LQR control for long-but-finite strings and LQR control for a class of infinite dimensional systems: similarities and differences

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Summary

- **Motivation**
  - **The attractive mathematical features of the class of spatially invariant systems and their applications.**
  - **Many possible applications: MEMS, flow control, veh. platoons, pde's etc.**
- **a class of spatially invariant systems**

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  - **Fourier-transform formulation**

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  - **Some results.**
  - **Special cases.**

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  - **Some results.**
  - **Special cases.**
- **Conclusions.**

Motivation

- Automatica (2010): **Curtain, Iftime and Zwart**,

A comparison: scalar case

- IEEE Trans. A-C (2011): **Curtain**,

Comments on Distributed control of spatially invariant systems, by Bamieh, Paganini and Dahleh, IEEE Trans. A-C (2002).

- Automatica (2009): **Curtain, Iftime and Zwart**,

System theoretic properties.

- IEEE Tr. A-C (2005): **Jovanovic and Bamieh** pointed out that the shortcomings of previous papers were due to lack of exponential stabilizability or detectability of the infinite platoon model.
- IEEE Tr. A-C (2002): **Bamieh, Paganini and Dahleh**,

Distributed control of spatially invariant systems.

- Levine and Athans (1966), **Melzer and Kuo** (1971), **J.L. Willems** (1971) studied the LQR control problem for very large and infinite platoons of vehicles.

Question: when spatially invariant systems serve as good models for long-but-finite strings?

A finite string model - scalar

$$\begin{aligned}\dot{z}_r(t) &= a_0 z_r(t) + b_0 u_r(t) + b_1 u_{r-1}(t), \quad -N+1 \le r \le N \\ \dot{z}_{-N}(t) &= a_0 z_{-N}(t) + u_{-N}(t), \\ y_r(t) &= c_0 z_r(t), \quad -N \le r \le N, \quad t \ge 0.\end{aligned}$$

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y_r(t) &= c_0 z_r(t), \quad -N \le r \le N, \quad t \ge 0.
\end{aligned}
$$

$$
\mathbf{A}_N = a_0 I, \quad \mathbf{C}_N = c_0 I, \quad \mathbf{B}_N = \begin{bmatrix} b_0 & 0 & 0 & \dots & 0 \\ b_1 & b_0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & b_1 & b_0 \end{bmatrix}.
$$

where $\mathbf{A}_N, \mathbf{B}_N, \mathbf{C}_N$ are Toeplitz matrices and we order from $-N$ to N,

$$
z^N = \begin{bmatrix} z_{-N} \\ z_{-N+1} \\ \cdot \\ \cdot \\ z_N \end{bmatrix}, \quad u^N = \begin{bmatrix} u_{-N} \\ u_{-N+1} \\ \cdot \\ \cdot \\ u_N \end{bmatrix}, \quad y^N = \begin{bmatrix} y_{-N} \\ y_{-N+1} \\ \cdot \\ \cdot \\ y_N \end{bmatrix}.
$$

A finite string model - matrix case

$$\dot{z}_r(t) = \sum_{l=-N}^{N} A_l z_{r-l}(t) + \sum_{l=-N}^{N} B_l u_{r-l}(t), \qquad (1)$$

$$y_r(t) = \sum_{l=-N}^{N} C_l z_{r-l}(t), \quad -N \leq r \leq N, \;\; t \geq 0,$$

finitely many nonzero $A_l,\ B_l,\ C_l \in \mathbb{C}^{2\times 2}$; col. vect. $z_r, y_r, u_r \in \mathbb{C}^2$

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Example - 2×2

The only the nonzero coefficients are

$$\begin{aligned} A_0 &= \begin{bmatrix} 0 & 1 \\ 0 & -\kappa \end{bmatrix}, B_0 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \\ C_0 &= h_0 I_2 \text{ and } \quad C_1 = h_1 I_2, \end{aligned} \tag{2}$$

$z_r(t) = [x_r(t),\ \dot{x}_r(t)]^T$, $y_r(t) = [y_{r,1}(t),\ y_{r,2}(t)]^T$, $u_r(t) = [0,\ u_{r,2}(t)]^T$ for $t \geq 0$.

