Interpolation, Carleson measures and controllability

Jonathan R. Partington (Leeds, UK)

CDPS, Wuppertal, July 2011

Joint work with Birgit Jacob and Sandra Pott

Contents

- 1. Interpolation in Hardy spaces.
- 2. Controllability of linear systems.
- 3. Model spaces and controllability.

Hardy spaces and Blaschke sequences

Recall the Laplace transform $\mathcal{L} : L^2(0,\infty) \to H^2(\mathbb{C}_+)$ is an isomorphism (Paley–Wiener), where
$H^2(\mathbb{C}_+)$ is the Hardy space on the right-hand half-plane $\mathbb{C}_+$, i.e., analytic functions f such that

$$\|f\|^2 := \sup_{x>0} \int_{-\infty}^{\infty} |f(x+iy)|^2 \, dy < \infty.$$

Two types of sequences of interest:
Blaschke sequences satisfy

$$\sum_n \frac{\operatorname{Re} z_n}{1+|z_n|^2} < \infty,$$

zero sets of functions in $H^2(\mathbb{C}_+)$, i.e., we can find a nonzero function vanishing at all (z_n) if and only if (z_n) is Blaschke.

Carleson sequences (interpolating sequences)

Carleson sequences satisfy

$$\inf_n d_n > 0,$$

where

$$d_n = \prod_{k \neq n} \left| \frac{z_k - z_n}{z_k + \overline{z_n}} \right|.$$

Much stronger than Blaschke, means that for every bounded sequence (c_n) we can solve $f(z_n) = c_n$ in $H^\infty(\mathbb{C}_+)$, bounded analytic functions.

Old question: find a condition on (b_n) and (z_n) such that for every $(c_n) \in \ell^2$ there is a function $g \in H^2(\mathbb{C}_+)$ such that

$$b_n g(z_n) = c_n.$$

This is a Carleson-type interpolation problem.

Can we always solve $b_n g(z_n) = c_n$?

Exact result needed given by McPhail (1990) based on ideas of Shapiro–Shields (1961).

The NSC is that

$$\nu := \sum_{n=1}^{\infty} \frac{|\operatorname{Re} z_n|^2}{|b_n|^2 d_n^2} \delta_{z_n}$$

is a Carleson measure on $\mathbb{C}_+$.

Here δ_z denotes a Dirac (point mass) at z.

That is, $\nu(Q_h) \leq \text{const.} h$, where $Q_h = (0, h) \times (y, y + h)$ is any Carleson square.

We don't need (z_n) to be a Carleson sequence, but it must be a Blaschke sequence.

Linear semigroup systems

H a complex Hilbert space, $(T(t))_{t\geq 0}$ a strongly continuous semigroup of bounded operators, i.e.,

$$T(0) = I, \quad T(t+u) = T(t)T(u), \quad \text{and}$$

$$t \mapsto T(t)x \quad \text{is norm-continuous for each } x \in H.$$

Let A be the infinitesimal generator, defined on domain $\mathrm{dom}(A) \subseteq H$:

$$Ax = \lim_{t\to 0} \frac{1}{t}(T(t) - I)x.$$

The Cauchy problem

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0$$

has mild solution

$$x(t) = T(t)x_0.$$

Controlled semigroups

We begin with a continuous-time infinite-dimensional linear system in state form, including control operator.

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t),$$

with $\quad x(0) = x_0, \quad$ say.

Here u is the input, x the state.

In general B (the control operator) is unbounded.

Diagonal semigroups

An important special case where interpolation ideas come in (includes some heat equations, vibrating structures, etc.): suppose that

$$A\phi_n = \lambda_n \phi_n,$$

with (ϕ_n) normalized eigenvectors forming a Riesz basis in H. Eigenvalues $\lambda_n = -z_n$. So every $x \in H$ can be written

$$x = \sum_{n=1}^{\infty} c_n \phi_n$$

with

$$K_1 \sum |c_n|^2 \le \|x\|^2 \le K_2 \sum |c_n|^2.$$

We lose no generality by supposing (ϕ_n) is an orthonormal basis.

