ISS Lyapunov functions for infinite dimensional systems

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$$\partial_t y + \Lambda(y)\partial_x y = 0\ , \quad x \in [0,1], t \geq 0 \tag{1}$$

where y: $[0,1] \times [0,\infty) \to \mathbb{R}^n$.

Assumptions: Λ: $\varepsilon\mathbb{B} \to \mathbb{R}^{n\times n}$ is a C^1 function such that $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, and

$$\lambda_1(0) < \ldots < \lambda_m(0) < 0 < \lambda_{m+1}(0) < \ldots < \lambda_n(0)$$

Notation: $y = \begin{pmatrix} y_- \\ y_+ \end{pmatrix} \in \mathbb{R}^{m\times(n-m)}$

$\mathbb{B} \subset \mathbb{R}^n$ is the unit open ball centered at 0

The boundary conditions are

$$\begin{pmatrix} y_-(1,t) \\ y_+(0,t) \end{pmatrix} = k \begin{pmatrix} y_-(0,t) \\ y_+(1,t) \end{pmatrix}, \qquad (2)$$

where $k\colon \varepsilon\mathbb{B} \to \mathbb{R}^n$ is C^1 s.t. $k(0) = 0$.

Many technics to derive sufficient conditions on k so that (1)-(2) is Locally Exponentially Stable in H^2, or in C^1...

This kind of models appear in many various applications such as

- the traffic flow control [Bressan, Han, 11], [Garavello, Piccoli, 06], [Gugat, Herty, Klar, Leugering, 06]
- the open-channel regulation
- ....

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Non-homogenous hyperbolic systems in $\mathbb{R}^n$

More recent problem

$$\partial_t y + \Lambda(y)\partial_x y = f(y,t) \ , \quad x \in [0,1], t \geq 0 \qquad (3)$$

where $f : \varepsilon\mathbb{B} \times [0,\infty) \to \mathbb{R}^n$ is an external function.

Some motivations:

- f may model a reaction phenomena, vanishing at the equilibrium: $f(0,t) = 0$
- f may be a perturbation or an model error: $f(0,t) \neq 0$ even when t is large

In this context, can we find sufficient conditions for local asymptotic stability of (3) when f vanishes at $y = 0$?
or at least so that y converges to a neighborhood of the origin when f is bounded only.
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Outline

1 Motivations

The hyperbolic system is Loc Exp Stable

$\not\Longrightarrow$ Loc Exp Stable in presence of source terms (even stable ones)

2 Related works: Robust Loc Expo Stability in presence of vanishing perturbations

using a Riemann coordinates approach

3 ISS = Sensitivity with respect to large perturbations

using a Lyapunov function

4 Related work: ISS for parabolic PDE

5 Two applications

Conclusion

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As a first example, let us consider the following linear hyperbolic system:

$$\begin{array}{c} \partial_t y + \Lambda \partial_x y = 0 \ , \quad x \in [0,1], \ t \geq 0 \\ \Lambda \text{ has positive eigenvalues} \\ y(0) = K y(1) \end{array} \tag{4}$$

Notation:

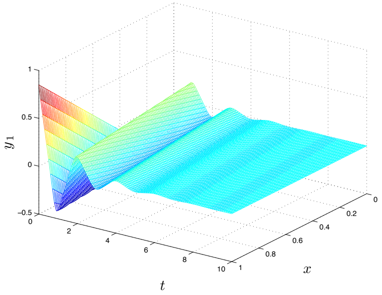
$$\begin{array}{c} \|K\| = \max\{|Kx|, \ x \in \mathbb{R}^n, \ |x| = 1\} \\ \rho_1(K) = \inf\{\|\Delta K \Delta^{-1}\|, \ \Delta \in \mathcal{D}_{n,+}\} \\ \rho(K) = \text{spectral radius of } |K| \end{array}$$

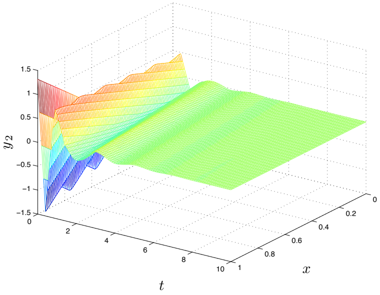
[Coron *et al*, 08]: if $\rho_1(K) < 1$ then the system (4) is Exp. Stable. This sufficient condition is weaker that the one of [Li Ta-tsien, 94].

