Kernel and eigenfunction estimates for some second order elliptic operators

Abdelaziz Rhandi
(Department of Mathematics, University of Salerno, Italy)
(Joint work with E.M. Ouhabaz)

CDPS 2011, 18-22 July, 2011

Schrödinger Operators

Consider

$$\begin{aligned} Au &= -\sum_{k,j=1}^{n} \partial_k(a_{kj}\partial_j u) + Vu, \\ Bu &= -\Delta u + Vu. \end{aligned}$$

The associated quadratic forms

$$\begin{aligned} \mathfrak{a}(u,u) &:= \sum_{k,j=1}^{n} \int_{\mathbb{R}^n} a_{kj}\partial_k u \partial_j u + \int_{\mathbb{R}^n} V|u|^2 \\ \mathfrak{b}(u,u) &:= \int_{\mathbb{R}^n} |\nabla u|^2 + \int_{\mathbb{R}^n} V|u|^2, \end{aligned}$$

$u \in D(\mathfrak{a}) = D(\mathfrak{b}) = \{u \in W^{1,2}(\mathbb{R}^n);\ \int_{\mathbb{R}^n} V|u|^2 < \infty\}.$

Assumptions

$$(H)\begin{cases} a_{kj} = a_{jk} \in W^{1,\infty}_{loc}(\mathbb{R}^n,\mathbb{R}),\ \partial_j a_{kj} = \mathrm{o}(|x|^{\frac{\alpha}{2}}) \text{ as } |x| \to \infty, \\ \eta|\xi|^2 \le \sum_{j,k=1}^n a_{kj}(x)\xi_k\xi_j \le \Lambda|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^n, \\ V \in L^1_{loc}(\mathbb{R}^n) \text{ such that } V(x) \ge |x|^\alpha, \quad \alpha > 2. \end{cases}$$

Spectrum

$\lim_{|x|\to+\infty} V(x) = +\infty$ implies that A and B have compact resolvents. Thus,

$$\begin{aligned} \sigma(A) &= \{\mu_i;\ i = 0, 1, \ldots\} \\ \sigma(B) &= \{\lambda_i;\ i = 0, 1, \ldots\}. \end{aligned}$$

Let $(\psi_i)_{i\geq 0}$ and $(\varphi_i)_{i\geq 0}$ the corresponding normalized eigenfunctions of A and B, respectively.

Gaussian estimates 1

It is known that A and B have heat kernels $k_t(x,y)$ and $p_t(x,y)$ satisfying

$$p_t(x,y) \leq \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}}, \quad k_t(x,y) \leq Ct^{-n/2} e^{-c\frac{|x-y|^2}{t}}$$

for $t > 0$ and constants c, $C > 0$

Gaussian estimates 2

See [E.M. Ouhabaz: Proc. Amer. Math. Soc. **134**, 2006].

$$p_t(x,y) \leq \frac{C}{t^{n/2}} e^{-\lambda_0 t} e^{-\frac{|x-y|^2}{4t}} \left[1 + \lambda_0 t + \frac{|x-y|^2}{t}\right]^{\frac{n}{2}}$$

and

$$k_t(x,y) \leq \frac{C}{t^{n/2}} e^{-\mu_0 t} e^{-\frac{\rho^2(x,y)}{4t}} \left[1 + \mu_0 t + \frac{\rho^2(x,y)}{t}\right]^{\frac{n}{2}}, \quad t > 0,$$

where

$$\begin{aligned} \rho(x,y) \quad &:= \quad \sup\{\phi(x) - \phi(y) : \ \phi \in C_c^\infty(\mathbb{R}^n), \\ &\qquad \sum_{k,j=1}^n a_{kj} \partial_k \phi \partial_j \phi \leq 1 \text{ a.e. on } \mathbb{R}^n\}. \end{aligned}$$

Intrinsic ultracontractivity for B

E.B. Davies in 1984 showed

$$p_t(x,y) \leq Ce^{ct^{-b}}\varphi_0(x)\varphi_0(y), \tag{1}$$

$x,\ y \in \mathbb{R}^n$, $0 < t \leq 1$, where C, c are constants and $b > \frac{\alpha+2}{\alpha-2}$. Using Lyapunov functions techniques Metafune and Spina [JEE **7**, 2007] obtained (1) with $b = \frac{\alpha+2}{\alpha-2}$.

