Optimal control of fractional systems: numerics under diffusive formulation

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Outline

Fractional guys ? almost everywhere !

Fractional differential systems have become quite popular in the recent decades, giving rise to a wide literature, both on the theoretical and on the applied sides :

- monogaphs,
- international journals : *Fractional Calculus and Applied Analysis*,*Fractional Dynamics and Applications*,
- special issues of international journals,

and also

- international conferences : *Fractional Differentiation and its Applications*
- workshops of international conferences,

are now devoted to this active research field !

What about optimal control, then ?

However, even if different scientific communities seem to have been involved in these questions, still very few papers are concerned with the question of optimal control of fractional differential systems.

In e.g. [Tricaud & Chen (2010)] or [Defterli (2010)],

1. ad hoc finite-dimensional approximations of fractional derivatives are used in the first place,
2. classical optimal control methods are being applied in the second place ;

But no proof of convergence of the process is provided.

Why is it so ?

Possible answers :

1. optimal control of infinite-dimensional systems is a quite involved and technical field,
2. the very nature of fractional operators itself : causal, but highly non-local in time ; hence their adjoint becomes necessarily anti-causal and still... non-local in time.

Thus, we will be left with coupled forward and backward fractional dynamics in order to solve the optimal control problem for fractional differential systems.
$\Longrightarrow$ at first glance, it seems very unlikely that Riccati equations could be either analysed or even solved (not to speak of adequate numerical schemes for these) in such a complicated setting !

So, what ?

In order to overcome this intrinsic difficulty, we propose to use the equivalent diffusive representations of fractional systems, and to work on it, as for infinite dimensional systems of integer order !

Let us recall diffusive representations of fractional operators and their adjoints and see how these can be useful for optimal control problems, on a series of models of decreasing complexity, namely :

1. Webster-Lokshin Wave equation,
2. A Fractionally Damped Oscillator,
3. An Oscillator Damped by Memory Variables.

Outline

Some useful identities

Let $\beta \in (0,\ 1)$, in the frequency domain, we have :

$$\begin{aligned} H_\beta : \quad \mathbb{C} \setminus \mathbb{R}^- &\to \mathbb{C} \\ s &\mapsto \int_0^\infty \mu_\beta(\xi)\, \frac{1}{s+\xi}\, \mathrm{d}\xi = \frac{1}{s^\beta}\,, \text{with } \mu_\beta(\xi) \propto \xi^{-\beta}\,. \end{aligned}$$

So to speak, fractional transfer functions H_β are nothing but a superposition of first-order systems, with appropriate weight μ_β.

Equivalently, in the time domain, this reads :

$$\begin{aligned} h_\beta : \quad \mathbb{R}^+ &\to \mathbb{R} \\ t &\mapsto \int_0^\infty \mu_\beta(\xi)\, e^{-\xi\, t}\, \mathrm{d}\xi = \frac{1}{\Gamma(\beta)} t^{\beta-1}\,. \end{aligned}$$

So to speak, fractional kernels h_β are nothing but a superposition of decaying exponential, with weight μ_β.

$\Longrightarrow$ Input-output & State-space representations can be derived for fractional integrals I^β and derivatives D^α.

Input-output representation

Let u and $y = I^\beta u$ be the input and output of the *causal* fractional integral of order β, defined by the Riemann-Liouville formula $y = h_\beta \star u = \int_0^t h_\beta(t-\tau)\, u(\tau)\, \mathrm{d}\tau$ in the time domain, which reads $Y(s) = H_\beta(s)\, U(s)$ in the Laplace domain :

$$y(t) = \int_0^\infty \mu_\beta(\xi)\, [e_\xi \star u](t)\, \mathrm{d}\xi\,,$$

with $e_\xi(t) := e^{-\xi\, t}\, 1_{t \geq 0}$, and $[e_\xi \star u](t) = \int_0^t e^{-\xi\,(t-\tau)}\, u(\tau)\, \mathrm{d}\tau$.