Particular 2D system:

$$\partial_t y + \Lambda \partial_x y = 0 \ , \quad x \in [0,1], \ t \geq 0$$
$$y(0) = K y(1)$$

where $K = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}$ and $\Lambda = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$, their eigenvalues are 1 and 2. The condition of [Coron *et al*, 08] (and thus of [Li Ta-tsien, 94]) is satisfied. Then this system is exponentially stable. Using a Lax-Friedrichs method, we may check the attractivity:





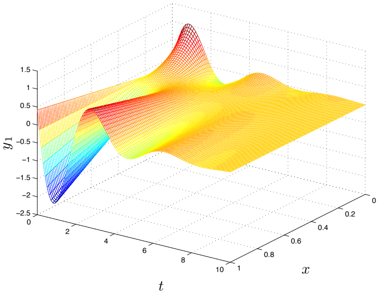
Moreover let us consider the following finite-dimensional system:

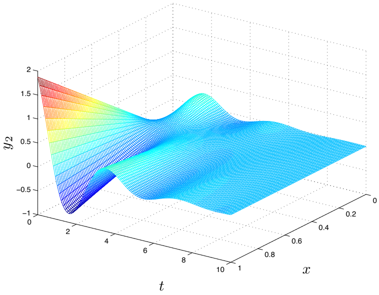
$$\partial_t y \qquad = Fy \ , \ t \geq 0$$

No boundary condition (x is a parameter).

where $F = \begin{pmatrix} 0 & -3 \\ 1 & -1 \end{pmatrix}$ (with eigenvalues having a negative real part).

It is Exp. Stable. With the same initial condition

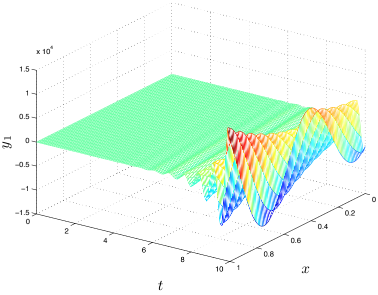


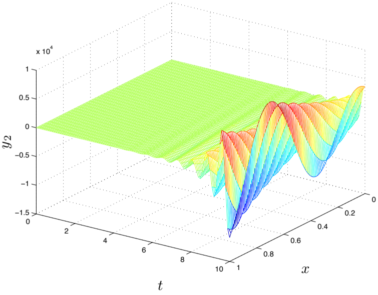


Now combining the two previous systems leads to

$$\partial_t y + \Lambda \partial_x y = Fy \ , \ x \in [0,1], \ t \geq 0$$
$$y(0) = Ky(1)$$

which is unstable.
Indeed, with the same initial condition:





For general non-homogeneous hyperbolic system

Let us consider the non-homogeneous case:

$$\partial_t y + \Lambda(y)\partial_x y = f(y) \ , \quad x \in [0,1], t \geq 0 \tag{5}$$

$$\begin{pmatrix} y_-(1,t) \\ y_+(0,t) \end{pmatrix} = k \begin{pmatrix} y_-(0,t) \\ y_+(1,t) \end{pmatrix} \tag{6}$$

Thus

When the homogeneous system (5)-(6) is stable then with a $f \equiv 0$, the non-homogeneous system (5)-(6) may be unstable.

In [Li, 94], and in [Coron *et al*, 08]
the unperturbed case ($f \equiv 0$) is considered for the system

$$\partial_t y + \Lambda(y)\partial_x y = f(y) \ , \quad x \in [0,1], t \geq 0 \tag{7}$$

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In presence of f, the 2-D case in considered in [Vazquez *et al*, 11]

Following an analogous approach of [Li, 94] on Riemann coordinates, we may study the sensitivity for small perturbations:

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Following an analogous approach of [Li, 94] on Riemann coordinates, we may study the sensitivity for small perturbations:

Stability analysis for small perturbations

Theorem [CP, Winkin, Bastin, 08]

If $\rho(\nabla k(0)) < 1$, then there exist $\varepsilon > 0$, and $H > 0$ such that, for all C^1-functions $f : \varepsilon\mathbb{B} \to \mathbb{R}^n$ such that $f(0) = 0$ and

$$\|\nabla f(0)\| \leq H \ ,$$

for all y^0, $\|y^0\|_{C^1(0,1)} \leq \varepsilon$ satisfying some compatibility conditions there exists one and only one solution $y \in C^1([0,1] \times [0,+\infty) \,; \mathbb{R}^n)$ satisfying (7), (8) and

$$y(x,0) = y^0(x) \ , \forall x \in [0,1].$$

Moreover, there exist $\mu > 0$ and $C > 0$ such that

$$\|y(.,t)\|_{C^1(0,1)} \leq Ce^{-\mu t}\|y^0\|_{C^1(0,1)} \ , \forall t \geq 0.$$

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And for large perturbations?

Back to the 2D example

$$y_t + \Lambda y_x = Fy \ , \ y \in \mathbb{R}^2 \ , \ x \in [0,1], \ t \geq 0$$
$$y(0) = Ky(1)$$

The condition $\rho(\nabla k(0)) < 1$ is satisfied. Thus with $F = 0$, the system is Exp. Stable
However since the system is unstable, the condition $\|F\| \leq H$ of the previous theorem does not hold.