Intrinsic ultracontractivity for B

Davies showed also

$$c_1|x|^{-\beta}e^{-\frac{|x|^\gamma}{\gamma}} \le \varphi_0(x) \le c_2|x|^{-\beta}e^{-\frac{|x|^\gamma}{\gamma}} \tag{2}$$

for large $|x|$, $\beta = \frac{\alpha}{4} + \frac{n-1}{2}$, $\gamma = 1 + \frac{\alpha}{2}$. By (1),

$$p_t(x,y) \le Ce^{ct^{-b}}(|x||y|)^{-\beta}e^{-\frac{|x|^\gamma}{\gamma}}e^{-\frac{|y|^\gamma}{\gamma}} \tag{3}$$

for large $|x|$, $|y|$, and $0 < t \le 1$.

The main theorem

Theorem 1. Assume (H) with $\Lambda < 1$. Then,

$$k_t(x,y) \le Ce^{-\mu_0 t} e^{ct^{-b}} \left(|x||y|\right)^{-\beta} e^{-\frac{|x|^\gamma}{\gamma} - \frac{|y|^\gamma}{\gamma}}$$

for large $|x|$, $|y|$ and all $t > 0$. Here
$C,\ c > 0,\ b > \frac{\alpha+2}{\alpha-2},\ \beta = \frac{\alpha}{4} + \frac{n-1}{2}$ and $\gamma = 1 + \frac{\alpha}{2}$.

Sketch of the proof

On $L^2_\varphi := L^2(\mathbb{R}^n, \varphi^2 dx)$ define

$$\widetilde{\mathfrak{a}}(u,v) := \mathfrak{a}(\varphi u, \varphi v),\ \widetilde{\mathfrak{b}}(u,v) := \mathfrak{b}(\varphi u, \varphi v)$$
$$D(\widetilde{\mathfrak{a}}) = D(\widetilde{\mathfrak{b}}) = \{u \in L^2_\varphi;\ \varphi u \in D(\mathfrak{a}) = D(\mathfrak{b})\}.$$

their associated kernels are

$$\widetilde{k}_t(x,y) = \frac{k_t(x,y)}{\varphi(x)\varphi(y)}, \quad \widetilde{p}_t(x,y) = \frac{p_t(x,y)}{\varphi(x)\varphi(y)}.$$

Using $\varphi \approx \varphi_0$, $|\nabla\varphi| \approx |\nabla\varphi_0|$ and the Beurling-Deny criterion for $(\varphi_0^{-1} e^{-tB} \varphi_0)$ on $L^2_{\varphi_0}$ we deduce

$$1 \wedge u \in D(\widetilde{\mathfrak{a}}), \quad \forall 0 \le u \in D(\widetilde{\mathfrak{a}}).$$

Sketch of the proof

We have

$$\tilde{\mathfrak{a}}(u,v) = \sum_{j,k=1}^{n} \int_{\mathbb{R}^n} a_{kj}\partial_k u \partial_j v \varphi^2\, dx + \int_{\mathbb{R}^n} W_{\mathfrak{a}} uv\varphi^2\, dx,$$

where $W_{\mathfrak{a}} = V - \sum_{j,k=1}^{n} \partial_j a_{kj} \frac{\partial_k \varphi}{\varphi} - \sum_{j,k=1}^{n} a_{kj} \frac{\partial_k \partial_j \varphi}{\varphi}$. Using (H) and $\Lambda < 1$ we deduce

$$W_{\mathfrak{a}}(x) \geq -\lambda_{\mathfrak{a}}.$$

Sketch of the proof

Thus,

$$\begin{aligned}\widetilde{\mathfrak{a}}(1\wedge u,(u-1)^+) &= \int_{\mathbb{R}^n} W_{\mathfrak{a}}(1\wedge u)(u-1)^+\varphi^2dx\\ &\geq -\lambda_{\mathfrak{a}}\int_{\mathbb{R}^n}(1\wedge u)(u-1)^+\varphi^2dx.\end{aligned}$$