More on diagonal semigroups

Note that

$$T(t)\sum_{n=1}^{\infty} c_n\phi_n = \sum_{n=1}^{\infty} c_n e^{\lambda_n t}\phi_n.$$

We often suppose exponential stability, i.e.,

$$\|T(t)\| \le Me^{-\lambda t}, \qquad (t \ge 0),$$

for some $M > 0$ and $\lambda > 0$.
That is, $\sup_n \mathrm{Re}\,\lambda_n < 0$.

Controllability

Look at the equation

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t),$$

with solution

$$x(t) = T(t)x_0 + \int_0^t \widetilde{T}(t-s)Bu(s)\,ds,$$

interpreted (if B is unbounded) by extending $T(t)$ to a larger space $\widetilde{H}$ spanned by the (ϕ_n), i.e., completion under a smaller norm.
An important operator to consider is

$$\mathcal{B}_\infty u = \int_0^\infty \widetilde{T}(t)Bu(t)\,dt,$$

where $\widetilde{T}$ is an extension of T to $\widetilde{H}$.

Exact controllability

The system is **exactly controllable**, if $\operatorname{Im} \mathcal{B}_\infty \supseteq H$,

Let $\mathcal{U}$ be the space in which $u(t)$ lies, i.e., on which B is defined. If $\dim \mathcal{U} = 1$ (which we'll do in detail), then $B : \mathbb{C} \to \widetilde{H}$ so can be written

$$Bw = w \sum_{n=1}^{\infty} b_n \phi_n.$$

Then a short calculation gives

$$\mathcal{B}_\infty u = \sum_{n=1}^{\infty} b_n \widehat{u}(-\lambda_n) \phi_n,$$

where the hat is a Laplace Transform, $\widehat{u} = \mathcal{L}u$.

Interpolation and exact controllability

Thus **exact controllability** is equivalent to:
for every $(c_n) \in \ell^2$ there is a function $g \in H^2(\mathbb{C}_+)$ such that

$$b_n g(-\lambda_n) = c_n.$$

This is the Carleson-type interpolation problem seen already.
The NSC is that

$$\nu = \sum_{n=1}^{\infty} \frac{|\mathrm{Re}\,\lambda_n|^2}{|b_n|^2 d_n^2} \delta_{-\lambda_n}$$

is a Carleson measure on $\mathbb{C}_+$. As usual,

$$d_n = \prod_{k \neq n} \left| \frac{\lambda_k - \lambda_n}{\lambda_k + \overline{\lambda_n}} \right|.$$

Exact and null controllability

We don't need $(-\lambda_n)$ to be a Carleson sequence, but it must be a Blaschke sequence, e.g., $\lambda_n = -n^\beta$ with $0 < \beta \le 1$ is never exactly controllable.

A weaker concept is **null controllability**: we ask $\operatorname{Im} \mathcal{B}_\infty \supseteq T(\tau)H$ for some $\tau \ge 0$.

The NSC now becomes that

$$\nu = \sum_{n=1}^{\infty} \frac{|\operatorname{Re} \lambda_n|^2}{|b_n|^2 d_n^2} e^{2\tau \operatorname{Re} \lambda_n} \delta_{-\lambda_n}$$

is a Carleson measure on $\mathbb{C}_+$.

Some extensions of the theory

1. The vectorial case. Solved by more sophisticated interpolation theory (Blaschke–Potapov products). Gives results when $N = \dim \mathcal{U} < \infty$.
2. A Riesz basis of eigenvectors is not necessary. We can allow for finitely-many non-trivial Jordan blocks (no more than one per eigenvalue). Similar results hold.
3. Perturbation results. The eigenvalues can be perturbed without destroying the Carleson measure property, and controllability properties are preserved. For example if $\lambda_n = -n^2$ (heat equation), then we can move λ_n by as much as $O(n^{1/2})$.

Finite-time controllability

What follows is well-known and much easier if the $z_n := -\lambda_n$ are a Carleson sequence. We do not make this assumption.

Question: **when does infinite-time exact controllability imply finite-time exact controllability?**

The system is *finite-time controllable* in time $\tau > 0$ if for every $x \in H$ there exists a control $u \in L^2(0, \tau; \mathbb{C}^N)$ such that

$$x = \sum_{n=1}^{\infty} \int_0^{\tau} e^{-\lambda_n(\tau - s)} \langle u(s), b_n \rangle \, ds \, \phi_n.$$

Exact controllability, with u restricted to $L^2(0, \tau; \mathbb{C}^N)$.

