Decay of Hankel singular values with applications to model reduction

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Model reduction

Model Reduction

By mathematical means replace an elaborate model with a simpler one that is close to the original.

- $$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = 0, \quad y(t) = Cx(t),$$
- Simpler means: of the same form, but with smaller state space dimension.
- Close means: input-output maps $u \mapsto y$ close in the $\mathcal{L}(L^2(0,\infty;\mathscr{U}), L^2(0,\infty;\mathscr{Y}))$ norm.
  (or in the gap metric)

Example: 1D heat equation

$$
\begin{aligned}
&\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2}, && t > 0, \quad x \in (0,1), \\
&w(0,x) = 0, && \qquad\quad\; x \in (0,1), \\
&w_x(t,0) = u(t), && t > 0, \\
&w(t,1) = 0, && t > 0, \\
&y(t) = -w(t,0), && t > 0.
\end{aligned}
$$

$$\mathsf{G}(s) = \frac{\tanh\sqrt{s}}{\sqrt{s}}.$$

Some reduction methods

- Three standard numerical PDE methods
  - Finite Element Method
  - Eigenvector based method (Modal truncation)
  - Chebyshev Collocation Method
- Lyapunov balanced truncation

Petrov–Galerkin methods

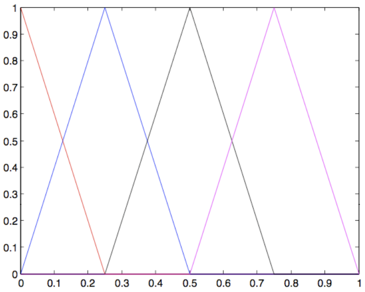
Weak form:

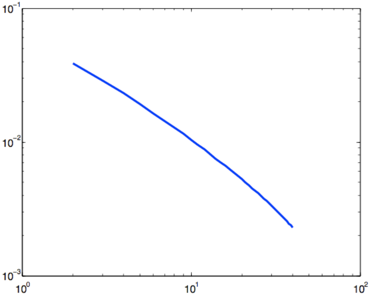
$$\langle \dot{x}(t), v \rangle = \langle Ax(t) + Bu(t), v \rangle$$

- Approximate solution: seek $x : [0, \infty) \to \mathscr{W}$ such that weak form holds for all $v \in \mathscr{V}$.
- $\mathscr{W}$ trial space,
- $\mathscr{V}$ test space.

Finite Element Method

- Trial space $\mathscr{W}$ and test space $\mathscr{V}$ piecewise polynomial functions.





- Full order input-output map: $\mathcal{D}$,
- Reduced order input-output map: $\mathcal{D}_n$.

Error estimate for heat equation example

$$\|\mathcal{D} - \mathcal{D}_n\|_{\mathcal{L}(L^2(0,\infty;\mathscr{U}),L^2(0,\infty;\mathscr{Y}))} \le \frac{C}{n}.$$

Better known estimate for FEM

The well-known error bound

$$\|\mathcal{D} - \mathcal{D}_n\|_{\mathcal{L}(L^2(0,\infty;\mathscr{U}),L^2(0,\infty;\mathscr{Y}))} \leq \frac{C}{n^2}.$$

is obtained for the more familiar interior-interior case (left).

$$\begin{aligned}
\frac{\partial w}{\partial t} &= \frac{\partial^2 w}{\partial x^2} + u(t,x), \\
w(0,x) &= 0, \\
w_x(t,0) &= 0, \\
w(t,1) &= 0, \\
y(t) &= w(t,\cdot).
\end{aligned}$$

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\end{aligned}$$

Eigenvector based method

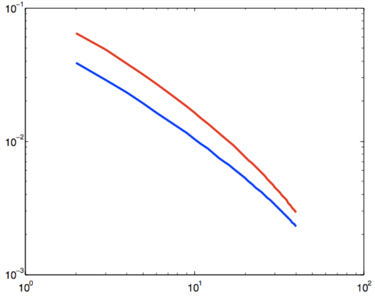
- Trial space $\mathscr{W}$ span of dominant eigenvectors of A,
- Test space $\mathscr{V}$ span of dominant eigenvectors of A^*,
- Notion of 'dominant' depends on B and C.