Applying Beurling-Deny

$$\|e^{-t\widetilde{A}}\|_{\mathcal{L}(L^\infty)}\leq e^{\lambda_{\mathfrak{a}}t},\quad t\geq 0.$$

Sketch of the proof

Since $\widetilde{\mathfrak{b}}$ satisfies a log-Sobolev inequality (see E.B. Davies) and by ellipticity, $\widetilde{\mathfrak{b}}(u,u) \leq \max\{\frac{1}{\eta}, 1\}\,\widetilde{\mathfrak{a}}(u,u)$, it follows that $\widetilde{\mathfrak{a}}$ satisfies the same log-Sobolev inequality. Hence, with the L^∞-contractivity, we deduce that $e^{-t\widetilde{A}}$ is ultracontractive and

$$\widetilde{k}_t(x,y) = \frac{k_t(x,y)}{\varphi(x)\varphi(y)} \leq Ce^{ct^{-b}}, \quad 0 < t \leq 1.$$

Sketch of the proof

For $t \geq 1$, by

$$k_{s+r}(x,y) = \int_{\mathbb{R}^n} k_s(x,z)k_r(y,z)\,dz, \quad s,\ r > 0,$$

$$\begin{aligned} k_t(x,x) &= \left\| e^{-\left(\frac{t}{2}-\frac{1}{2}\right)A} k_{\frac{1}{2}}(x,\cdot) \right\|_{L^2}^2 \\ &\leq e^{-2\mu_0\left(\frac{t}{2}-\frac{1}{2}\right)} \left\| k_{\frac{1}{2}}(x,\cdot) \right\|_{L^2}^2 \\ &\leq M e^{-\mu_0 t} \varphi^2(x). \end{aligned}$$

The result follows from

$$k_t(x,y) \leq \sqrt{k_t(x,x)}\sqrt{k_t(y,y)}, \quad x,\ y \in \mathbb{R}^n.$$

Estimates for the eigenfunctions

Corollary 2.

$$|\psi_j(x)| \leq C|x|^{-\beta} e^{-\frac{|x|^\gamma}{\gamma}}$$

for large $|x|$ a $C > 0$ with $\beta = \frac{\alpha}{4} + \frac{n-1}{2}$, $\gamma = 1 + \frac{\alpha}{2}$.

Proof

$$\begin{aligned} |\psi_j(x)|e^{-\mu_j t} &= |e^{-tA}\psi_j(x)| \\ &= |\int_{\mathbb{R}^n} k_t(x,y)\psi_j(y)\,dy| \\ &\leq \left(\int_{\mathbb{R}^n} k_t(x,y)^2 dy\right)^{1/2} \|\psi_j\|_2 \\ &= (k_{2t}(x,x))^{1/2}. \end{aligned}$$

The general case

If $\Lambda \geq 1$, we study first

$$H := -\Delta + \theta |x|^\alpha$$

with $0 < \theta < 1$. One proves that its ground state ϕ_0 satisfies

$$\phi_0(x) \approx \phi(x), \quad |\nabla \phi_0(x)| \approx |\nabla \phi(x)|,$$

where $\phi(x) = |x|^{-\beta} e^{-\frac{\sqrt{\theta}}{\gamma}|x|^\gamma}$ for large $|x|$.

The general case

Theorem 2. Assume (H) and $\theta > 0$ s.t. $\theta\Lambda < 1$. Then,

$$k_t(x,y) \le Ce^{-\mu_0 t}e^{ct^{-b}}\left(|x||y|\right)^{-\beta} e^{-\frac{\sqrt{\theta}}{\gamma}|x|^\gamma} e^{-\frac{\sqrt{\theta}}{\gamma}|y|^\gamma}, \quad t > 0$$

for large $|x|$, $|y|$. Here C, $c > 0$, $b > \frac{\alpha+2}{\alpha-2}$, $\beta = \frac{\alpha}{4} + \frac{n-1}{2}$ and $\gamma = 1 + \frac{\alpha}{2}$.