Now for fractional *derivative* of order $\alpha \in (0,1)$ in the sense of distributions of Schwartz, we have $\widetilde{y} = D^\alpha u = D[I^{1-\alpha} u]$, and a careful computation shows that :

$$\widetilde{y}(t) = \int_0^\infty \mu_{1-\alpha}(\xi)\, \left[u - \xi\, e_\xi \star u\right](t)\, \mathrm{d}\xi\,.$$

State-space representations

Let $\varphi(\xi,.) := [e_\xi \star u](t)$ be the state, parametrized by ξ.

$$\partial_t \varphi(\xi,t) = -\xi\,\varphi(\xi,t) + u(t),\ \ \varphi(\xi,0) = 0\,, \quad (1)$$

$$y(t) = \int_0^\infty \mu_\beta(\xi)\,\varphi(\xi,t)\,\mathrm{d}\xi\,; \quad (2)$$

and

$$\partial_t \widetilde{\varphi}(\xi,t) = -\xi\,\widetilde{\varphi}(\xi,t) + u(t),\ \ \widetilde{\varphi}(\xi,0) = 0\,, \quad (3)$$

$$\widetilde{y}(t) = \int_0^\infty \mu_{1-\alpha}(\xi)\,[u(t) - \xi\,\widetilde{\varphi}(\xi,t)]\,\mathrm{d}\xi\,. \quad (4)$$

are state-space representations for I^β and D^α, respectively.

Note : functional spaces must be specified for these representations to make sense ; more precisely :

- φ belongs to $\mathcal{H}_\beta := \{\varphi \ s.t.\ \int_0^\infty \mu_\beta(\xi)|\varphi|^2\,\mathrm{d}\xi < \infty\}$,
- $\widetilde{\varphi}$ belongs to $\widetilde{\mathcal{H}}_\alpha := \{\widetilde{\varphi} \ s.t.\ \int_0^\infty \mu_{1-\alpha}(\xi)|\widetilde{\varphi}|^2\,\xi\,\mathrm{d}\xi < \infty\}$;

see e.g. [Haddar and M. (2008)].

Adjoints of fractional integrals

On $\mathcal{H} = L^2(0, T)$, the adjoint of the causal fractional integrator $I^{\beta}_{0+} : u \mapsto h_\beta \star u$, defined by

$$y(t) := I^{\beta}_{0+} u(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1}\, u(\tau)\, \mathrm{d}\tau\,,$$

is $I^{\beta}_{T-} : z \mapsto v$, the anti-causal fractional integral, defined by

$$v(\tau) := I^{\beta}_{T-} z(\tau) = \frac{1}{\Gamma(\beta)} \int_\tau^T (t-\tau)^{\beta-1}\, z(t)\, \mathrm{d}t\,.$$

$\Longrightarrow$ quite difficult to handle, especially in coupled situations of optimal control !
$\Longrightarrow$ Need to make it easier.
$\Longrightarrow$ Extend diffusive representation to anti-causal context !
(see e.g. [M. (2009)] for a first definition of those).

The backward diffusive realization (1)

Let the *backward* dynamical system :

$$\begin{aligned} \partial_t \psi(\xi,\tau) &= +\xi\, \psi(\xi,\tau) - z(\tau), \quad \text{for} \quad 0 < \tau < T, && (5) \\ \text{with} \quad \psi(\xi, T) &= 0 \quad \text{as final condition}\,; && (6) \end{aligned}$$

together with the output, defined by :

$$v(\tau) = \int_0^{\infty} \mu_\beta(\xi)\, \psi(\xi,\tau)\, \mathrm{d}\xi\,;$$

they provide a realization for $v = I^\beta_{T-} z$.

Moreover, the fundamental equality holds :

$$(I^\beta_{0+} u, z)_{L^2(0,T)} = (u, I^\beta_{T-} z)_{L^2(0,T)}\,. \tag{7}$$

Proof : straightforward, simply relies on properties of real-valued exponentials.

Adjoints of fractional derivatives

On $\mathcal{H} = L^2(0, T)$, the adjoint of the causal fractional derivative $D^\alpha_{0+} : u \mapsto \frac{d}{dt}(h_{1-\alpha} \star u)$, defined *on its domain* by

$$\widetilde{y}(t) := D^\alpha_{0+} u(t) = \frac{d}{dt}\left[\frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha}\, u(\tau)\, \mathrm{d}\tau\right],$$

is $D^\alpha_{T-} : z \mapsto \widetilde{v}$, the anti-causal fractional derivative, defined *on its domain* by

$$\widetilde{v}(\tau) := D^\alpha_{T-} z(\tau) = -\frac{d}{d\tau}\left[\frac{1}{\Gamma(1-\alpha)} \int_\tau^T (t-\tau)^{-\alpha}\, z(t)\, \mathrm{d}t\right].$$

Note : the derivatives are to be understood in the sense of distributions of Schwartz.