Condition (JZ)

The following condition on the scalar- or matrix-valued Blaschke (or Blaschke–Potapov) product β will be central.

Condition (JZ). There are constants a, $\delta > 0$ such that $\|(\beta(s))^{-1}\| \le \frac{1}{a}$ on the strip $S_\delta = \{s \in \mathbb{C} : 0 < \mathrm{Re}\, s < \delta\}$.

JZ is Jacob and Zwart, who were doing something else entirely!
Scalar condition (JZ) is equivalent to the conditions that $\inf \mathrm{Re}\, z_n > 0$ and

$$\sup_{y\in\mathbb{R}} \sum_{n=1}^{\infty} \frac{\mathrm{Re}\, z_n}{(\mathrm{Re}\, z_n)^2 + (y - \mathrm{Im}\, z_n)^2} < \infty.$$

Note $\inf \mathrm{Re}\, z_n > 0$ already implied by exponential stability.
Example: $z_n = 1 + in^{3/4}$ is Blaschke but not (JZ).

Model spaces

Write

$$K_\beta = H^2(\mathbb{C}_+) \ominus \beta H^2(\mathbb{C}_+)$$

and

$$K_{\theta_\tau} = \mathcal{L}L^2(0,\tau) = H^2(\mathbb{C}_+) \ominus \theta_\tau H^2(\mathbb{C}_+),$$

where $\theta_\tau(s) = \exp(-\tau s)$.

Classical Lemma The following conditions are equivalent:

1. the operator $P_{K_{\theta_\tau}|K_\beta}$ is bounded below;
2. the Toeplitz operator $T_{\overline{\beta}\theta_\tau}$ is bounded below on $H^2(\mathbb{C}_+)$;
3. the Hankel operator $\Gamma_{\overline{\beta}\theta_\tau}$ has norm strictly less than 1, i.e., $\mathrm{dist}(\overline{\beta}\theta_\tau, H^\infty) < 1$;
4. given $F \in H^2(\mathbb{C}_+)$ there exists a function $G \in K_{\theta_\tau}$ such that $F(z_n) = G(z_n)$ for all n.

Main Theorem

Condition (JZ) holds **if and only if** the equivalent conditions of the lemma hold for some $\tau > 0$. More precisely, there exists a constant $m > 0$ such that given a, δ from Condition (JZ), the equivalent conditions of the lemma hold for each

$$\tau > \frac{2}{\delta} \log(4\sqrt{2e}(a^{-1} + 1)/\sqrt{\pi}) \qquad \text{if } N = 1$$

or for each

$$\tau > \frac{2}{\delta} \log(m(\log(N+1))(a^{-1} + 1)/\sqrt{\pi}) \qquad \text{if } N > 1.$$

Reproducing kernel theses

One of the tricks here is to estimate the norms of Hankel operators using the following quantitative version of Bonsall's reproducing kernel thesis.

Theorem. Let $k_\lambda(s) = 1/(s+\lambda)$ for $s \in \mathbb{C}_+$, and $K_\lambda = k_\lambda/\|k_\lambda\|_2$. For a Hankel operator $\Gamma : H^2(\mathbb{C}_+) \to H^2(\mathbb{C}_-)$ we have

$$\|\Gamma\| \leq M \sup_{\lambda \in \mathbb{C}_+} \|\Gamma K_\lambda\|,$$

where M can be taken to be $4\sqrt{2e}$.

Proved via a link to Carleson embeddings, for which a similar result is known (constants due to Petermichl, Treil and Wick).

Finite-time controllability

Theorem. Suppose that Condition (JZ) holds and that the system is exactly controllable in infinite time. Then it is exactly controllable in any time τ satisfying

$$\tau > \begin{cases} \frac{2}{\delta}\log(4\sqrt{2e}(a^{-1}+1)/\sqrt{\pi}) & \text{if } N = 1, \\ \frac{2}{\delta}\log(m(\log(N+1))(a^{-1}+1)/\sqrt{\pi}) & \text{if } N > 1, \end{cases}$$

for some constant $m > 0$ which does not depend on N.

THE END (in finite time)