What happen for such perturbations?

Question: for an asymptotically hyperbolic stable system

Do bounded perturbations result bounded states?

Let us consider a linear, space-dependent hyperbolic system:

$$\partial_t y + \Lambda(x,t)\partial_x y = F(x,t)y + \delta(x,t) \ , \qquad (9)$$

up to a change of variables, we assume that
$\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, and $0 < \lambda_1(x,t) < \ldots < \lambda_n(x,t)$
The boundary condition

$$y(0,t) = Ky(1,t) \ . \qquad (10)$$

F is a source term. δ is an unkown perturbation

Assumption 1

Λ, F and δ are T-periodic with respect to t
F, Λ, and δ are C^1

If Λ is constant, nonnegative, and $\rho_1(K) < 1$,
then $\exists$ a diag. pos. def. matrix Δ such that $\mathrm{Sym}(\Delta K \Delta^{-1}) < Id$.
then $\exists$ a diag. pos. def. matrix $Q := \Delta^2\Lambda^{-1}$, and $\varepsilon > 0$ such that

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$\exists$ a sym. pos. def. matrix Q, $\alpha \in (0,1)$, a C^0, $r : [0,\infty) \to \mathbb{R}$, periodic of period $T > 0$ with a positive mean value, i.e. such that

$$R = \int_0^T r(m)dm > 0$$

such that, for all $t \geq 0$ and for all $x \in [0,1]$, it holds

$$\mathtt{Sym}\left(\alpha Q\Lambda(L,t) - K^\top Q\Lambda(L,t)K\right) \geq 0 \ , \tag{11}$$

$$\mathtt{Sym}\left(Q\Lambda(x,t)\right) \geq r(t)Id \ , \tag{12}$$

$$\mathtt{Sym}\left(Q\partial_x\Lambda(x,t) + 2QF(x,t)\right) \leq 0 \tag{13}$$

Remark: If If Λ is constant, nonnegative, and $\rho_1(K) < 1$, then $\exists$ a diag. pos. def. matrix Q, and $\varepsilon > 0$ such that

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Under Assumption 2, let $\mu \in (0, \ln(\alpha))$ and $q(t) := \frac{\mu}{\|Q\|}\left(r(t) - \frac{B}{2T}\right)$.

Theorem : [CP, Mazenc, 11]

Under Assumptions 1 and 2, letting $V : L^2(0,1) \times [0,+\infty) \to [0,+\infty)$ defined, for all $y \in L^2(0,1)$ and $t \geq 0$, by

$$V(y,t) := e^{\frac{1}{T}\int_{t-T}^{t}\int_{\ell}^{t} q(m)dmd\ell} \int_0^1 y(x)^\top Q y(x) e^{-\mu x} dx \ ,$$

we have, along the solutions of (9) and (10), for all $t \geq 0$,

$$\dot{V} \leq -c_1 V(y,t) + c_2 \|\delta(.,t)\|^2_{L^2(0,1)}$$
$$c_3 \|y(.,t)\|^2_{L^2(0,1)} \leq V(y,t) \leq c_4 \|y(.,t)\|^2_{L^2(0,1)}$$

for suitable constant values $c_i > 0$.

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About the expression of the Lyapunov function

Time varying positive definite function

$$V(y,t) := e^{\frac{1}{T}\int_{t-T}^{t}\int_{\ell}^{t} q(m)\,dm\,d\ell} \int_0^1 y(x)^\top Q y(x) e^{-\mu x} dx \ ,$$

Introduction of μ:

- [Coron, 98] for the stabilization of the Euler equation.
- [Xu, Sallet, 02] for symmetric linear hyperbolic systems.

Introduction of the time-varying term

- Quite usual for nonlinear finite dimensional systems [Mazenc, Nesic, 07] among others
- but not so usual for PDEs?

Input-to-State Stable Lyapunov function for hyperbolic systems

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This implies

- exponential stability when $\delta \equiv 0$
- along the solutions of (9) and (10), for all $t \geq 0$,

$$\|y(.,t)\|_{L^2(0,1)} \leq C_1 e^{-t\varepsilon} \|y(.,0)\|_{L^2(0,1)} + C_2 \sup_{s\in[0,t]} \|\delta(.,s)\|_{L^2(0,1)}$$

[Logemann, 11] in other words

$$\delta \text{ bounded} \Rightarrow y \text{ bounded}$$

- similarly we may prove

$$\delta \to 0 \;\Rightarrow\; y \to 0, \text{ as } t \to \infty$$

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Sketch of the proof of our result on hyperbolic systems

$$\partial_t y + \Lambda(x,t)\partial_x y = F(x,t)y + \delta(x,t)\ , \qquad (14)$$

$$y(0,t) = Ky(1,t)\ . \qquad (15)$$

First Step: $\dot{W} \leq 0$???

