Null Controllability of Coupled Parabolic Degenerate Equations

L. Maniar, University Cadi ayyad, Marrakesh

M. Ait Ben hassi, Marrakesh,
F. Ammar Khodja, Besançon,
A. Hajjaj, Marrakesh

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Parabolic Systems

$$u_t - (a(x)u_x)_x + c(x)u = h_1 1_\omega,$$

$$+ \text{ Boundary Conditions}$$

$$u(0,x) = u_0(x), \quad x \in (0,1),$$

$$\omega \subset (0,1).$$

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$$u(T) = 0.$$

- Null controllability $\Longleftrightarrow$ Observability Inequality :=
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$$\int_0^1 U^2(T,x)dx \le C \int_0^T \int_\omega U^2(t,x)dxdt$$

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Carleman Estimate for parabolic equations

$$\int_0^T \int_0^1 \rho_1 \, z^2 + \int_0^T \int_0^1 \rho_2 \, z_x^2 \leq C \int_0^T \int_0^1 \rho_3 \, f^2.$$

- Non degenerate Case : $a(x) \geq m > 0$.
- Fursikov-Imanuvilov
- Albano, Cannarsa, Zuazua, Yamamoto, Zhang, Guerrero, Benabdellah, Dermenjian, Le Rousseau, ...

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Degenerate Carleman estimate

- Degenerate case :
- e.g. $a(0) = 0$ and $a > 0, x \in (0, 1]$
- $a(x) = x^{\alpha}, x \in [0, 1], \alpha > 0.$
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- assumption (H_1) : Weak degeneracy :

$$(i)\ a \in \mathcal{C}[0,1] \cap \mathcal{C}^1(0,1];\ \ a(0) = 0;\ \ a > 0 \text{ on } (0,1].$$
$$(ii)\ \exists K \in [0,1);\ \ xa'(x) \leq Ka(x);\ \ \forall x \in [0,1].$$

- assumption (H_2) : Strong degeneracy :

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A-B. C., F. Degenerate Carleman estimate

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$$\int_0^T \int_0^1 [s\Theta(t)a(x)z_x^2 + s^3\Theta^3(t)\frac{x^2}{a(x)}z^2]e^{2s\varphi(t,x)}\,dxdt$$
$$\leq C(\int_0^T \int_0^1 f^2 e^{2s\varphi}\,dxdt + a(1)\int_0^T z_x^2(t,1)e^{2s\varphi(t,1)}\,dxdt$$

$$\varphi(t,x) = \Theta(t)\psi(x), \qquad \Theta(t) := \frac{1}{[t(T-t)]^4}$$
$$\psi(x) := c_1\left(\int_0^x \frac{y}{a(y)}dy - c_2\right) \qquad c_2 \geq \frac{1}{a(1)(2-K)},\ c_1 > 0.$$

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Coupled Parabolic Systems

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$$v_t - (a_2(x)v_x)_x + c_2(x)v + b_2(x)u = 0,$$

$$+ \text{ Boundary Conditions}$$

$$u(0,x) = u_0(x),\ v(0,x) = v_0(x), \quad x \in (0,1),$$

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- Nondegenerate Case : $a_i(x) \geq m > 0$
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Coupled Degenerate Parabolic Systems

Cannarsa and De teresa (2009)

$$a_1 = a_2, \qquad b_1 = 0$$

$$u_t - (a(x)u_x)_x + c_1(x)u = h_1 1_\omega,$$
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$$u(t,1) = v(t,1) = 0$$

$$u(t,0) = v(t,0) = 0 \qquad (H_1)$$

or

$$(au_x)(0) = (av_x)(0) = 0 \quad (H_2)$$
$$u(0,x) = u_0(x),\; v(0,x) = v_0(x), \quad x \in (0,1),$$

Coupled Degenerate Parabolic Adjoint System

$$U_t - (a_1(x)U_x)_x + c_1(x)U + b_2(x)V = 0,$$
$$V_t - (a_2(x)V_x)_x + c_2(x)V = 0,$$

$$u(t,1) = v(t,1) = 0$$

$$u(t,0) = 0 \qquad (H_1) \text{ or } (au_x)(0) = 0 \qquad (H_2)$$
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$$U(0,x) = U_0(x),\ V(0,x) = V_0(x), \quad x \in (0,1),$$