For the heat equation example

$$\|\mathcal{D} - \mathcal{D}_n\|_{\mathcal{L}(L^2(0,\infty;\mathscr{U}),L^2(0,\infty;\mathscr{Y}))} = \sum_{k=n+1}^{\infty} \frac{2}{(\frac{\pi}{2} + k\pi)^2}.$$

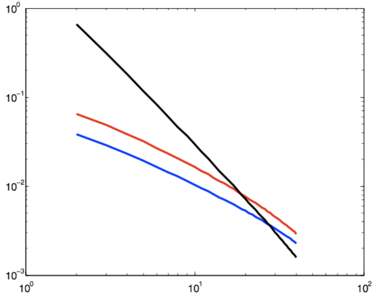
So

$$\frac{c}{n} \leq \|\mathcal{D} - \mathcal{D}_n\|_{\mathcal{L}(L^2(0,\infty;\mathscr{U}),L^2(0,\infty;\mathscr{Y}))} \leq \frac{C}{n}.$$



Chebyshev Collocation Method

- Trial space $\mathscr{W}$ consists of Chebyshev polynomials,
- Test space $\mathscr{V}$ consists of Dirac delta's at collocation points,
- Apply boundary bordering to include boundary conditions.



Numerically

$$\|\mathcal{D} - \mathcal{D}_n\|_{\mathcal{L}(L^2(0,\infty;\mathscr{U}),L^2(0,\infty;\mathscr{Y}))} \approx \frac{C}{n^2}.$$

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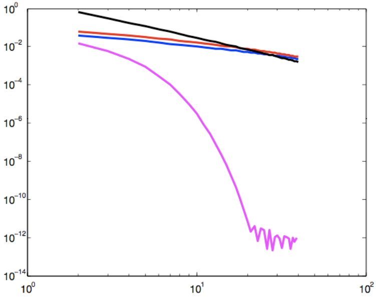
- Lyapunov balanced truncation



Lyapunov balanced truncation

- Lyapunov balanced truncation (Moore '81).
- $(\sigma_k)_{k=1}^{\infty}$ singular values of the Hankel operator of the system.
- The following error-bound holds

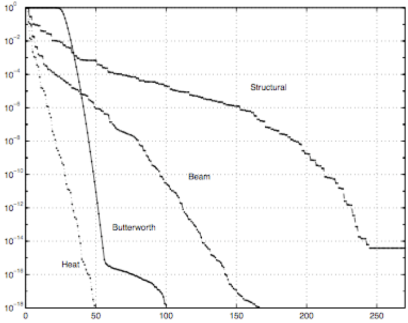
$$\|\mathcal{D} - \mathcal{D}_n\| \leq 2 \sum_{k=n+1}^{\infty} \sigma_k,$$

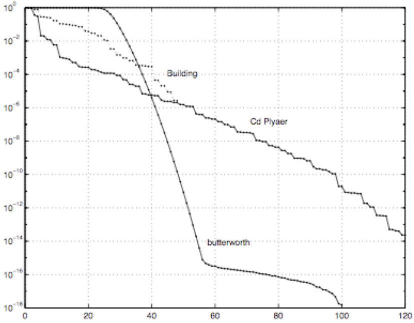
  (Glover and Enns '84, Glover–Curtain–Partington '88, Guiver–Opmeer '11).
- For any input-output map $\mathcal{D}_n$ of a system with an n-dimensional state space:

$$\sigma_{n+1} \leq \|\mathcal{D} - \mathcal{D}_n\|.$$

- Decay rate of σ_k gives information about the decay of the error.

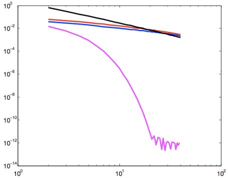
Taken from an article by Antoulas:





Decay rate

- Any nonincreasing sequence of nonnegative numbers $(\sigma_k)_{k=1}^{\infty}$ can be the sequence of singular values of a Hankel operator (Ober, Treil '90).
- Estimates for delay differential equations (Glover–Lam–Partington '91): decay rate of k^{-p} for $(\sigma_k)_{k=1}^{\infty}$ can occur for any $p \in \mathbb{N}_0$.
- Opmeer (SCL 2010): for analytic systems (e.g. parabolic PDEs) for all $p > 0$
$$\sum_{k=1}^{\infty} \sigma_k^p < \infty,$$
so that for all $q > 0$ we have $k^q \sigma_k \to 0$.
- It follows that for $r \geq 0$ there exists a $C_r > 0$ such that
$$\|\mathcal{D} - \mathcal{D}_n\| \leq \frac{C_r}{n^r}.$$



- If A bounded (and exponentially stable), then exponential decay.

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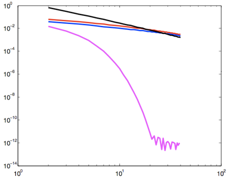

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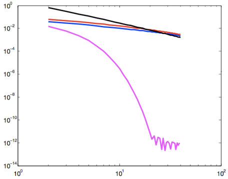

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Analytic control systems

- Analytic control systems:
  - A generates an exponentially stable analytic C_0 semigroup,
  - B and C are jointly no more unbounded than A
    - $B \in \mathcal{L}(\mathscr{U}, \mathscr{X}_\beta)$,
    - $C \in \mathcal{L}(\mathscr{X}_\alpha, \mathscr{Y})$,
    - $\beta - \alpha < 1$.
- Analytic control systems with either $\mathscr{U}$ or $\mathscr{Y}$ finite-dimensional have a S_p Hankel operator ($p > 0$).
- Proof uses
  - Peller–Semmes characterization of Schatten class Hankel operators in terms of their symbols belonging to the Besov space $B_{pp}^{1/p}$ ('84),
  - The generation theorem for analytic semigroups to show that the transfer function is in the Besov space $B_{pp}^{1/p}$.
- Improves result of Curtain and Sasane 2001 who assumed both $\mathscr{U}$ and $\mathscr{Y}$ finite-dimensional and proved only the case $p \geq 1$ (by very different methods).