Prove that the function $W(y) = \int_0^1 y(x)^\top Q y(x) e^{-\mu x} dx$, is a weak Lyapunov function when δ is identically equal to zero

With Assumption 2 and our choice for μ (sufficiently small), we get

$$\dot{W} \leq -\mu r(t) \int_0^1 |y(x,t)|^2 e^{-\mu x} dx + 2 \int_0^1 y(x,t)^\top Q \delta(x,t) e^{-\mu x} dx\ ,$$

with $r(t) \geq 0$.

It follows that, for all $\kappa > 0$,

$$\begin{aligned}\dot{W} &\leq -\frac{\mu}{||Q||} r(t) W(y) + 2\|Q\|\kappa \int_0^1 |y(x,t)|^2 e^{-\mu x} dx \\ &\quad + \frac{\|Q\|}{2\kappa} \int_0^1 |\delta(x,t)|^2 e^{-\mu x} dx \\ &\leq -q_\kappa(t) W(y) + \frac{\|Q\|}{2\kappa} \int_0^1 |\delta(x,t)|^2 dx \ ,\end{aligned}$$

with $q_\kappa(t) = \frac{\mu}{\|Q\|} r(t) - \frac{2\|Q\|\kappa}{\lambda_Q}$.

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Second Step

Use the positive mean value of r to modify W.

Let us consider the time-varying candidate Lyapunov function

$$V(t,y) = e^{s_\kappa(t)} W(y) \ ,$$

with $s_\kappa(t) = \frac{1}{T}\int_{t-T}^{t}\int_{\ell}^{t} q_\kappa(m)dmd\ell$.

One get

$$\begin{aligned}\dot{V} &\leq -e^{s_\kappa(t)} q_\kappa(t) W(y) + \frac{\|Q\|}{2\kappa} e^{s_\kappa(t)} \int_0^1 |\delta(x,t)|^2 dx \\ &\quad + e^{s_\kappa(t)} \left[q_\kappa(t) - \frac{1}{T}\int_{t-T}^{t} q_\kappa(m)dm \right] W(y) \ .\end{aligned}$$

Since r is periodic of period T, we have

$$\int_{t-T}^{t} q_\kappa(m)dm = \frac{\mu}{\|Q\|} R - \frac{2T\|Q\|\kappa}{\lambda_Q} \ ,$$

where R is the mean value of r.

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4 – ISS property for parabolic semilinear equation

It parallels what is known for parabolic systems with a nonlinearity. More precisely consider

$$\partial_t y(x,t) = \partial_{xx} y(x,t) + f(y(x,t))$$

Assumption # 1

- $\exists$ a sym. pos. def. Q such that, letting $\mathcal{V}(y) = \frac{1}{2} y^\top Q y$ $-W_1(y) := \partial_x \mathcal{V}(y) f(y) \leq 0$
- either Dirichlet conditions or the Neumann conditions or $y(0,t) = y(1,t)$ and $\partial_x y(0,t) = \partial_x y(1,t)$

[Krstic, Smyshlyaev, 08] and [Coron, Trélat, 04] for instance

The function $V(y) = \int_0^1 \mathcal{V}(y(x))dx$ is a weak Lyapunov function:

$$\dot{V} = -\int_0^1 \partial_x y(x,t)^\top Q \partial_x y(x,t) dx - \int_0^1 W_1(y(x,t)) dx$$

Assumption # 2

$\exists\ c_a > 0,\ c_b > 0$, a C^2 $M : \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$, $M(0) = 0$ and $\partial_y M(0) = 0$, and a C^0 $W_2 : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that $W_1 + W_2$ is pos. def. and

$$\partial_y M(y) f(y) \leq -W_2(y)\ ,\ |\partial_{yy} M(y)| \leq c_a\ ,\ \forall y \in \mathbb{R}^2\ ,$$
$$W_1(y) + W_2(y) \geq c_b |y|^2\ ,\ \forall y \in \mathbb{R}^2 : |y| \leq 1$$

Theorem [Mazenc, CP, 11]

Then $\exists$ a def. pos. function $k : \mathbb{R} \to \mathbb{R}$ such that

$$\overline{V}(y) = \int_0^1 k(\mathcal{V}(y(x)) + M(y(x)))dx$$

is a strict Lyapunov function for

$$\partial_t y(z,t) = \partial_{xx} y(x,t) + f(y(x,t))$$

Useful for

$$\partial_t y(x,t) = \partial_{xx} y(x,t) + f(y(x,t)) + \delta(x,t)$$

where $\delta(x,t)$ is an unknown continuous function.

