A Review on Old and New Results on Robust Regulation of Distributed Parameter Systems with Infinite-Dimensional Exosystems

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Robustness

- Robustness is a property that allows systems to maintain its functions despite external and internal perturbations.
- A system must be robust to function in unpredictable environments using unreliable components.
- Robustness is a fundamental feature of evolvable complex systems.

Kitano: Biological Robustness, Nature (2004).
Sontag: Molecular Systems Biology and Control, European J. Control (2009).

Introduction

- For linear finite-dimensional systems we have the Internal Model Principle (IMP) by Francis and Wonham from the 1970's: A stabilizing feedback controller solves the robust output regulation problem if and only if it incorporates a suitably reduplicated model (a p-copy) of the signal generator.
- Regulation: Output will asymptotically track given reference signals and reject given disturbance signals.
- Robust Regulation: Regulation occurs despite perturbations in the system's parameters.
- The purpose of this presentation is the generalization of the IMP to infinite-dimensional systems.

Previous work

- Structurally stable synthesis (Bhat 1976). Finite dimensional exosystem, mainly for time-delay systems.
- Yamamoto and Hara (1988): frequency domain generalization of the Internal Model Principle for systems having a pseudorational impulse response.
- Robust control: Logemann, Townley, Pohjolainen, Hämäläinen.
- Regulator theory without robustness: Schumacher (1983), Byrnes et. al. (2000).
- Internal model for infinite-dimensional systems Immonen (2006,2007), Hämäläinen (2010), Paunonen (2010), jointly with Pohjolainen.

Plant, exosystem, and controller

The plant is described by the state-space equations

$$\begin{aligned} \dot{x} &= Ax + Bu + F_s v, \qquad x(0) \in X \\ y &= Cx + Du + F_m v, \end{aligned}$$

where v is generated by the exosystem

$$\dot{v} = Sv, \qquad v(0) \in W.$$

The controller is described by the equations

$$\begin{aligned} \dot{z} &= \mathcal{G}_1 z + \mathcal{G}_2 e, \qquad z(0) \in Z \\ u &= Kz, \end{aligned}$$

where $e = y - r = y - F_r v$.

Assumptions

- The state spaces X, W, and Z, the input space U and the output space Y are Hilbert spaces.
- $A, S, \mathcal{G}_1$ generate C_0-semigroups.
- All other operators are bounded.
- The pair (A, B) is exponentially stabilizable.
- The pair (A, C) is exponentially detectable.
- $\sigma(A) \cap \sigma(S) = \emptyset$ and the plant transfer function $P(s) = C(sI - A)^{-1}B + D$ is a bijection for $s \in \sigma(S)$.

Standard form

The plant state and the tracking error $e = y - r = y - F_r v$ can put into the standard form

$$\begin{aligned} \dot{x} &= Ax + Bu + Ev, \qquad x(0) \in X \\ e &= Cx + Du + Fv, \end{aligned}$$

where $E = F_s \in \mathcal{L}(W, X)$ and $F = F_m - F_r \in \mathcal{L}(W, Y)$.

Closed Loop System

The closed loop system consists of the plant and the controller

$$\dot{x}_e = A_e x_e + B_e v, \qquad x_e(0) \in X_e$$
$$e = C_e x_e + D_e v,$$

on the Hilbert space $X_e = X \times Z$ where

$$A_e = \begin{bmatrix} A & BK \\ \mathcal{G}_2 C & \mathcal{G}_1 + \mathcal{G}_2 DK \end{bmatrix}, \quad B_e = \begin{bmatrix} E \\ \mathcal{G}_2 F \end{bmatrix} \in \mathcal{L}(W, X_e).$$

with $\mathcal{D}(A_e) = \mathcal{D}(A) \times \mathcal{D}(\mathcal{G}_1)$ and

$$C_e = \begin{bmatrix} C & DK \end{bmatrix} \in \mathcal{L}(X_e, Y), \quad D_e = F \in \mathcal{L}(W, Y)$$

Output Regulation Problem

Definition

The controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ solves the Output Regulation Problem (ORP) if

(i) The closed-loop system operator A_e generates a (strongly/weakly) stable C_0-semigroup.

(ii) For all initial states $x_e(0) \in X_e$ and $v(0) \in W$

$$\lim_{t \to \infty} e(t) = 0$$

either strongly or weakly.