Optimality of finite elements

$$\begin{aligned} \frac{\partial w}{\partial t} &= \frac{\partial^2 w}{\partial x^2} + u(t,x), \\ w(0,x) &= 0, \\ w_x(t,0) &= 0, \\ w(t,1) &= 0, \\ y(t) &= w(t,\cdot). \end{aligned}$$

$$\|\mathcal{D}-\mathcal{D}_n\|_{\mathcal{L}(L^2(0,\infty;\mathscr{U}),L^2(0,\infty;\mathscr{Y}))} \leq \frac{C}{n^2}.$$

- Peller–Semmes theorem: $\mathsf{G} \in B_{pp}^{1/p}(\mathbb{C}_0^+; S_p(\mathscr{U}, \mathscr{Y}))$.
- $\mathsf{G}(s) = (sI - A)^{-1} \in S_p(L^2(0,1), L^2(0,1))$,
- $p > 1/2$,
- $\sigma_k \approx \frac{1}{k^2}$.

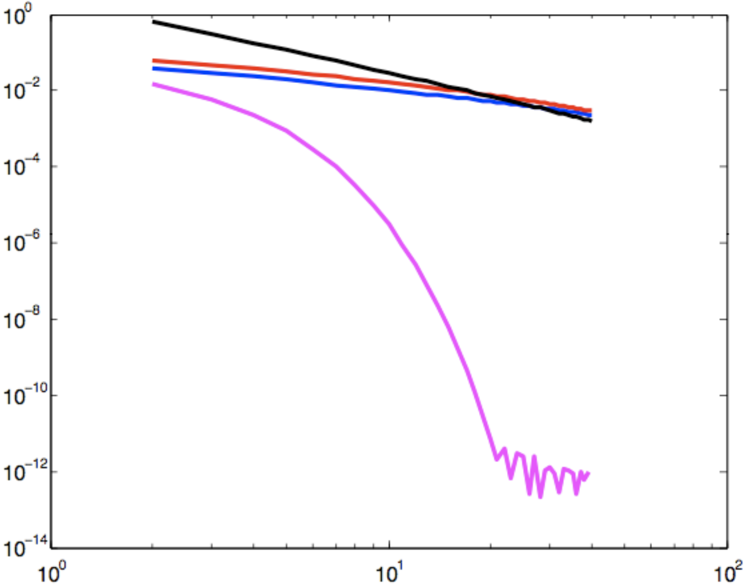
Computation

- The Lyapunov balanced truncation cannot be analytically computed for a PDE.
- What is done:
  1. Apply a numerical method to the PDE $\to \mathcal{D}^N$ and aim to obtain a Lyapunov balanced truncation $\mathcal{D}_n^N$ of $\mathcal{D}^N$.
  2. Use numerical linear algebra to obtain an approximation of $\mathcal{D}_n^N$.
- Under conditions valid for most numerical PDE methods ($h^N \to^{L^1} h$):

$$\lim_{N\to\infty} \|\mathcal{D}_n^N - \mathcal{D}_n\| = 0,$$

(Singler '09 for bounded B and C; Guiver and Opmeer '11).

Thank you



Besov spaces

- Bergman kernel (for the right half-plane):

$$K(z,w) := \frac{1}{(z+\bar{w})^2}.$$

- Weighted Bergman space $A^{p,r}(\mathbb{C}_0^+;\mathscr{B})$ with $p > 0$ and $r > -\frac{1}{2}$: $f : \mathbb{C}_0^+ \to \mathscr{B}$ analytic and

$$\int_0^\infty \int_{-\infty}^\infty \|f(x+iy)\|_{\mathscr{B}}^p \, K(x+iy, x+iy)^{-r} \, dy \, dx < \infty.$$

- Besov space $B_{pp}^{1/p}(\mathbb{C}_0^+;\mathscr{B})$: $f : \mathbb{C}_0^+ \to \mathscr{B}$ analytic and

$$f^{(n)} \in A^{p,\frac{np}{2}-1}(\mathbb{C}_0^+;\mathscr{B}),$$

for some integer $n > \frac{1}{p}$ (equivalently: for all integers $n > \frac{1}{p}$).