Assumption #3

$\exists$ a C^2 $M : \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ such that $M(0) = 0$, $-\partial_y M(y) f(y) =: W_2(y) \geq 0$, and $\exists$ $c_a > 0$, $c_b > 0$ and $c_c > 0$ such that, for all $y \in \mathbb{R}^2$

$$|\partial_y M(y)| \leq c_a |y| \ , \ |\partial_{yy} M(y)| \leq c_b \ , c_c |y|^2 \leq [W_1(y) + W_2(y)]$$

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ISS property for semilinear parabolic equation

Theorem : [Mazenc, CP, 11]

Assume that Assumptions #1 and #3 with periodic boundary conditions

$$y(L,t) = y(0,t) \text{ and } \partial_x y(1,t) = \partial_x y(0,t)\ ,\ \forall t \geq 0\ .$$

Then, $\exists\ K > 0$ such that

$$\widetilde{V}(y) = \int_0^L [K\mathcal{V}(y(x)) + M(y(x))]dx$$

is an ISS Lyapunov function for

$$\partial_t y(x,t) = \partial_{xx} y(x,t) + f(y(x,t)) + \delta(x,t)$$

Applications of the design of ISS Lyapunov functions

- Hyperbolic systems
- Parabolic systems

Saint-Venant–Exner equation, [Graf, 84], [Diagne, Bastin, Coron, 11]:

$$\begin{aligned} &\partial_t \mathcal{H} + \mathcal{V}\partial_x \mathcal{H} + \mathcal{H}\partial_x \mathcal{V} = \delta_1 \ , \\ &\partial_t \mathcal{V} + \mathcal{V}\partial_x \mathcal{V} + g\partial_x \mathcal{H} + g\partial_x \mathcal{B} = gS_b - C_f \frac{\mathcal{V}^2}{\mathcal{H}} + \delta_2 \ , \\ &\partial_t \mathcal{B} + a\mathcal{V}^2 \partial_x \mathcal{V} = \delta_3 \ , \end{aligned} \tag{16}$$

where

- $\mathcal{H} = \mathcal{H}(x, t)$ is the water height at x in $[0, L]$
- $\mathcal{V} = \mathcal{V}(x, t)$ is the water velocity
- $\mathcal{B} = \mathcal{B}(x, t)$ is the bathymetry, i.e. the sediment layer
- g is the gravity constant
- S_b is the slope (which is assumed to be constant)
- C_f is the friction coefficient (also assumed to be constant)
- a is the effects of the porosity and of the viscosity
- $\delta(x, t) = (\delta_1(x, t), \delta_2(x, t), \delta_3(x, t))^\top$ is a disturbance, e.g. it can be a supply of water or an evaporation along the channel (see [Graf, 98]).

Saint-Venant–Exner equation, [Graf, 84], [Diagne, Bastin, Coron, 11]:

$$\begin{cases} \partial_t \mathcal{H} + \mathcal{V}\partial_x \mathcal{H} + \mathcal{H}\partial_x \mathcal{V} = \delta_1 \ , \\ \partial_t \mathcal{V} + \mathcal{V}\partial_x \mathcal{V} + g\partial_x \mathcal{H} + g\partial_x \mathcal{B} = gS_b - C_f \frac{\mathcal{V}^2}{\mathcal{H}} + \delta_2 \ , \\ \partial_t \mathcal{B} + a\mathcal{V}^2\partial_x \mathcal{V} = \delta_3 \ , \end{cases} \tag{16}$$

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Let us consider a steady-state $\mathcal{H}^\star$, $\mathcal{V}^\star$ and $\mathcal{B}^\star$ which is constant with respect to the x-variable.
(It should satisfy $gS_b\mathcal{H}^\star = C_f\mathcal{V}^{\star 2}$.)
The linearization of (16) is:

$$\begin{aligned} \partial_t h + \mathcal{V}^\star\partial_x h + \mathcal{H}^\star\partial_z v &= \delta_1 \ , \\ \partial_t v + \mathcal{V}^\star\partial_x v + g\partial_x h + g\partial_x b &= C_f\frac{\mathcal{V}^{\star 2}}{\mathcal{H}^{\star 2}} - 2C_f\frac{\mathcal{V}^\star}{\mathcal{H}^\star}u + \delta_2 \ , \\ \partial_t b + a\mathcal{V}^{\star 2}\partial_x v &= \delta_3 \ . \end{aligned}$$

In Riemann coordinates we get, for $k \in \{1, 2, 3\}$,

$$\partial_t y_k + \lambda_k\partial_x y_k + \sum_{s=1}^{3}(2\lambda_s - 3\mathcal{V}^\star)\theta_s y_s = \delta_k \ , \qquad (17)$$

where λ_k are some (distinct) constant values

θ_k are some physical values.

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This system is

$$\partial_t y + \Lambda \partial_x y = F y + \delta(x,t) \ ,$$

where $y = (y_1,\ y_2,\ y_3)^\top$, $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$, and, for all $x \in [0, L],\ t \geq 0,\ F = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{pmatrix}$

Λ and F are not simultaneously diagonalizable.