Example: a second order system

$$
\begin{aligned}
\ddot{x}_r(t) &= -\kappa\dot{x}_r(t) + u_{r,2}(t),\ -N \le r \le N, && (3)\\
y_{r,1}(t) &= h_1 x_{r-1}(t) + h_0 x_r(t), \\
y_{r,2}(t) &= h_1 \dot{x}_{r-1}(t) + h_0 \dot{x}_r(t), \\
& \quad -N+1 \le r \le N, && (4)\\
y_{-N,1}(t) &= h_0 x_{-N}(t), \\
y_{-N,2}(t) &= h_0 \dot{x}_{-N}(t), \quad t \ge 0.
\end{aligned}
$$

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\begin{aligned}
A_0 &= \begin{bmatrix} 0 & 1 \\ 0 & -\kappa \end{bmatrix}, B_0 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, && (5)\\
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A finite string model - matrix case

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\dot{z}_r(t) &= \sum_{l=-N}^{N} A_l z_{r-l}(t) + \sum_{l=-N}^{N} B_l u_{r-l}(t), \\
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\end{aligned}
$$

A finite string model: compact form $\Sigma(\mathbf{A}_N, \mathbf{B}_N, \mathbf{C}_N, 0)$

$$
\begin{aligned}
\dot{\mathbf{z}}_N(t) &= \mathbf{A}_N \mathbf{z}_N(t) + \mathbf{B}_N \mathbf{u}_N(t), \\
\mathbf{y}_N(t) &= \mathbf{C}_N \mathbf{z}_N(t), \ \ t \geq 0,
\end{aligned}
\tag{6}
$$

where, $\mathbf{u}_N(t), \mathbf{y}_N(t), \mathbf{z}_N(t)$ are column vectors in $\mathbb{C}^{2(2N+1)}$, e.g.,

$$
\mathbf{z}_N(t) = \begin{bmatrix} z_{-N}(t)^T & z_{-N+1}(t)^T & \cdots & z_N(t)^T \end{bmatrix}^T
$$

and $\mathbf{A}_N, \mathbf{B}_N, \mathbf{C}_N$ are $2(2N+1) \times 2(2N+1)$ banded block Toeplitz matrices.

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For example

$$\mathbf{A}_N = \begin{bmatrix} A_0 & A_{-1} & 0 & 0 & \cdots & \cdots & 0 \\ A_1 & A_0 & A_{-1} & 0 & \cdots & \cdots & 0 \\ A_2 & A_1 & A_0 & A_{-1} & 0 & \cdots & 0 \\ 0 & A_2 & A_1 & A_0 & A_{-1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & A_2 & A_1 & A_0 & A_{-1} \\ 0 & \cdots & \cdots & 0 & A_2 & A_1 & A_0 \end{bmatrix} \tag{8}$$

when only A_0, $A_{\pm 1}$, A_2 are nonzero.

Infinite strings = spatially invariant systems

$$
\begin{aligned}
\dot{z}_r(t) &= \sum_{l=-\infty}^{\infty} A_l z_{r-l}(t) + \sum_{l=-\infty}^{\infty} B_l u_{r-l}(t), r \in \mathbb{Z}, \\
y_r(t) &= \sum_{l=-\infty}^{\infty} C_l z_{r-l}(t) + \sum_{l=-\infty}^{\infty} D_l u_{r-l}(t).
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$\Sigma(A, B, C, D)$: Infinite matrix formulation

$$\begin{aligned} \dot{z}(t) &= (Az)(t) + (Bu)(t), \\ y(t) &= (Cz)(t) + (Du)(t), \quad t \geq 0, \end{aligned}$$

where A, B, C, D are infinite banded matrices and bounded operators on the infinite-dimensional spaces $Z = \ell_2(\mathbb{C}^2) = U = Y$.
For simplicity, assume for the moment that
$A_l = a_l, B_l = b_l, C_l = c_l, D_l = d_l$ are real scalar and only finitely many are nonzero.