Dynamical steady-state operator H_{ss}

Lemma

Suppose $H_{ss} \in \mathcal{L}(W, X_e)$ is an operator satisfying $H_{ss}\mathcal{D}(S) \subset \mathcal{D}(A_e)$. The following are equivalent:

(a) *H_{ss} satisfies the Sylvester equation*

$$H_{ss}S - A_e H_{ss} = B_e \quad \text{on } \mathcal{D}(S). \tag{1}$$

(b) *For $x_e(0) \in X_e$, $v(0) \in W$ the mild solution of the closed-loop system can be written as*

$$x_e(t) = T_{A_e}(t)(x_e(0) - H_{ss}v(0)) + H_{ss}v(t),$$

where $v(t) = T_S(t)v(0)$.

Importance of the dynamical steady-state operator

The extended state decomposes into two parts

$$x_e(t) = T_{A_e}(t)(x_e(0) - H_{\mathsf{ss}} v(0)) + H_{\mathsf{ss}} v(t).$$

For stable A_e, as $t \to \infty$

$$\begin{aligned} x_e(t) &\approx H_{\mathsf{ss}} v(t), \\ e(t) &\approx (C_e H_{\mathsf{ss}} + D_e) v(t). \end{aligned}$$

Hence regulation = stabilization + tracking.

Sylvester Equation $\implies$ ORP

Theorem

If A_e generates a stable C_0-semigroup and there exists an operator $H_{ss} \in \mathcal{L}(W, X_e)$ which satisfies $H_{ss}\mathcal{D}(S) \subset \mathcal{D}(A_e)$ and the constrained Sylvester equation

$$H_{ss}S - A_e H_{ss} = B_e, \text{ on } \mathcal{D}(S) \tag{2a}$$

$$C_e H_{ss} + D_e = 0, \tag{2b}$$

then the controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ solves the ORP.
Equation (2b) is called the regulation constraint.

Robust Output Regulation Problem (RORP)

Definition

The controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ solves the Robust Output Regulation Problem (RORP) if

(i) The controller solves the ORP.

(ii) (i) holds for a class of perturbations on the system parameters (A, B, C, D, E, F).

Robust Output Regulation Problem (RORP)

As

$$e(t) = C_e T_{A_e}(t)(x_e(0) - H_{\mathsf{ss}} v(0)) + (C_e H_{\mathsf{ss}} + D_e) T_S(t) v(0)$$

the RORP is divided into two parts: Finding a controller that

(i) is robustly stabilizing

(ii) robustly satisfies the regulation constraint.

Internal Model Structure

Definition

The controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ has Internal Model Structure (IMS) (Immonen), if for every $\Gamma \in \mathcal{L}(W, Z)$ and $\Delta \in \mathcal{L}(W, Y)$ with $\Gamma \mathcal{D}(S) \subset \mathcal{D}(\mathcal{G}_1)$ the following implication holds:

$$\Gamma S = \mathcal{G}_1 \Gamma + \mathcal{G}_2 \Delta \quad \text{on } \mathcal{D}(S) \implies \Delta = 0. \tag{3}$$

Robust Regulation Constraint

Suppose that

$$H_{ss} = \begin{bmatrix} \Pi \\ \Gamma \end{bmatrix}.$$

satisfies the Sylvester equation $H_{ss}S - A_e H_{ss} = B_e$ on $\mathcal{D}(S)$.

Now assume the system parameters (A, B, C, D, E, F) are perturbed to (A, B, C, D, E, F) so that the perturbed Sylvester equation $H_{ss}S - A_e H_{ss} = B_e$ has the solution H_{ss}. Then

$$\Pi S = A\Pi + BK\Gamma + E,$$

$$\Gamma S = \mathcal{G}_1 \Gamma + \mathcal{G}_2 (C\Pi + DK\Gamma + F).$$

Robust Controller

If the controller has IMS, this implies that

$$C\Pi + DK\Gamma + F = C_e H_{ss} + D_e = 0.$$

Hence the perturbed regulation constraint is also satisfied.

Theorem

Assume that the controller has IMS. If A_e generates a strongly/weakly (robustly) stable C_0-semigroup and there exists an operator $H_{ss} \in \mathcal{L}(W, X_e)$ satisfying $H_{ss}\mathcal{D}(S) \subset \mathcal{D}(A_e)$ and the Sylvester equation $H_{ss}S - A_e H_{ss} = B_e$ on $\mathcal{D}(S)$, then the controller solves the RORP.

The exosystem

Suppose the exosystem is given by

$$Sv = \sum_{n=-\infty}^{\infty} \mathrm{i}\omega_n \langle v, \phi_n \rangle \phi_n,$$

with

$$\mathcal{D}(S) = \left\{ v \in W \;\middle|\; \sum_{n=-\infty}^{\infty} \omega_n^2 |\langle v, \phi_n \rangle|^2 < \infty \right\}$$

$(\phi_n)_{n\in\mathbb{Z}}$ is an orthonormal basis of W and $(\mathrm{i}\omega_n)_{n\in\mathbb{Z}}$ has no finite accumulation points.