The backward diffusive realization (2)

Let the *backward* dynamical system :

$$\begin{aligned} \partial_t \widetilde{\psi}(\xi,\tau) &= +\xi\,\widetilde{\psi}(\xi,\tau) - z(\tau), \quad \text{for} \quad 0 < \tau < T, && (8)\\ \text{with} \quad \widetilde{\psi}(\xi,T) &= 0 \quad \text{as final condition}\,; && (9) \end{aligned}$$

together with the extended output, defined by :

$$\widetilde{v}(\tau) = \int_0^{\infty} \mu_{1-\alpha}(\xi)\,\left[z(\tau) - \xi\,\widetilde{\psi}(\xi,\tau)\right]\,\mathrm{d}\xi\,;$$

they provide a realization for $\widetilde{v} = D^{\alpha}_{T-} z$.

Moreover, the fundamental equality holds :

$$(D^{\alpha}_{0+} u, z)_{L^2(0,T)} = (u, D^{\alpha}_{T-} z)_{L^2(0,T)}\,, \qquad (10)$$

Proof : less straightforward, but still relies on properties of real-valued exponentials.

Outline

Webster-Lokshin Wave equation

1. Webster-Lokshin Wave equation :

$$\partial_t^2 w + \varepsilon(x) D_{0+}^{1/2}[\partial_t w] + d(x)\, \partial_t w + \eta(x)\, I_{0+}^{1/2}[\partial_t w] - \partial_x^2 w = 0\,,$$

for $0 < x < L$ and $t > 0$, with boundary control $u_e(t)$ at $x = 0$, and initial conditions.
Provided $\varepsilon(x) > 0$, $d(x) \geq 0$ and $\eta(x) > 0$, once the diffusive reformulation has been used, we can prove :

- existence and uniqueness, see e.g. [Haddar & M. (2008)],
- asymptotic stability, see [M. (2006)], [M. & Prieur (2011)],
- consistent and accurate numerical schemes, see e.g. [Haddar, Li & M. (2009)], also [Li (2010)].

A finite horizon optimal control problem, reformulated in the new framework presented above, will become tractable with the theory of optimal control for linear PDEs, because the system is now no more than the coupling of a 1D wave equation with two 1D diffusion equations... still to be done !

A Fractionally Damped Oscillator

2. A Fractionally Damped Oscillator : together with dynamic boundary conditions of Robin type, the Lokshin model has a Riesz basis of eigenvectors (studied in [M. (1996)], see also [Kergomard, Debut & M. (2006)]) : the projection of the PDE onto one mode gives rise to a fractionally damped oscillator, studied in [M. & Prieur (2005)], and for which elementary propreties and numerical simulations have been presented in e.g. [Deü & M. (2010)].

$$\ddot{x} + D^{\alpha}_{0+}[\dot{x}] + \dot{x} + I^{\beta}_{0+}[\dot{x}] + \omega^2 x = u_e \, ,$$

The above framework is well suited to the formulation of an optimal control problem of this system in a classical setting, with no more fractional operators and no more heredity : only the diffusive subsystems are infinite dimensional.

Oscillator Damped by Memory Variables

3. An Oscillator Damped by Memory Variables (toy model) : Discretizing the diffusive representations of $y = I_{0^+}^{\beta} u$ on K points, and $\tilde{y} = D_{0^+}^{\alpha} u$ on L points, in a consistent way, we get :

$$\ddot{x} + \tilde{y} + \dot{x} + y + \omega^2 x = u_e \,,$$

with three types of damping :

- $\dot{x} = u$, instantaneous w.r.t u ;
- $y(u)$, with memory and low-pass behaviour : measure μ consists of finitely many (K) Dirac measures located at some ξ_k with positive weights μ_k ;
- $\tilde{y}(u)$ with memory and high-pass behaviour : measure ν consists of finitely many (L) Dirac measures located at some ξ_l with positive weights ν_l.