Design of a stabilizing boundary control

Let us explain how out theorem can be applied to design a stabilizing boundary feedback control.

Boundary conditions

1) Operation of the gate at outflow of the reach:

$$\mathcal{H}(L,t)\mathcal{V}(L,t) = k_g\sqrt{[\mathcal{H}(L,t) - u_1(t)]^3}$$

2) Value of the channel inflow rate

$$\mathcal{H}(0,t)\mathcal{V}(0,t) = u_2(t)$$

3) Physical constraint on the bathymetry

$$\mathcal{B}(0,t) = \mathcal{B}^\star$$

Two boundary control laws u_1 and u_2

By linearizing these boundary conditions,
with suitable choice of the u_i we get in Riemann coordinates:

$$\begin{aligned} y_1(L,t) &= k_{12} y_2(L,t) + k_{13} y_3(L,t) \\ y_2(0,t) &= k_{21} y_1(0,t) \end{aligned}$$

for tuning parameters k_{12}, k_{13} and k_{21} in $\mathbb{R}$.
The last boundary condition is:

$$\sum_i [(\lambda_i - \mathcal{V}^\star)^2 - g\mathcal{H}^\star] y_i(0,t) = 0$$

How to compute k_{12}, k_{13} and k_{21}?
How to compute an ISS Lyapunov function?

To summarize we get:

$$\begin{gathered}\partial_t y + \Lambda \partial_x y = Fy + \delta(x,t) \\ y(0,t) = Ky(L,t)\end{gathered}$$

with

$$K = \begin{pmatrix} 0 & k_{12} & k_{13} \\ k_{21} & 0 & 0 \\ \xi(k_{21}) & 0 & 0 \end{pmatrix},$$

and

$$\xi(k_{21}) = -\frac{[(\lambda_1 - \mathcal{V}^\star)^2 - g\mathcal{H}^\star] + k_{21}[(\lambda_2 - \mathcal{V}^\star)^2 - g\mathcal{H}^\star]}{(\lambda_3 - \mathcal{V}^\star)^2 - g\mathcal{H}^\star}.$$

Assumption 1 is ok.

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Assumption 1 is ok.

Assumption 2 holds as soon as there exists a symmetric positive definite matrix Q such that

$$\begin{array}{c} \mathtt{Sym}(Q\Lambda - K^\top Q\Lambda K) \geq 0 \ , \\ \mathtt{Sym}(QF) \leq 0 \ . \end{array} \tag{18}$$

Note that, given K, computing Q is a convex problem in a cone

Numerically tractable problem

The equilibrium is chosen as in [Dos Santos, CP, 08]:
$\mathcal{H}^\star = 0.13$ [m], $\mathcal{V}^\star = 15$ [ms^{-1}], and $\mathcal{B}^\star = 0$ [m].
We use $\lambda_1 = -10$, $\lambda_2 = 7.72 \times 10^{-4}$, $\lambda_3 = 13$. With K given by

$$k_{12} = 0 \ , \quad k_{13} = 0 \ , \quad k_{21} = -0.095 \ ,$$

we compute a solution of (18):

$$Q = \begin{pmatrix} 8.1 \times 10^7 & -2.7 \times 10^3 & -7.2 \times 10^7 \\ \star & 2.9 \times 10^2 & 2.1 \times 10^3 \\ \star & \star & 6.5 \times 10^7 \end{pmatrix}$$

which ensures that Assumption 2 holds.

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Final remarks on this application

Thus selecting $\mu = 1.5 \times 10^{-2}$, we compute the following ISS Lyapunov function, defined by, for all y in $L^2(0, L)$,

$$V(y) = \int_0^L y(x) Q y(x) e^{-\mu x} \, dx$$

for the Saint-Venant–Exner system.

Note that the computed controller is a locally stabilizing boundary control.

It depends only on the height at both ends of the channel and the bathymetry of the water.

Does not depend on all the state.

Output feedback law only.

More details in [CP, Mazenc, 11]

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Design of an ISS Lyapunov function for a parabolic PDE
Magnetic flux in a Tokamak plasma: With [Blum, 1989], or [E. Witrant, *et al*, 2007], we have to consider

$$\partial_t z = \partial_r \left[\frac{\eta}{r} \partial_r [rz] \right] + \partial_r [\eta u] \,, \;\; r \in (0,1), \;\; t \geq 0 \qquad (19)$$

where

- r in the normalized position in the small disc.
- Tokamak = Torus
  but no dependence wrt the angle and to the height variable
- z is the inverse of the "safety factor" that should be controlled
- $\eta = \eta(r,t)$ is the diffusion
- u is the control from the ECCD[1] antennas.