The exosystem

S generates the C_0-group

$$T_S(t)v_0 = \sum_{n=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega_n t} \langle v_0, \phi_n \rangle \phi_n, \quad v_0 \in W.$$

The reference signal is given by

$$r(t) = F_r T_S(t) v_0 = \sum_{n=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega_n t} \langle v_0, \phi_n \rangle F_r \phi_n,$$

for some operator $F_r \in \mathcal{L}(W, Y)$.

The exosystem

- The behaviour of the sequence $(\langle v_0, \phi_n \rangle F_r \phi_n)_{n \in \mathbb{Z}}$ determines the class of allowable reference signals.
- Hence the choice of v_0 and F_r can be used to control the smoothness of r.
- Similar considerations hold for the disturbance signals.
- Results can be extended for exosystems with Jordan block structure (Paunonen, Pohjolainen (2010))

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Solvability of the Sylvester Equation : Convergence condition

Lemma

Suppose that $\mathrm{i}\omega_n \in \rho(A_e)$ *for* $n \in \mathbb{Z}$. *There exists a unique* $H_{ss} \in \mathcal{L}(W, X_e)$ *satisfying* $H_{ss}\mathcal{D}(S) \subset \mathcal{D}(A_e)$ *and the Sylvester equation* $H_{ss}S - A_eH_{ss} = B_e$ *iff*

$$\sup_{\|x_e\|\leq 1} \sum_{n=-\infty}^{\infty} |\langle R(\mathrm{i}\omega_n; A_e)B_e\phi_n, x_e\rangle|^2 < \infty. \tag{4}$$

IMS and $\mathcal{G}$-conditions

Definition

The controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ satisfies $\mathcal{G}$-conditions if

(1) $\mathcal{N}(\mathcal{G}_2) = \{0\}$.

(2) $\mathcal{R}(\mathcal{G}_2) \cap \mathcal{R}(\mathcal{G}_1 - \mathrm{i}\omega_n I) = \{0\} \quad \forall n \in \mathbb{Z}$.

Theorem (Paunonen)

A controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ satisfies the $\mathcal{G}$-conditions if and only if it has IMS.

$$\Gamma S = \mathcal{G}_1 \Gamma + \mathcal{G}_2 \Delta \quad \text{on } \mathcal{D}(S) \implies \Delta = 0.$$

$\mathcal{G}$-conditions and p-copy internal model

Theorem (Paunonen)

Let $\dim(Y) = p$ *and* $\sigma(S) \cap \sigma(A_e) = \emptyset$ *. The controller* $(\mathcal{G}_1, \mathcal{G}_2, K)$ *satisfies the* $\mathcal{G}$*-conditions iff*

(1) $i\omega_k \in \sigma_p(\mathcal{G}_1)$

(2) *The geometric multiplicity of* $i\omega_k$ *as an eigenvalue of* $\mathcal{G}_1$ *is at least* p*.*

IMS and $\mathcal{G}$-conditions

Lemma

The controller satisfies the $\mathcal{G}$-conditions on $Z = Z_1 \times Z_2$ if $\mathcal{G}_1$ and $\mathcal{G}_2$ are of the form

$$\mathcal{G}_1 = \begin{bmatrix} R_1 & R_2 \\ 0 & G_1 \end{bmatrix}, \qquad \mathcal{G}_2 = \begin{bmatrix} R_3 \\ G_2 \end{bmatrix},$$

where G_1 and G_2 satisfy the $\mathcal{G}$-conditions

(1) $\mathcal{N}(G_2) = \{0\}$.

(2) $\mathcal{R}(G_2) \cap \mathcal{R}(G_1 - i\omega_n I) = \{0\} \quad \forall n \in \mathbb{Z}$.