Outline

Methodology

The objective is to minimize the energy functional

$$J(u_e) = \frac{1}{2}\int_0^T X^t(\tau)\, Q\, X(\tau) + u_e(\tau)^2 \,\mathrm{d}\tau + \frac{1}{2} X^t(T)\, D_T\, X(T)$$

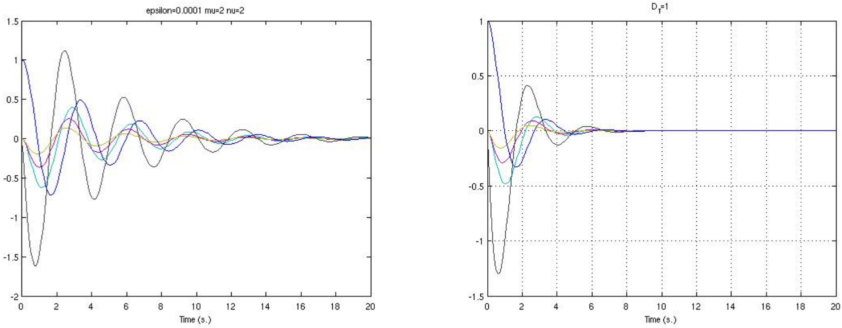
with an external input u_e on the toy model, a controlled dynamical systems.

Why ? Because the diffusive components with small ξ_k (big τ_k) display a long-memory behaviour that is typical for fractional systems ! Thus, the objective is to make the convergence to equilibrium much faster.

$\Longrightarrow$ solve Dynamic Riccati Equation on $S(t)$, of dimension $(2+K+L)\times(2+K+L)$, thanks to a Runge-Kutta method, then apply the time-varying feedback on the state X.

Time domain simulations

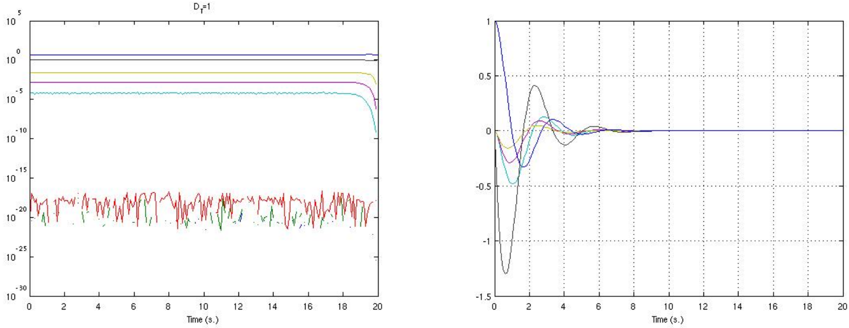
Parameters : $K = L = 3$, and $T = 20$, $D_T = 1$.



Left : Open Loop, Right : Closed Loop (feedback from DRE).

A plot of the SVD of the Riccati matrix

Parameters : $K = L = 3$, and $T = \infty$.



Left : SVD plot, Right : Closed Loop (feedback from ARE).

An interesting idea ?

An interesting idea follows from the plot the singular values of the Riccati matrix versus time : why not apply the infinite-time feedback, solution of the Algebraic Riccati equation, then ? (much easier, allows greater values of K and L).

But... this heuristics cannot be used to prove any convergence results, since the diffusive approximations converge on finite horizons only.

$\Longrightarrow$ there is indeed a need for low dimensional representation of complexity, by :

- interpolation methods,
- optimization methods.

Outline

Re-interpreting Sobolev spaces

- Optimization in the frequency domain, stemming from

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- Norms in L^2, or Sobolev spaces H^s, are defined as :

$$\|h\|^2_{H^s(\mathbb{R}_t)} = \int_{\mathbb{R}_f} w_s(f)\,|H(2i\pi f)|^2\,\mathrm{d}f\,, \text{ with } w_s(f) = (1+4\pi^2 f^2)^s\,.$$

  where $s \in \mathbb{R}$ tunes the balance between low and high frequencies.

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- For specific applications, more general frequency dependent weights can be used : bounded frequency range, logarithmic scale, relative error measurement, bounded dynamics ... see e.g. [Hélie & M. (2006)].