Fourier Transformed system

$$
\begin{aligned}
\dot{z}_r(t) &= \sum_{l=-\infty}^{\infty} a_l z_{r-l}(t) + \sum_{l=-\infty}^{\infty} b_l u_{r-l}(t), \quad -\infty \leq r \leq \infty, \\
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Take Fourier transforms:

$$\check{z}(t,\theta) = \sum_{r=-\infty}^{\infty} z_r(t) e^{-\jmath\theta}, \quad \check{D}(\theta) := \sum_{l=-\infty}^{\infty} d_l e^{-\jmath l\theta} \text{ for } 0 \leq \theta \leq 2\pi.$$

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Fourier transformed formulation: $\Sigma(\check{A}, \check{B}, \check{C}, \check{D})$

$$\begin{aligned} \frac{\partial}{\partial t}\check{z}(\theta,t) &= \check{A}(\theta)\check{z}(\theta,t) + \check{B}(\theta)\check{u}(\theta,t) \\ \check{y}(\theta,t) &= \check{C}(\theta)\check{z}(\theta,t) + \check{D}(\theta)\check{u}(\theta,t), \quad t \geq 0, \quad 0 \leq \theta \leq 2\pi. \end{aligned}$$

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A 2×2 MIMO system parametrized by $\theta \in [0, 2\pi]$ and an ∞-dim. system $\Sigma(\check{A}, \check{B}, \check{C}, \check{D})$ with $\check{Z} = \mathbf{L}_2(\partial\mathbb{D}, \mathbb{C}^2) = \check{U} = \check{Y}$.

Key features of these spatially invariant systems.

- $\Sigma(\check{A},\check{B},\check{C},\check{D})$ and $\Sigma(A,B,C,D)$ are isometrically isomorphic systems:

$$\Sigma(\mathfrak{F}A\mathfrak{F}^{-1},\mathfrak{F}B\mathfrak{F}^{-1},\mathfrak{F}C\mathfrak{F}^{-1},\mathfrak{F}D\mathfrak{F}^{-1}) = \Sigma(\check{A},\check{B},\check{C},\check{D})$$

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- $\check{A}(\theta), \check{B}(\theta), \check{C}(\theta), \check{D}(\theta)$ have only finitely many nonzero terms are so they are all continuous periodic functions on $[0, 2\pi]$.
- The analysis for $\Sigma(\check{A}, \check{B}, \check{C}, \check{D})$: 2×2 MIMO systems with parameter θ.

Exponential stability

$e^{\check{A}t}$ is exponentially stable iff $\exists\, M > 0$ and $\alpha > 0$ such that

$$\|e^{\check{A}t}\|_\infty \leq Me^{-\alpha t} \text{ for all } t \geq 0.$$

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Exponential stabilizability and detectability

$\Sigma(\check{A}, \check{B}, \check{C}, 0)$ is exponentially stabilizable if and only if $(\check{A}(\theta), \check{B}(\theta))$ is stabilizable for each $\theta \in [0, 2\pi]$.

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$\Sigma(\check{A}, \check{B}, \check{C}, 0)$ is exponentially detectable if and only if $(\check{A}(\theta), \check{C}(\theta))$ is detectable for each $\theta \in [0, 2\pi]$.

Corresponding Riccati equations

The control Riccati equation and the closed-loop generators corresponding to $\Sigma(\mathbf{A}_N, \mathbf{B}_N, \mathbf{C}_N, 0)$, $\Sigma(A, B, C, 0)$ and $\Sigma(\check{A}, \check{B}, \check{C}, 0)$ are respectively

$$\mathbf{A}_N^* Q_N + Q_N \mathbf{A}_N - Q_N \mathbf{B}_N \mathbf{B}_N^* Q_N + \mathbf{C}_N^* \mathbf{C}_N = 0, \tag{9}$$

$$A^* Q + QA - QBB^* Q + C^* C = 0, \tag{10}$$

$$\check{A}^* \check{Q} + \check{Q}\check{A} - \check{Q}\check{B}\check{B}^* \check{Q} + \check{C}^* \check{C} = 0. \tag{11}$$

Denote $A_{Q_N} := \mathbf{A}_N - \mathbf{B}_N \mathbf{B}_N^* Q_N$, $A_Q := A - BB^* Q$, $\check{A}_Q := \check{A} - \check{B}\check{B}^* \check{Q}$. A closed-loop operator A_{cl} has a *growth bound* which equals the spectral bound $\omega_{cl} = \sup\{\mathrm{Re}(\lambda) : \lambda \in \sigma(A_{cl})\}$ (since A_{cl} is bounded). Denote by ω_∞ and ω_N the growth bounds of the infinite systems and its Toeplitz approximants, respectively.

