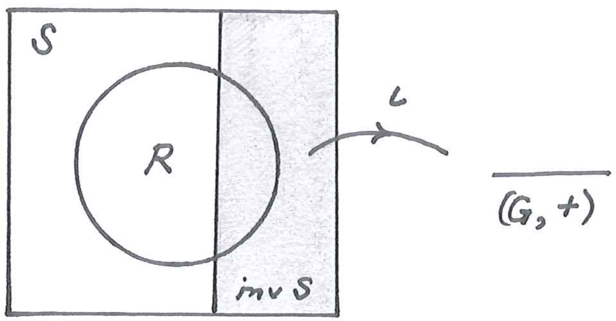
If c stabilizes p and $d_\nu(p,\tilde{p})$ small enough,
then c also stabilizes $\tilde{p}$ and guarantees a certain performance.

Classical ν-metric

What if $R \neq RH^\infty$? For *example*, $R = \mathcal{A}^+$?

Extension of the ν-metric

Abstract set-up:



R commutative integral domain with identity

S commutative complex semisimple Banach algebra with an involution $\cdot^*$ and with identity

$\text{inv } S :=$ set of invertible elements of S

$(G, +)$ Abelian group with identity o

Index function $\iota : \text{inv } S \longrightarrow (G, +)$ s.t

(I1) $\iota(ab) = \iota(a) + \iota(b)$

(I2) $\iota(a^*) = -\iota(a)$

(I3) ι is locally constant (G has discrete topology)

(I4) $x \in R \cap (\text{inv } S)$ invertible in R iff $\iota(x) = o$.

What is the extension of d_ν?

$X := \{ p \in \mathbb{F}(R) :$ p has a normalized coprime factorization$\}$

$p = \frac{n}{d}$ s.t. (1) $n, d \in R$, $d \neq 0$

(2) $\exists\, x, y \in R$ s.t. $nx + dy = 1$

(3) $n^* n + d^* d = 1$ in S.

For $p_1, p_2 \in X$,
$$d_\nu(p_1, p_2) := \begin{cases} \|n_2 d_1 - n_1 d_2\|_\infty & \text{if } n_1 n_2^* + d_1 d_2^* \in \text{inv } S \text{ and } \iota(n_1 n_2^* + d_1 d_2^*) = 0, \\ 1 & \text{otherwise} \end{cases}$$

$\|\cdot\|_\infty$? $M(S)$ = maximal ideal space of the Banach algebra S

$x \in S$; $\hat{x} \in C(M(S); \mathbb{C})$ Gelfand transform

$\hat{x}(\varphi) := \varphi(x) \quad (\varphi \in M(S))$

$$\|x\|_\infty := \sup_{\varphi \in M(S)} |\hat{x}(\varphi)| .$$

Theorem 1 d_ν is a metric on X.

Theorem 2 $\mu_{\tilde{p},c} \geq \mu_{p,c} - d_\nu(p,\tilde{p})$.

Here $\mu_{p,c} := \dfrac{1}{\|H(p,c)\|_\infty}$ if p is stabilized by c.

Examples

(1) $R = RH^\infty$

$S = C(\mathbb{T})$

$G = \mathbb{Z}$

ι = winding number $w : \text{inv}\, C(\mathbb{T}) \longrightarrow \mathbb{Z}$

Then d_ν = classical ν-metric.

Also $R = A(\mathbb{D})$, $W^+(\mathbb{D})$, $\widehat{L^1(0,\infty) + \mathbb{C}}$, $\ldots$.

$$(2) \qquad R = \mathcal{A}^+ = \left\{ \hat{\mu} : \begin{array}{l} \mu \text{ complex Borel measure on } \mathbb{R}, \\ \operatorname{supp} \mu \subset [0,\infty) , \text{ without singular nonatomic part} \end{array} \right\}$$

$$= \left\{ \hat{f_a} + \sum_{k \geq 0} f_k e^{-\cdot t_k} : \begin{array}{l} f_a \in L^1(0,\infty) \\ (f_k)_{k\geq 0} \in \ell^1 \\ t_0 = 0 < t_1, t_2, t_3, \dots \end{array} \right\}$$

$$S = \mathcal{A} = \left\{ \hat{f_a} + \sum_{k \in \mathbb{Z}} f_k e^{-\cdot t_k} : \begin{array}{l} f_a \in L^1(\mathbb{R}) \\ (f_k)_{k\in\mathbb{Z}} \in \ell^1 \end{array} \right\}$$

$$F = \hat{f_a} + \underbrace{\sum_{k \in \mathbb{Z}} f_k e^{-\cdot t_k}}_{F_{AP}} \quad ; \qquad \|F\|_{\mathcal{A}} = \|f_a\|_{L^1} + \|(f_k)_{k\in\mathbb{Z}}\|_{\ell^1}$$

$$G = \mathbb{R} \times \mathbb{Z}$$

$$\iota \qquad \iota(F) = \iota\left(\hat{f_a} + F_{AP}\right) = \left(\omega_{av}(F_{AP}),\ \omega\left(1 + F_{AP}^{-1} \hat{f_a}\right)\right)$$

$$\text{for} \quad F = \hat{f_a} + f_{AP} \in \operatorname{inv} \mathcal{A} .$$

(3) $R = A(\mathbb{D}^n)$ polydisk algebra

$S = C(\mathbb{T}^n) \times C(\mathbb{T})$

$f \in A(\mathbb{D}^n)$ — $f_d : \overline{\mathbb{D}} \to \mathbb{C}$

$f_d(z) := f(z, \dots, z) \quad (z \in \overline{\mathbb{D}})$

$f_d \in C(\mathbb{T})$

$f \in A(\mathbb{D}^n)$ — $f|_{\mathbb{T}^n} \in C(\mathbb{T}^n)$

$$A(\mathbb{D}^n) \longrightarrow C(\mathbb{T}^n) \times C(\mathbb{T})$$

$$f \longmapsto (f|_{\mathbb{T}^n}, f_d)$$

$G = \mathbb{Z}$

$\iota = \big((g, h) \mapsto \omega(h) \big)$

Relation to the gap topology

Theorem The ν-metric topology coincides with the gap metric topology for stabilizable plants over $\mathcal{A}^+$.