The circle criterion, balls of stabilizing gains and input-to-state stability

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Further development of joint work (2008) with

B Jayawardhana (Groningen) and E P Ryan (Bath)

Contents

1 Introduction

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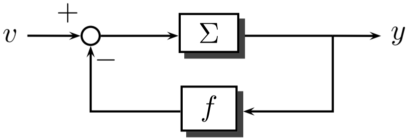
absolute stability, input-to-state stability, and boundedness properties of a feedback interconnection of a well-posed infinite-dimensional MIMO linear system Σ and a static nonlinearity f



Lure system

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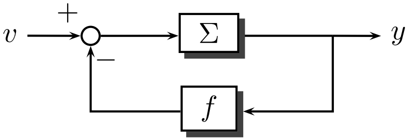

Lure system

Well-posed linear systems: Curtain, Salamon, Staffans, Weiss.

Feedback system

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$$\left.\begin{aligned} \dot{x} &= Ax + Bu, \quad x(0) = x^0 \in X, \\ y &= C_\Lambda \left(x - (s_0 I - A)^{-1} Bu\right) + \mathbf{G}(s_0)u, \\ u &= v - f(y) \end{aligned}\right\} \qquad \text{(FS)}$$

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where

$$\operatorname{Re} s_0 > \text{exponential growth constant of } A.$$

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Let $0 < \sigma \leq \infty$. A solution of (FS) on $[0,\sigma)$ is a pair

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The set of all global solutions of (FS) is denoted by $\mathcal{S}(x^0, v)$.

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- We are mainly concerned with stability properties of (FS): existence of solutions is not the main concern here.
- The question of existence requires addressing on a less general basis, taking into account relevant features of the particular system or subclass of systems under consideration.
- **Special case:** if C is bounded, $\dim Y < \infty$, feedthrough is equal to 0 and f is continuous, then, for every $(x^0, v) \in X \times L^2_{\text{loc}}(\mathbb{R}_+, Y)$, (FS) has solutions.

The feedback system (FS) is said to be input-to-state stable (ISS) if there exist functions $\gamma_1 \in \mathcal{KL}$ and $\gamma_2 \in \mathcal{K}$ such that, for each $x^0 \in X$, each $v \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, Y)$ and all solutions in $\mathcal{S}(x^0, v)$,

$$\|x(t)\| \leq \gamma_1(t, \|x^0\|) + \gamma_2(\|v\|_{L^\infty_{\mathrm{loc}}(0,t)}), \quad \forall\, t \in \mathbb{R}_+.$$

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- $\mathcal{KL}$ = all $\gamma : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ which are decreasing and converging to 0 in the first variable and of class $\mathcal{K}$ in the second variable.

2 Balls of stabilizing gains: Aizerman version of circle criterion & ISS

- We say that $K \in \mathcal{B}(Y)$ is an admissible feedback operator for Σ if there exists $\alpha \in \mathbb{R}$ such that

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Notation. For $K \in \mathcal{B}(Y)$ and $r > 0$, define

$$\mathbb{B}(K, r) := \{T \in \mathcal{B}(Y) : \|T - K\| < r\}.$$

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Moreover, if $v \in L^\infty_{\rm loc}(\mathbb{R}_+, Y)$, then, in the above estimate for x, the L^2-norm of v on $[0, t]$ may be replaced by the L^∞-norm of v on $[0, t]$ (yielding an ISS result).

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- The assumptions of the above Theorem guarantee that maximal defined solutions are global, provided that (FS) has the blow-up property.

- (FS) has the blow-up property if, for every maximally defined solution (x, y) with finite interval of existence $[0, \omega)$,

$$\max\left\{\limsup_{t\uparrow\omega}\|x(t)\|,\ \lim_{t\uparrow\omega}\int_0^t\|y(\tau)\|^2\mathrm{d}\tau\right\} = \infty.$$

- If $f \circ w \in L^2_{\mathrm{loc}}(\mathbb{R}_+, Y)$ for all $w \in L^2_{\mathrm{loc}}(\mathbb{R}_+, Y)$, then the condition

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- **Special case:** blow-up property holds if C is bounded, $\dim Y < \infty$, feedthrough is equal to 0 and f satisfies the "ball condition" of the Theorem.