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First Carleman Estimate

Theorem 1

$\exists\, C > 0, \quad \exists\, s_0 > 0 \quad / \quad \forall (U, V)\ ,\ \forall s \geq s_0\ :$

$$\int_0^T \int_0^1 \left[s\Theta a_1 U_x^2 + s^3 \Theta^3 \frac{x^2}{a_1(x)} U^2 \right] e^{2s\varphi_1} dxdt$$
$$+ \int_0^T \int_0^1 \left[s\Theta a_2 V_x^2 + s^3 \Theta^3 \frac{x^2}{a_2(x)} V^2 \right] e^{2s\varphi_2} dxdt$$
$$\leq C \int_0^T \int_\omega \left[U^2 + V^2 \right] e^{2s\Phi_2} dxdt.$$

Weight functions

$$\Theta(t) := \frac{1}{[t(T-t)]^4},$$

$$\psi_i(x) := \lambda_i \left(\int_0^x \frac{y}{a_i(y)} dy - d_i \right),$$

$$\varphi_i(t,x) = \Theta(t)\psi_i(x), \quad i = 1, 2,$$

$$\Psi_i(x) := e^{r_i \zeta_i(x)} - e^{2\rho_i},$$

$$\zeta_i(x) := \int_x^1 \frac{1}{\sqrt{a_i(y)}} dy, \qquad \rho_i := r_i \zeta_i(0),$$

$$\Phi_i(t,x) := \Theta(t)\Psi_i(x) \quad i = 1, 2$$

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Proof

$$\omega = (a, b) \subset\subset (0, 1).$$

Estimate in $(0, a)$

ξ = cut-off function ;

$w := \xi U, \qquad z := \xi V,$

$$w_t - (a_1 w_x)_x + c_1 w = b_2 z - \xi_x a_1 U_x - (a_1 \xi_x U)_x := f_1,$$
$$z_t - (a_2 z_x)_x + c_2 z = -\xi_x a_2 V_x - (a_2 \xi_x V)_x := f_2,$$
$+$ Boundary conditions
$$w(0, x) = w_0(x), z(0, x) = z_0(x).$$

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Proof

$$\int_0^T \int_0^1 [s\Theta a_1 w_x^2 + s^3\Theta^3 \frac{x^2}{a_1(x)} w^2] e^{2s\varphi_1} dxdt$$
$$\leq C \int_0^T \int_0^1 [b_2^2 z^2 + (\xi_x a_1 U_x + (a_1 \xi_x U)_x)^2] e^{2s\varphi_1} dxdt$$

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- $d_i \geq max\{\frac{1}{a_i(1)(2-K)}, 4\int_0^1 \frac{y}{a_i(y)}dy\}$,
- $e^{\rho_2} \geq 4\frac{d_2-\int_0^1 \frac{y}{a_2(y)}dy}{d_2-4\int_0^1 \frac{y}{a_2(y)}dy}$,
- $\rho_1 = 2\rho_2$,
- $\lambda_1 = \frac{e^{2\rho_1}-1}{d_1-\int_0^1 \frac{y}{a_1(y)}dy}$,
- $\lambda_2 = \frac{4}{3d_2}(e^{2\rho_2} - e^{\rho_2})$.

Weight functions comparaison

For this choice we have :

- $\varphi_1 \leq \varphi_2$,
- $\Phi_1 \leq \Phi_2$,
- $\varphi_i \leq \Phi_i$.

Hardy-Poincaré Inequality

$$\int_0^1 \frac{a_i(x)}{x^2} g^2(x)dx \ \leq \ C \int_0^1 a_i(x)|g_x|^2 dx$$

Estimation in $(a,1)$

$$W := (1-\xi)U, \quad Z := (1-\xi)V.$$

Albano-Cannarsa Carleman estimate :

$$\int_0^T \int_0^1 s^3\Theta^3 e^{3r\zeta_i(x)} y^2 e^{2s\Phi_i}\,dxdt + \int_0^T \int_0^1 s\Theta e^{r\zeta_i(x)} y_x^2 e^{2s\Phi_i}\,dxdt \leq C\int_0^T \int_0^1 |y_t - (a_i y_x)_x + c_i y|^2 e^{2s\Phi_i}\,dxdt - C\int_0^T \left[r_i s\Theta(t) e^{r_i\zeta_i(x)} e^{2s\Phi_i(t,x)} a_i(x) y_x^2(t,x)\right]_{x=0}^{x=1} dt.$$

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$$W := (1-\xi)U, \quad Z := (1-\xi)V.$$