Choosing G_1 and G_2

The following choice of G_1 and G_2 gives a controller with IMS: $Z_2 = \ell^2(Y)$, $G_1 (y_n)_{n\in\mathbb{Z}} = (\mathrm{i}\omega_n y_n)_{n\in\mathbb{Z}}$ with domain

$$\mathcal{D}(G_1) = \left\{ (y_n)_{n\in\mathbb{Z}} \in \ell^2(Y) \;\middle|\; \sum_{n=-\infty}^{\infty} \omega_n^2 \|y_n\|^2 < \infty \right\}$$

and $G_2 \in \mathcal{L}(Y, \ell^2(Y))$, $G_2 y = (G_{2n} y)_{n\in\mathbb{Z}}$ where $G_{2n} \in \mathcal{L}(Y)$ are bijections that satisfy

$$\sum_{n=-\infty}^{\infty} \omega_n^2 \|G_{2n}\|^2 < \infty.$$

G_1 has a p-copy of S

S and G_1 can be viewed as infinite diagonal matrices

$$S = \begin{bmatrix} \ddots & & 0 \\ & i\omega_n & \\ 0 & & \ddots \end{bmatrix}, \quad G_1 = \begin{bmatrix} \ddots & & 0 \\ & i\omega_n I & \\ 0 & & \ddots \end{bmatrix}$$

If $\dim Y = p < \infty$, then each $i\omega_n$ occurs p times in G_1, i.e., G_1 contains a p-copy of S. If $\dim Y = \infty$, G_1 has an ∞-copy of S.

A Stabilizing Controller

Let $Z = Z_1 \times Z_2 = X \times \ell^2(Y)$, $K = [K_1 \;\; K_2]$, G_1 and G_2 as before and

$$\mathcal{G}_1 = \begin{bmatrix} A + BK_1 + L(C + DK_1) & (B + LD)K_2 \\ 0 & G_1 \end{bmatrix},$$

$$\mathcal{G}_2 = \begin{bmatrix} -L \\ G_2 \end{bmatrix}, \qquad L \in \mathcal{L}(Y, X).$$

Choice of K and L

- Choose L so that $A + LC$ is exponentially stable.
- Let $K_1 = K_{11} + K_{12}$ and choose K_{11} so that $A + BK_{11}$ is exponentially stable.
- Choose $K_{12} = K_2 H$ where H is the solution of the Sylvester equation

$$G_1 H - H(A + BK_{11}) = G_2(C + DK_{11}).$$

- Finally choose $K_2 = -B_1^*$ where

$$B_1 = HB + G_2 D.$$

A Stabilization Theorem

Theorem

The above controller weakly stabilizes the closed-loop operator A_e *and satisfies the* $\mathcal{G}$*-conditions. If* Y *is finite-dimensional, then* A_e *is strongly stable. (Hämäläinen, Pohjolainen (2010), Benchimol (1978))*

Conditions on E and F

With the above choice of controller parameters the condition $\sup_{\|x_e\|\leq 1} \sum_{n=-\infty}^{\infty} |\langle R(i\omega_n; A_e)B_e\phi_n, x_e\rangle|^2 < \infty$ holds if

$$\sum_{n=-\infty}^{\infty} \|R(i\omega_n; A_e)\|^2 < \infty.$$

Asymptotically

$$\|R(i\omega_n; A_e)\|^2 \approx \|P(i\omega_n)^{-1}\|^2 \|(P_K(i\omega_n)^* G_{2n}^*)^{-1}\|^2 \{\|E\phi_n\|^2, \|F\phi_n\|^2\}$$

Conditions on E and F

Since the reference and disturbance signals are of the form

$$\sum_{n=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega_n t} \langle v_0, \phi_n \rangle E\phi_n$$

these conditions put constraints on the sequences $(E\phi_n)_{n\in\mathbb{Z}}$ and $(F\phi_n)_{n\in\mathbb{Z}}$ and on the behavior of the system's transfer functions at the infinity (Laakkonen (2011)).

Conclusions

- A review of robust controllers for infinite dimensional systems with infinite-dimensional input and output spaces and infinite-dimensional exosystems was given .
- Easily checkable conditions for the robustness of the regulation constraint are given
- Necessary and sufficient conditions for the boundedness of the dynamic steady state operator are given. These are related to the behaviour of the transfer function of the plant on the spectrum of the exosystem and the smoothness of the reference and disturbance signals.

Conclusions

- An observer based controller incorporating an infinite dimensional internal model that strongly/weakly stabilizes the closed loop system was constructed.
- If the input and output spaces are p-dimensional, then the controller contains a p-copy of the infinite-dimensional exosystem. For infinite-dimensional input and output spaces this generalizes the concept of p-copy to ∞-copy.

Further work

- Robustness of strong/weak stability.
- Robustness of the solution of Sylvester equation
- Robustness of the condition $\mathrm{i}\omega_n \in \rho(A_e)$ and the convergence condition

$$\sup_{\|x_e\|\leq 1} \sum_{n=-\infty}^{\infty} |\langle R(\mathrm{i}\omega_n; A_e) B_e \phi_n, x_e \rangle|^2 < \infty.$$

- More general plant, e.g., unbounded B and C.