Proof of Aizerman version of circle criterion - main ideas

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- It is important that Y is a complex space - Lemma does not hold in a real setting.
- Apply loop shifting with K.
- Apply small-gain ideas together with exponential weighting technique to output equation and use Lemma to obtain estimate for y.
- Use results from theory of well-posed linear systems to obtain estimate for state. □

- Exponential weighting/small gain: idea is old - goes back to papers by Sandberg and Zames from the 1960s. Was used in input-output setting, but not in state-space context.

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- Aizerman conjecture over the complex field: was studied (in a different context) by Hinrichsen & Pritchard (1992).

Let us look at the Lemma again.

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Proof of Lemma

Set $\mathbf{G}_K := \mathbf{G}(I+K\mathbf{G})^{-1}$.

Choose s_n with $\operatorname{Re} s_n > 0$ such that

$$\|\mathbf{G}_K\|_{H^\infty} - \|\mathbf{G}_K(s_n)\| \leq 1/n.$$

Can construct operators $Z_n \in \mathcal{B}(Y)$ (of rank 1, in general complex, even if the underlying system is real) such that

$$0 \leq \|Z_n\| - 1/\|\mathbf{G}_K(s_n)\| \leq 1/n$$

and

$$I + Z_n \mathbf{G}_K(s_n) \text{ is not invertible.}$$

Hence $Z_n \notin S(\mathbf{G}_K)$ and so $Z_n + K \notin S(\mathbf{G})$. By hypothesis, this implies that $Z_n + K \notin \mathbb{B}(K, r)$ and therefore $\|Z_n\| \geq r$. By the above construction, $\|Z_n\| \to 1/\|\mathbf{G}_K\|_{H^\infty}$ as $n \to \infty$, showing that

$$\frac{1}{\|\mathbf{G}_K\|_{H^\infty}} \geq r,$$

or, equivalently,

$$\|\mathbf{G}(I + K\mathbf{G})^{-1}\|\|_{H^\infty} = \|\mathbf{G}_K\|_{H^\infty} \leq \frac{1}{r}.$$

□

- Lemma remains true for real data, provided that

$$\|\mathbf{G}_K\|_{H^\infty} = \sup_{s \in R} \|\mathbf{G}_K(s)\|, \qquad (\star)$$

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$$R := \{s \in \mathbb{C}_0 : \mathbf{G}_K(s) \text{ real}\}.$$

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- In the SISO case, $(\star)$ means that the maximal distance of the Nyquist diagram of $\mathbf{G}_K$ to the origin is achieved when it "intersects" the real axis.
- Under the additional assumption that $(\star)$ holds, Aizerman version of the circle criterion remains true in a real setting.

3 “Standard” version of circle criterion & ISS

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- K_1 is an admissible feedback operator,
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Moreover, assume that there exists $\delta > 0$ such that the sector condition

$$\text{Re}\langle f(z) - K_1 z, f(z) - K_2 z\rangle \leq -\delta\|z\|^2 \quad \forall\, z \in Y$$

holds.

Then there exist positive γ and Γ such that, for each $(x, y) \in \mathcal{S}(x^0, v)$,

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Moreover, if $v \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, Y)$, then, in the above estimate for x, the L^2-norm of v on $[0, t]$ may be replaced by the L^∞-norm of v on $[0, t]$ (yielding an ISS result).