LQR RICCATI EQUATIONS AND TOEPLITZ APPROXIMANTS

THEOREM

The system $\Sigma(A, B, C, 0)$ is exponentially stabilizable (detectable) if and only if $(\check{A}(e^{j\theta}), \check{B}(e^{j\theta}), \check{C}(e^{j\theta}), 0)$ is stabilizable (detectable) for each $\theta \in [0, 2\pi]$. If the above holds, then the control Riccati equation (10) has a unique nonnegative solution Q and A_Q generates an exponentially stable semigroup. Moreover (11), the control Riccati equation for $\Sigma(\check{A}, \check{B}, \check{C}, 0)$ has a unique nonnegative solution $\check{Q} \in \mathbf{L}_\infty(\partial\mathbb{D}; \mathbb{C}^{2\times 2})$ and $\check{A}_Q$ generates an exponentially stable semigroup. Furthermore, $\check{Q}(e^{j\theta})$ is continuous in θ on $[0, 2\pi]$.

Note that the input and output spaces are infinite-dimensional. The strongest convergence results for approximating solutions to operator Riccati equations (Ito 1987) are achieved only if the input and output spaces are finite-dimensional.

LQR RICCATI EQUATIONS AND TOEPLITZ APPROXIMANTS

THEOREM

Suppose $\Sigma(A, B, C, 0)$ is exponentially stabilizable and detectable and the sequence of finite-dim. approximating systems $\Sigma(\mathbf{A}_N, \mathbf{B}_N, \mathbf{C}_N, 0)$ is uniformly stabilizable and detectable. Let $Q \in \mathcal{L}(\ell_2(\mathbb{C}^2))$ and $Q_N \in \mathcal{L}(Z^N)$ be the unique nonnegative solutions of the Riccati equations (10) and (9). Then Q_N converges strongly to Q, i.e.,

$$Qz = \lim_{N \to \infty} i^N Q_N \pi^N z, \quad \forall z \in \ell_2(\mathbb{C}^2),$$

and consequently $\|Q_N\|$ are uniformly bounded in N. Moreover, A_{Q_N} converges strongly to A_Q, i.e.,

$$i^N e^{A_{Q_N} t} \pi^N z \to e^{A_Q t} z, \quad \forall z \in \ell_2(\mathbb{C}^2)$$

as $N \to \infty$ uniformly on compact time intervals. There exist $\overline{M} > 0$ and $\mu > 0$ such that $\|e^{A_Q t}\| \leq \overline{M} e^{-\mu t}$, $\|e^{A_{Q_N} t}\| \leq \overline{M} e^{-\mu t}$ for all $t \geq 0$. Moreover, ...

FURTHER ANALYSIS

We present now an example in which $\Sigma(\check{A}, \check{B}, \check{C}, 0)$ is exponentially stabilizable and detectable, $\Sigma(\mathbf{A}_N, \mathbf{B}_N, \mathbf{C}_N 0)$ are uniformly stabilizable but not uniformly detectable and ω_N does not converge to ω_∞.

Consider $\mathbf{A}_N = \text{diag}\{A_0\}$ and $\mathbf{B}_N = \text{diag}\{B_0\}$, (where $a_1 = 0$, $\kappa = 1$), $c_i(e^{j\theta}) = h(e^{j\theta}) = h_0 + h_1 e^{j\theta}$, for $i = 1, 2$, $\theta \in [0, 2\pi]$. Let h_0, $h_1 \in \mathbb{R}$ positive numbers, $h_0 \neq h_1$ and $|h(e^{j\theta})| > \delta > 0$. The $\mathbf{C}_N-$matrix is lower triangular block Toeplitz with

$$C_0 = \begin{bmatrix} h_0 & 0 \\ 0 & h_0 \end{bmatrix} \text{ and } C_1 = \begin{bmatrix} h_1 & 0 \\ 0 & h_1 \end{bmatrix}.$$

The infinite-dimensional system $\Sigma(\check{A}, \check{B}, \check{C}, 0)$ associated to the above Toeplitz approximant system is exponentially stabilizable and detectable and has the growth bound $\omega_\infty = \max_{\theta \in [0,2\pi]} \{-|h(e^{j\theta})|\}$. The growth bounds for the Toeplitz approximants are given by $\omega_N = -\min_{k=0,..,2N} \gamma_k(N)$.