[1] ECCD=Electron Cyclotron Current Drive

5.2 – Control of the flux in a Tokamak plasma

Design of an ISS Lyapunov function for a parabolic PDE
Magnetic flux in a Tokamak plasma: With [Blum, 1989], or [E. Witrant, *et al*, 2007], we have to consider

$$\partial_t z = \partial_r \left[\frac{\eta}{r}\partial_r [rz]\right] + \partial_r [\eta u], \;\; r \in (0,1), \;\; t \geq 0 \qquad (19)$$

where

- r in the normalized position in the small disc.
- Tokamak = Torus
  but no dependence wrt the angle and to the height variable
- z is the inverse of the "safety factor" that should be controlled
- $\eta = \eta(r,t)$ is the diffusion
- u is the control from the ECCD[1] antennas.

[1] ECCD=Electron Cyclotron Current Drive

The Dirichlet boundary conditions

$$z(0,t) = z(1,t) = 0,\ \forall t \in [0,T) \tag{20}$$

and initial condition:

$$z(r,0) = z_0(r),\ \forall r \in (0,1) \tag{21}$$

Control Lyapunov function candidate:

$$V(z) = \frac{1}{2}\int_0^1 f(r) z^2 dr;\ \ f(r) > 0\ \forall r \in [0,1]$$

with some function $f : [0,1] \to (0,\infty)$ twice continuously differentiable.

Theorem [Bribiesca, CP *et al*, 11]

If there exist a C^1 f and $\alpha > 0$ such that, $\forall r \in [0,1]$, $\forall t \geq 0$,

$$f''(r)\eta + f'(r)\left[\partial_r \eta - \frac{\eta}{r}\right] + f(r)\left[\frac{\partial_r \eta}{r} - \frac{\eta}{r^2}\right] \leq -\alpha f(r),$$

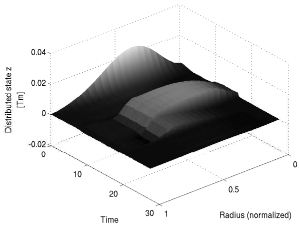
then, along the solutions of (19), (20), (21),

$$\dot{V} \leq -\alpha V(z) + \int_0^1 f(r)\partial_r\left[\eta u\right] z dr, \ \forall t \geq 0$$

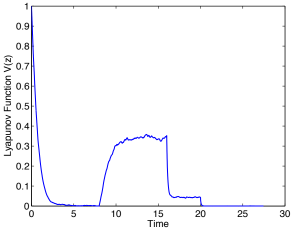
and thus with $u = -\frac{\gamma}{\eta}\int_0^r z(\rho,t)d\rho$, where $\gamma \geq 0$ is a tuning parameter, the system is globally exponentially stable.

Illustration of ISS property

Full-physics simulator to describe the evolution of $\eta = \eta(r, t)$
Experimental data drawn from Tore Supra shot 35109
Actuator perturbation for $t \in [8, 20]$ s
control action for $t \geq 16$ s ($\gamma = 0.75$).



(a) Solution of the PDE.



(b) Normalized evolution of the Lyapunov function.

See [Bribiesca, CP *et al*, 11] for more informations

Conclusion and open questions

We have considered two problems

For Locally Exp. Stable hyperbolic system, the attractivity may be lost in presence of perturbations

- **1** Stability analysis of non-homogeneous non-linear hyperbolic system

estimating the influence of the perturbations

on the Riemann coordinates

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- **2** Sensitivity of stable non-homogeneous hyperbolic system wrt perturbations

perturbations are bounded $\Rightarrow$ state is bounded

Input-to-State Stability

Sensitivity of linear space-dependent time-varying hyperbolic systems wrt perturbations

It parallels what have been done for a class of semilinear parabolic PDEs

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Open questions

- ISS for nonlinear hyperbolic systems.
  We are working on the Lyapunov function that is derived in [Coron, Bastin, and d'Andréa-Novel, 08]
- Applications of ISS?
  Does it give the offset that we have seen on an experimental channel?
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$$\partial_t y + \Lambda(x,t)\partial_x y = F(x,t)y + \delta(x,t)\ , \tag{22}$$

$$y(0,t) = Ky(1,t)\ . \tag{23}$$

First Step: $\dot{W} \leq 0???$

Prove that the function $W(y) = \displaystyle\int_0^1 y(x)^\top Q y(x) e^{-\mu x} dx$, is a weak Lyapunov function when δ is identically equal to zero

We note first that, for all $y \in L^2(0,1)$,

$$\frac{1}{\beta}\int_0^1 |y(x)|^2\ dx \leq W(y) \leq \beta \int_0^1 |y(x)|^2\ dx \tag{24}$$

with $\beta = \max\left\{\|Q\|, \frac{e^\mu}{\lambda_Q}\right\}$, and λ_Q is the smallest eigenvalue of Q.