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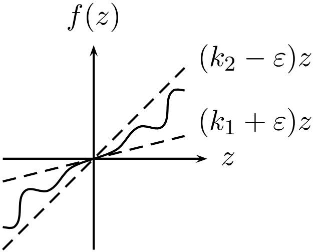

Proof of "standard" version of circle criterion - main ideas

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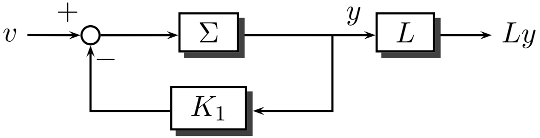

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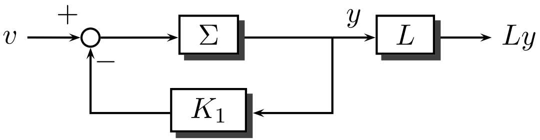

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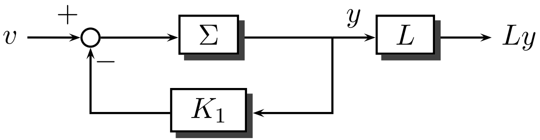

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$$\tilde{f}(z) := f(L^{-1}z) - K_1 L^{-1} z \quad \forall z \in Y.$$

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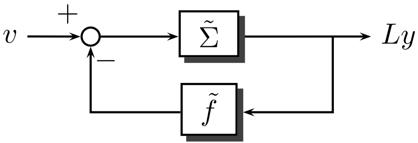

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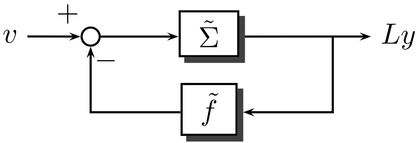

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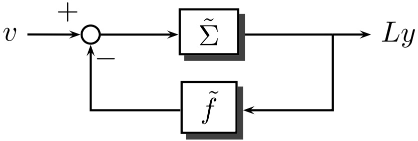

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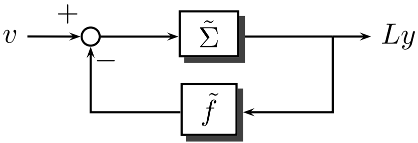

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Aizerman version of circle criterion (with $K = I$ and $r = 1$) applies to above system, proving the claim. $\square$

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  $\triangleright$ For finite-dimensional case, see Arcak & Teel (2002) and Jayawardhana, L & Ryan (2009).

4 ISS with bias (practical ISS)

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Moreover, assume that there exist $\delta > 0$ and a bounded set $E \subset Y$ such that the generalized sector condition

$$\text{Re}\langle f(z) - K_1 z, f(z) - K_2 z\rangle \leq -\delta \|z\|^2 \quad \forall\, z \in Y \backslash E$$

holds.

Then there exist $\beta \geq 0$, $\gamma > 0$ and $\Gamma \geq 1$ such that, for each $(x, y) \in \mathcal{S}(x^0, v)$,

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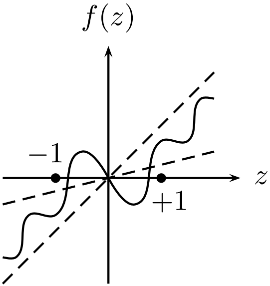
where the bias β depends on f, E, K_1 and K_2.

The bias β is a measure of the extent of the violation of the sector condition on the set E. A bound for β is given by

$$\beta \leq \sup_{z \in E} \left\| f(z) - \frac{1}{2}(K_1 + K_2)z \right\|.$$



SISO nonlinearity f satisfying a generalized sector condition with $E = [-1, 1]$.

Proof of Theorem - main ideas

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- Construct nonlinearity $\tilde{f}$ such that $\tilde{f}$ satisfies a "proper" sector condition with sector data K_1 and K_2,

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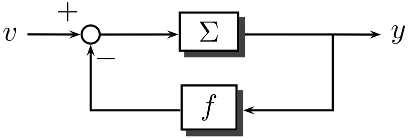
- Apply ISS version of circle criterion to Lure system with nonlinearity $\tilde{f}$.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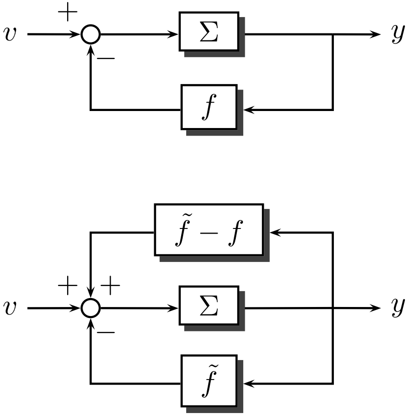