FURTHER ANALYSIS

Table: The growth bounds ω_N and $\widetilde{\omega}_N$ when $1 = h_0 < h_1 = 2$ ($\omega_\infty = -1$)

$N =$	1	2	3	4	5	6
$\omega_N =$	-0.1378	-0.0333	-0.0083	-0.0021	-0.0005	-0.0001
$\widetilde{\omega}_N =$	-1.1688	-1.0789	-1.0444	-1.0281	-1.0193	-1.0140

$$\lim_{N\to\infty} \omega_N = 0 > -1 = \omega_\infty$$

which is a significant gap.

Table: The growth bounds ω_N and $\widetilde{\omega}_N$ when $2 = h_0 > h_1 = 1$ ($\omega_\infty = -1$)

$N =$	1	2	3	4	5	10
$\omega_N =$	-1.0870	-1.0464	-1.0288	-1.0196	-1.0141	-1.0046
$\widetilde{\omega}_N =$	-1.1688	-1.0789	-1.0444	-1.0281	-1.0193	-1.0055

FURTHER ANALYSIS

Consider the system

$$\check{A}(e^{j\theta}) = \begin{bmatrix} 0 & 1 \\ -a_1 & -\kappa \end{bmatrix}, \; \check{B}(e^{j\theta}) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad (12)$$

are constant matrices in $\mathbb{R}^{2\times 2}$ and

$$\check{C}(e^{j\theta}) = \mathrm{diag}\left\{c_1(e^{j\theta}), c_2(e^{j\theta})\right\}, \; \theta \in [0, 2\pi], \qquad (13)$$

with $c_i(e^{j\theta})$, $i = 1, 2$, having finitely many nonzero Fourier coefficients. The system $\Sigma(\check{A}, \check{B}, \check{C}, 0)$ is exponentially stabilizable and, for $c_1(e^{j\theta}) \neq 0$ for all $\theta \in [0, 2\pi]$ it is also exponentially detectable (use Aut 2009).

FURTHER ANALYSIS

Proposition:
Consider the particular infinite-dimensional system where $\check{A}$, $\check{B}$ and $\check{C}$ are given by (12) and (13) with $c_1 = c_2 = h \in \mathbf{H}_\infty$, together with the corresponding large-but-finite system (1). Then there holds

$$\lim_{N\to\infty} \|Q_N\| = \|\check{Q}\|_\infty.$$

Proposition:
Consider the particular infinite-dimensional system where $\check{A}$, $\check{B}$ and $\check{C}$ are given by (12) and (13) with $c_1 = c_2 = h \in \mathbf{H}_\infty$, $h(e^{\mathrm{j}\theta}) \neq 0$ for all $\theta \in [0, 2\pi]$, together with the corresponding large-but-finite system (1). Assume also that $T(h)$ is invertible. Then there holds

1. $\lim_{N\to\infty} \omega_N \to \omega_\infty$.
2. There exists an $\alpha > 0$ such that $\|e^{\mathbf{A}_{Q_N} t}\| \leq e^{-\alpha t}$ for all $t \geq 0$ and all N.

FURTHER ANALYSIS

Tilli, P.,(1998) Singular Values and Eigenvalues of Non-Hermitian Block Toeplitz Matrices, *Linear Algebra and Its Applications* **272**.

Theorem:
Suppose that $\check{F} \in \mathbf{L}_{\infty}^{2\times 2}$. Then

$$\sigma_{\max}(\mathbf{F}_n) \leq \sigma_{\max}(\check{F}), \text{ for all } n \in \mathbb{N}.$$

A nontrivial lower bound for the singular values of $\{\mathbf{F}_n\}_n$ cannot be given in general even in the case when $\sigma_{\min}(\check{F}) > 0$ (see Remark 4.2, Tilli, P.,(1998)).