To do that, we compute the time-derivative of W along the solutions of (22) with (23):

$$\dot{W} = -R_\Lambda(y(.,t),t) + R_F(y(.,t),t) + R_\delta(y(.,t),t) ,$$

with

$$R_\Lambda(y,t) = 2\int_0^1 y(x)^\top Q\Lambda(x,t)\partial_x y(x) e^{-\mu x} dx ,$$

$$R_F(y,t) = 2\int_0^1 y(x)^\top QF(x,t)y(x) e^{-\mu x} dx ,$$

$$R_\delta(y,t) = 2\int_0^1 y(x)^\top Q\delta(x,t) e^{-\mu x} dx .$$

Now, observe that

$$\begin{aligned} R_\Lambda(y,t) &= \int_0^1 \partial_x (y(x)^\top Q\Lambda(x,t) y(x)) e^{-\mu x} dx \\ &\quad - \int_0^1 y(x)^\top Q \partial_x \Lambda(x,t) y(x) e^{-\mu x} dx \ . \end{aligned}$$

Performing an integration by part on the first integral and using the boundary condition we get:

$$\begin{aligned} \dot{W} &= -y(1,t)^\top Q\Lambda(1,t) y(1,t) e^{-\mu} + y(1,t)^\top K^\top Q\Lambda(1,t) K y(1,t) \\ &\quad + \tilde{R}_\Lambda(y,t) + R_F(y,t) + R_\delta(y,t) \ . \end{aligned}$$

with

$$\begin{aligned} \tilde{R}_\Lambda(y,t) &= -\mu \int_0^1 y(x)^\top Q\Lambda(x,t) y(x) e^{-\mu x} dx \\ &\quad + \int_0^1 y(x)^\top Q \partial_x \Lambda(x,t) y(x) e^{-\mu x} dx \ . \end{aligned}$$

By grouping the terms and using the notation

$$N(t) = K^\top Q\Lambda(1,t)K \ , \ M(x,t) = \mu\Lambda(x,t) - \partial_x\Lambda(x,t) - 2F(x,t)$$

we obtain

$$\begin{aligned}\dot{W} &= y(1,t)^\top \left[N(t) - e^{-\mu} Q\Lambda(1,t)\right] y(1,t) \\ &\quad - \int_0^1 y(x,t)^\top QM(x,t)y(x,t)e^{-\mu x}dx \\ &\quad + 2\int_0^1 y(x,t)^\top Q\delta(x,t)e^{-\mu x}dx \ .\end{aligned}$$

With Assumption 2 and our choice for μ (sufficiently small), we get

$$\dot{W} \leq -\mu r(t)\int_0^1 |y(x,t)|^2 e^{-\mu x}dx + 2\int_0^1 y(x,t)^\top Q\delta(x,t)e^{-\mu x}dx \ ,$$

with $r(t) \geq 0$.

It follows that, for all $\kappa > 0$,

$$\begin{aligned}\dot{W} &\leq -\frac{\mu}{||Q||} r(t) W(y) + 2\|Q\|\kappa \int_0^1 |y(x,t)|^2 e^{-\mu x} dx \\ &\quad + \frac{\|Q\|}{2\kappa} \int_0^1 |\delta(x,t)|^2 e^{-\mu x} dx \\ &\leq -q_\kappa(t) W(y) + \frac{\|Q\|}{2\kappa} \int_0^1 |\delta(x,t)|^2 dx \ ,\end{aligned}$$

with $q_\kappa(t) = \frac{\mu}{\|Q\|} r(t) - \frac{2\|Q\|\kappa}{\lambda_Q}$.

End of the first step

W is not exactly a weak Lyapunov function when $\delta \equiv 0$.
But the mean value of r is positive and κ can be arbitrarily small
Thus W is a weak Lyapunov function "by mean"

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Second Step

Use the positive mean value of r to modify W.

Let us consider the time-varying candidate Lyapunov function

$$V(t,y) = e^{s_\kappa(t)} W(y) \ ,$$

with $s_\kappa(t) = \frac{1}{T}\int_{t-T}^{t}\int_{\ell}^{t} q_\kappa(m) dm d\ell$.

One get

$$\begin{aligned} \dot{V} \ &\leq \ -e^{s_\kappa(t)} q_\kappa(t) W(y) + \frac{\|Q\|}{2\kappa} e^{s_\kappa(t)} \int_0^1 |\delta(x,t)|^2 dx \\ &\quad + e^{s_\kappa(t)} \left[q_\kappa(t) - \frac{1}{T}\int_{t-T}^{t} q_\kappa(m) dm \right] W(y) \ . \end{aligned}$$

Since r is periodic of period T, we have

$$\int_{t-T}^{t} q_\kappa(m) dm \ = \ \frac{\mu}{\|Q\|} R - \frac{2T\|Q\|\kappa}{\lambda_Q} \ ,$$

where R is the mean value of r.

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