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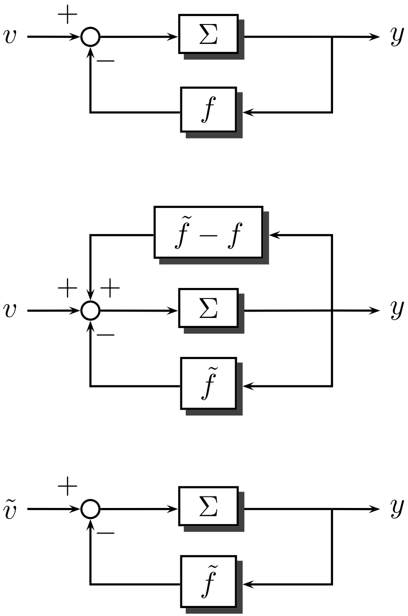

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5 Hysteretic nonlinearities

Replace static nonlinearity $f : Y \to Y$ by a causal nonlinear operator

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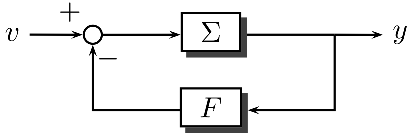

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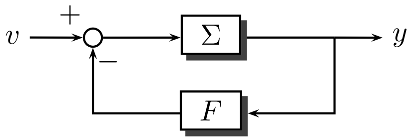

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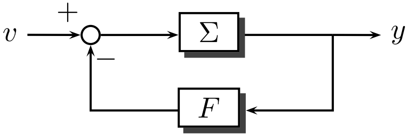

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In what sense?

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$$\forall\, (t, w) \in \mathbb{R}_+ \times \text{dom}(F) \text{ s.t. } w(t) \in Y \setminus E$$

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Yes: hysteretic nonlinearities!

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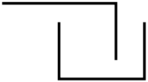
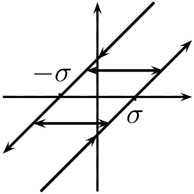
- A basic hysteresis operator is the backlash or play operator:





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- μ is a finite Borel measure on $\mathbb{R}_+$ such that $\int_0^\infty \sigma\mu(\mathrm{d}\sigma) < \infty$.

Illustration

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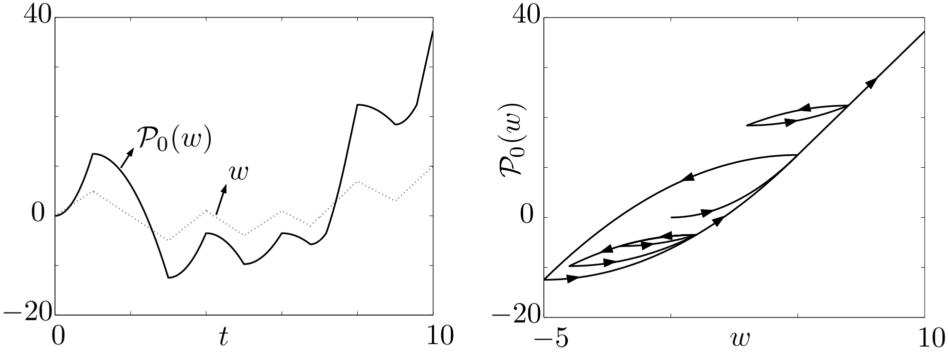
- $\mu(S) = \lambda(S \cap [0, 5])$ for every Borel set $S \subset \mathbb{R}_+$

Illustration

- $\xi(\sigma) \equiv 0$
- $\mu(S) = \lambda(S \cap [0,5])$ for every Borel set $S \subset \mathbb{R}_+$



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Conclusion: circle criterion (ISS with bias 2) applies.