Denote by $\sigma_{\min}(\check{F})$ and $\sigma_{\max}(\check{F})$ the smallest and the greatest singular values of the function $\check{F}$. For example

$$\sigma_{\min}(\check{F}) := \min_{\theta\in[0,2\pi]} \sigma_{\min}(\check{F}(e^{\mathrm{j}\theta})).$$

A finite string model - matrix case

$$\begin{aligned}\dot{z}_r(t) &= \sum_{l=-N}^{N} A_l z_{r-l}(t) + \sum_{l=-N}^{N} B_l u_{r-l}(t),\\ y_r(t) &= \sum_{l=-N}^{N} C_l z_{r-l}(t), \quad -N \leq r \leq N, \;\; t \geq 0,\end{aligned}$$

A finite string model: compact form $\Sigma(\mathbf{A}_N, \mathbf{B}_N, \mathbf{C}_N, 0)$

$$\begin{aligned}\dot{\mathbf{z}}_N(t) &= \mathbf{A}_N \mathbf{z}_N(t) + \mathbf{B}_N \mathbf{u}_N(t),\\ \mathbf{y}_N(t) &= \mathbf{C}_N \mathbf{z}_N(t), \;\; t \geq 0,\end{aligned} \tag{14}$$

Block circulant: compact form $\Sigma(\tilde{A}_N, \tilde{B}_N, \tilde{C}_N, 0)$

$$\begin{aligned}\dot{\mathbf{z}}_N(t) &= \tilde{A}_N \mathbf{z}_N(t) + \tilde{B}_N \mathbf{u}_N(t),\\ \mathbf{y}_N(t) &= \tilde{C}_N \mathbf{z}_N(t), \;\; t \geq 0,\end{aligned} \tag{15}$$

LQR RICCATI EQUATIONS AND CIRCULANT APPROXIMANTS

THEOREM

Consider the exponentially stabilizable and detectable system $\Sigma(A, B, C, 0)$ with Q the unique self-adjoint solution to the Riccati equation (10)

1. The Riccati equation

$$\tilde{A}_N^* \tilde{Q}_N + \tilde{Q}_N \tilde{A}_N - \tilde{Q}_N \tilde{B}_N \tilde{B}_N^* \tilde{Q}_N + \tilde{C}_N^* \tilde{C}_N = 0 \tag{16}$$

   has a unique self-adjoint stabilizing solution $\tilde{Q}_N$ which is the circular approximant of $\check{Q}$.

2. There holds $\limsup_{N\to\infty} \|\tilde{Q}_N\| = \|\check{Q}\|_\infty = \|Q\|$.

3. The growth bound $\tilde{\omega}_N$ of $e^{\tilde{A}_{Q_N} t}$ satisfies

$$\tilde{\omega}_N \le \omega_\infty, \quad \limsup_{N\to\infty} \tilde{\omega}_N = \omega_\infty.$$

LQR RICCATI EQUATIONS AND CIRCULANT APPROXIMANTS

We now relate the solutions to the Toeplitz Riccati equations to those to the circulant Riccati equations.

THEOREM

Assume that $\Sigma(A, B, C, 0)$ is stabilizable and detectable and $\Sigma(\mathbf{A}_N, \mathbf{B}_N, \mathbf{C}_N, 0)$ is uniformly stabilizable and detectable. Then the following hold

1. $(Q_N - \tilde{Q}_N)$ and $(A_{Q_N} - \tilde{A}_{Q_N})$ converge strongly to zero as $N \to \infty$.
2. $|Q_N - \tilde{Q}_N|_N \to 0$ and $|A_{Q_N} - \tilde{A}_{Q_N}|_N \to 0$ as $N \to \infty$ (see the Appendix for the definition of the $|\cdot|_N$ norm).
3. The closed-loop transfer functions satisfy $\||G^{cl}(\cdot) - G_N^{cl}(\cdot)|_N\|_{\mathbf{H}_\infty} \to 0$ and $\||G^{cl}(\cdot) - G_N^{cl}(\cdot)|_N\|_{\mathbf{H}_2} \to 0$.

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  - Thank you for your attention!