Building up specific weights for audio applications

For audio applications, $w(f)$ can be adapted and modified according to the following requirements :

1. a bounded frequency range $f \in [f^-, f^+]$: $w(f)\, 1_{[f^-,f^+]}(f)$;
2. a frequency log-scale : $w(f)/f$;
3. a relative error measurement : $w(f)/|H(2i\pi f)|^2$
4. a relative error on a bounded dynamics : $w(f)/\left(\mathrm{Sat}_{H,\Theta}(f)\right)^2$ where the saturation function $\mathrm{Sat}_{H,\Theta}$ with threshold Θ is defined by

$$\mathrm{Sat}_{H,\Theta}(f) = \begin{cases} |H(2i\pi f)| & \text{if } |H(2i\pi f)| \geq \Theta_H \\ \Theta_H & \text{otherwise} \end{cases}$$

Note : normalization of the samples is desirable in most audio applications, before the sequence is sent to DAC audio converters.

Regularized criterion with equality constraints

- Let $\widetilde{H_\mu}(s) = \sum_{k=1}^{K} \mu_k (s + \xi_k)^{-1}$; based on Bode diagrams, a heuristic choice for the $\{\xi_k\}_{1 \leq k \leq K}$ leads to a geometric sequence on a frequency range of interest.

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- The regularized criterion reads :

$$\mathcal{C}_R(\mu) = \int_{\mathbb{R}^+} \left| \widetilde{H_\mu}(2i\pi f) - H(2i\pi f) \right|^2 w(f) \mathrm{d}f + \sum_{k=1}^{K} \epsilon_k |\mu_k|^2,$$

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- Equality constraints for $\widetilde{H}_\mu^{(d_j)}$ at prescribed frequency points η_j, $1 \leq j \leq J$ are taken into account thanks to a Lagrangian $\mathcal{C}_{R,L}$ by adding to $\mathcal{C}_R$:

$$\Re\mathrm{e} \left(\ell^* \begin{bmatrix} H^{(d_1)}(2i\pi\eta_1) - \widetilde{H}_\mu^{(d_1)}(2i\pi\eta_1) \\ \vdots \\ H^{(d_J)}(2i\pi\eta_J) - \widetilde{H}_\mu^{(d_J)}(2i\pi\eta_J) \end{bmatrix} \right),$$

Discrete criterion

- Discrete version of the criterion for frequencies increasing from $f_1 = f_- $ to $f_{N+1} = f_+$ is, with $s_n = 2i\pi f_n$:

$$\mathcal{C}(\mu) \approx \sum_{n=1}^{N} w_n \left| \widetilde{H}_\mu(s_n) - H(s_n) \right|^2 \text{ with } w_n = \int_{f_n}^{f_{n+1}} w(f)\mathrm{d}f.$$

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- In matrix notations, this rewrites

$$\mathcal{C}_{R,L}(\boldsymbol{\mu}) = (\boldsymbol{M}\boldsymbol{\mu} - \boldsymbol{h})^* \boldsymbol{W} (\boldsymbol{M}\boldsymbol{\mu} - \boldsymbol{h}) + \boldsymbol{\mu}^t \boldsymbol{E} \boldsymbol{\mu} + \Re\mathrm{e}\left(\ell^* \left[\boldsymbol{k} - \boldsymbol{N}\boldsymbol{\mu} \right] \right),$$

$$\text{with} \begin{cases} \boldsymbol{M} : & \text{model} & N \times K \\ \boldsymbol{N} : & \text{constraint model} & J \times K \\ \boldsymbol{E} : & \text{regularization} & K \times K \\ \boldsymbol{W} : & \text{weights} & N \times N \\ \boldsymbol{h} : & \text{data} & N \times 1 \\ \boldsymbol{k} : & \text{constraints} & J \times 1 \end{cases}$$

Closed-form solutions !

- If $J = 0$ (no constraint), the solution reduces to

$$\boldsymbol{\mu} = \mathcal{M}^{-1}\mathcal{H}\,,$$

where $\mathcal{M} = \Re\mathrm{e}\left(\boldsymbol{M}^{*}\boldsymbol{W}\boldsymbol{M} + \boldsymbol{E}\right)$ and $\mathcal{H} = \Re\mathrm{e}\left(\boldsymbol{M}^{*}\boldsymbol{W}\boldsymbol{h}\right)$.