Albano-Cannarsa Carleman estimate :

$$\int_0^T \int_0^1 s^3\Theta^3 e^{3r\zeta_i(x)} y^2 e^{2s\Phi_i} dxdt + \int_0^T \int_0^1 s\Theta e^{r\zeta_i(x)} y_x^2 e^{2s\Phi_i} dxdt \le C \int_0^T \int_0^1 |y_t - (a_i y_x)_x + c_i y|^2 e^{2s\Phi_i} dxdt - C \int_0^T \left[r_i s\Theta(t) e^{r_i\zeta_i(x)} e^{2s\Phi_i(t,x)} a_i(x) y_x^2(t,x) \right]_{x=0}^{x=1} dt.$$

Estimation in $(a,1)$

$$W := (1-\xi)U, \quad Z := (1-\xi)V.$$

Albano-Cannarsa Carleman estimate :

$$\begin{aligned}
&\int_0^T \int_0^1 s^3\Theta^3 e^{3r\zeta_i(x)} y^2 e^{2s\Phi_i} dxdt \\
&+ \int_0^T \int_0^1 s\Theta e^{r\zeta_i(x)} y_x^2 e^{2s\Phi_i} dxdt \\
&\leq C \int_0^T \int_0^1 |y_t - (a_i y_x)_x + c_i y|^2 e^{2s\Phi_i} dxdt \\
&- C \int_0^T \left[r_i s\Theta(t) e^{r_i\zeta_i(x)} e^{2s\Phi_i(t,x)} a_i(x) y_x^2(t,x) \right]_{x=0}^{x=1} dt.
\end{aligned}$$

Theorem 2

If $b_2 \geq \mu \quad$ in ω_1,

$$\exists C > 0,\ \exists s_0 > 0 \quad / \quad \forall (U, V),\ \forall s \geq s_0$$

$$\int_0^T \int_0^1 \left[s\Theta a_1 U_x^2 + s^3 \Theta^3 \frac{x^2}{a_1(x)} U^2 \right] e^{2s\varphi_1} dxdt$$
$$+ \int_0^T \int_0^1 \left[s\Theta a_2 V_x^2 + s^3 \Theta^3 \frac{x^2}{a_2(x)} V^2 \right] e^{2s\varphi_2} dxdt$$
$$\leq C \int_0^T \int_\omega U^2 dxdt.$$

Observability Inequality

$$\int_0^1 [U(T,x))^2 + (V(T,x))^2)\,dx \\ \leq C\int_0^T\int_\omega U^2(t,x)\,dxdt.$$

Non Cascade degenerate Systems

$$U_t - (x^{\alpha_1} U_x)_x + c_1(x) U + b_2(x) V = 0,$$

$$V_t - (x^{\alpha_2} V_x)_x + c_2(x) V + b_1(x) U = 0,$$

$$+ \text{Boundary Conditions}$$

$$U(0,x) = U_0(x), \;\; V(0,x) = V_0(x), \quad x \in (0,1),$$

$$\int_0^T \int_0^1 [s\Theta a_1 w_x^2 + s^3\Theta^3 \frac{x^2}{a_1(x)} w^2] e^{2s\varphi_1} dxdt$$
$$\leq C \int_0^T \int_0^1 [b_2^2 z^2 + (\xi_x a_1 U_x + (a_1 \xi_x U)_x)^2] e^{2s\varphi_1} dxdt$$

$$\int_0^T \int_0^1 [s\Theta a_2 z_x^2 + s^3\Theta^3 \frac{x^2}{a_2(x)} z^2] e^{2s\varphi_2} dxdt$$
$$\leq C \int_0^T \int_0^1 \left(b_1^2 w^2 + \xi_x a_2 V_x + (a_2 \xi_x V)_x\right)^2] e^{2s\varphi_2} dxdt.$$

New Carleman estimate of Parabolic systems

$$y_t - (x^\alpha y_x)_x = f,$$
$$+ \text{boundary conditions}$$
$$y(0,x) = y_0, x \in (0,1).$$

$$\psi(x) = c_1(x^{2-\alpha} - c_2).$$

$$\psi(x) = c_1(x^{2-\beta} - c_2).$$

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New Carleman estimate of Parabolic systems

$$\int_0^T \int_0^1 \left(s\Theta(t)x^{2\alpha-\beta}y_x^2 + s^3\Theta^3(t)x^{2+2\alpha-3\beta}y^2 \right) e^{2s\varphi}\, dxdt$$
$$\leq C \int_0^T \int_0^1 f^2(t,x)e^{2s\varphi(t,x)}\, dxdt$$
$$+ \int_0^T s\Theta(t)y_x^2(t,1)e^{2s\varphi(t,1)}dt \qquad \textit{for all } s \geq s_0$$

Choukran=Danke=Thank you