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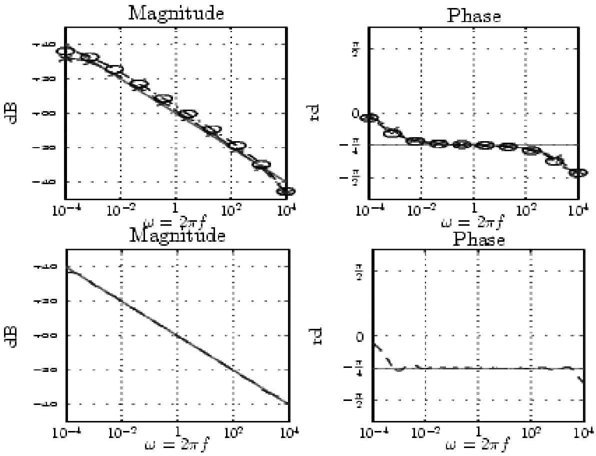
- For $J \geq 1$, the solution reads :

$$\boldsymbol{\mu} = \mathcal{M}^{-1}\left[\mathcal{H} + \underline{\boldsymbol{N}}^t\mathcal{N}^{-1}\left(\underline{\boldsymbol{k}} - \underline{\boldsymbol{N}}\mathcal{M}^{-1}\mathcal{H}\right)\right],$$

where $\mathcal{N} = \underline{\boldsymbol{N}}\mathcal{M}^{-1}\underline{\boldsymbol{N}}^t$ is invertible for non-redundant constraints, and $\left\{ \begin{array}{lll} \underline{\boldsymbol{N}}^t & \text{denotes} & [\Re\mathrm{e}(\boldsymbol{N}^t), \Im\mathrm{m}(\boldsymbol{N}^t)] \\ \underline{\boldsymbol{k}}^t & \text{denotes} & [\Re\mathrm{e}(\boldsymbol{k}^t), \Im\mathrm{m}(\boldsymbol{k}^t)] \end{array} \right.$.

Our example : $H_\beta(s) = s^{-\beta}, \mu_\beta(-\xi) \propto \xi^{-\beta}$



Top : Interpolation, $K = 16$. Bottom : Optimization, $K = 10$!

Identification in l^1 setting

Suppose we want to identify K values of μ_k with N prescribed measurements, and $N >> K$. Following [Boyd et al. (2004)],

1. Consider the following free optimization problem :

$$\min_{\boldsymbol{\mu}\in\mathbb{R}^K} \|\boldsymbol{M}\,\boldsymbol{\mu} - \boldsymbol{h}\|_{l^1(\mathbb{R}^N)}\,, \qquad \text{i.e.} \qquad \min_{\boldsymbol{\mu}\in\mathbb{R}^K} \sum_{n=1}^{N} |(\boldsymbol{M}\,\boldsymbol{\mu})_n - h_n|\,.$$

   It can be rewritten in an *equivalent* Linear Programming problem, as follows, where $\leq$ means componentwise :

$$\min_{\{\boldsymbol{\mu}\in\mathbb{R}^K,\quad \boldsymbol{t}\in\mathbb{R}^N_+,\quad -\boldsymbol{t}\leq \boldsymbol{M}\,\boldsymbol{\mu} - \boldsymbol{h} \leq \boldsymbol{t}\}} \mathbf{1}^t\boldsymbol{t}$$

2. Using the *simplex algorithm*, the LP problem can be solved efficiently. Moreover, the algorithm searches for vertices (corners of the polytope) as particular solutions : many equalities are fulfilled !

Outline

Many things are... still to be done !

1. Optimal weights ? Refine constrained l^2 methods, thourough study of l^1 methods, comparison of the results. Frequency domain versus time-domain formulation ?
2. Optimal control ? Solve dynamic Riccati equation through the Hamiltonian matrix, using symplectic numerical methods on an invariant manifold.
3. Top-down methodology instead of bottom-up strategy ? Derive the infinite-dimensional optimal control system in the first place, discretize the equations second place.

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