Some Compact Classes of Open Sets under Hausdorff Distance and Application to Shape Optimization

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Shape optimization: isoperimetric problem

- Many shape optimization problems can be seen in the larger framework of optimal control problems:

D. Bucur, G. Buttazzo,
Variational Methods in Shape Optimization Problems,
Birkhauser, 2005

- The first and certainly most classical example of a shape optimization problem is the isoperimetric problem:

Find, among all admissible domains with a given *perimeter*, the one whose Lebesgue measure is as large as possible.

Perimeter is fixed

Hurwitz,1902

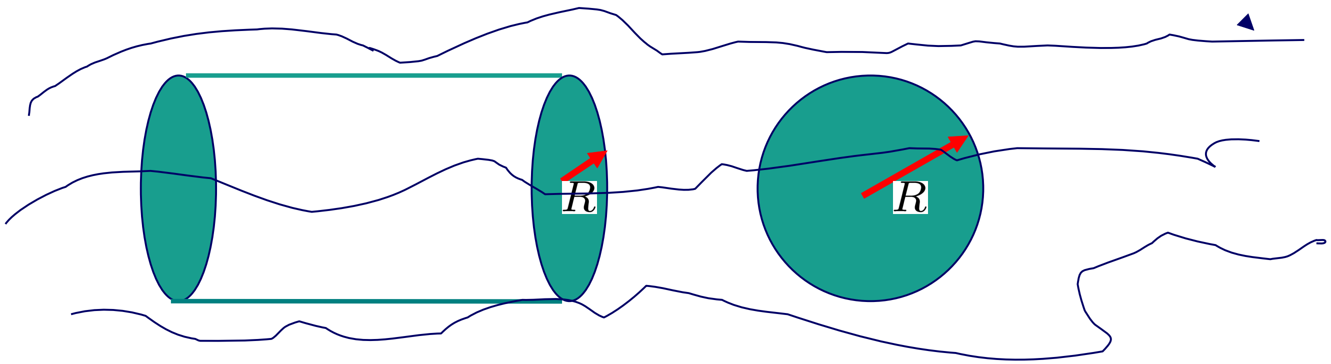
Ω is the circle

Shape optimization: minimal aerodynamical resistance

- The Newton problem of minimal aerodynamical resistance: The problem of finding the shape of a body which moves in a fluid with minimal resistance to motion

One of the first problems in the calculus of variations

Newton (1685, Principia Mathematica):
[an inviscid and incompressible medium]



Resistance of globe is half of cylinder!

Shape optimization: first eigenvalue of Laplacian

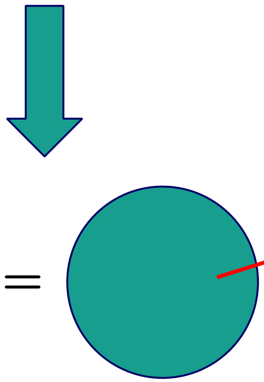
❖ The most famous shape optimization problem is on the first eigenvalue of Laplacian:

$$\begin{cases} -\Delta u(x) = \lambda u(x), x \in \Omega \subset \mathbb{R}^2, |\Omega| = c, \\ u|_{\partial\Omega} = 0. \end{cases}$$

Fixed constant

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$$

$$\lambda_1(\Omega^*) = \min_{|\Omega|=c} \lambda_1(\Omega) = \min_{|\Omega|=c} \min_{u \in H_0^1(\Omega), \|u\|_{L^2(\Omega)}=1} \int_\Omega |\nabla u|^2 dx.$$

$\Omega^* =$ 

Area c

Shape optimization: nonexistence

Solution does not always exist!

$$\min\left\{\int_D |u_A - c|^2 dx, -\Delta u_A = 1 \text{ in } A, u_A \in H_0^1(A)\right\}$$

where D is an open bounded set, $A \subset D \in \mathbb{R}^2$ is open set, u_A is defined on D with zero extension.

Conclusion:

Too small!

If c is small enough, No smooth A solves above problem!

Shape optimization

- For shape optimization: The control variable is domain!

Minimizing sequence: $\{\Omega_m\}_{m=1}^{\infty}$ has convergent subsequence $\{\Omega_{m_k}\}$

A topology for admissible sets to have compactness! (weak enough)

+

A topology should be to get continuity! (strong enough)

$u_{\Omega_{m_k}} \to u_{\Omega}$

Involves: PDE +Geometry

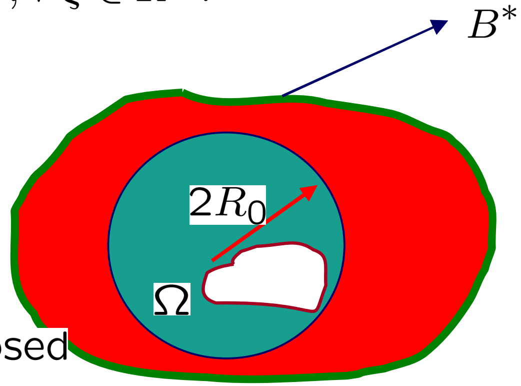
Our problem

Consider an elliptic problem:

$$\begin{cases} -\mathcal{A}u_\Omega = f \in H^{-1}(B^*), \ u_\Omega \in H_0^1(\Omega), \\ \mathcal{A} = \mathrm{div}(A\nabla), \alpha|\xi|^2 \le \langle A\xi, \xi\rangle, \forall\, \xi \in \mathbb{R}^N. \end{cases}$$

$B^* \subset \mathbb{R}^N$ bounded domain $B^* \subset \mathbb{R}^N$

$B^* \supset U(0, 2R_0)$



$U(x,r)$ the open ball, and $B(x,r)$ the closed ball of $\mathbb{R}^N$, both centered at x with radius r.

$$J(\Omega) = \frac{1}{2}\int_{B^*} |u_\Omega - g|^2 dx, \ J(\Omega^*) = \inf\{J(\Omega); \Omega \in \mathcal{C} \subset U(0, 2R_0)\}, g \in L^2(B^*)$$

What class $\mathcal{C}$ of open sets Ω can be found so that there exists at least one solution for above shape optimization?

Hausdorff distance

Open sets class $\mathcal{C}$ becomes a metric space under Hausdorff distance:

$$\rho(\Omega_1,\Omega_2)=\max\left\{\sup_{x\in\overline{B^*}\setminus\Omega_1}\text{dist}\left(x,\overline{B^*}\setminus\Omega_2\right),\ \sup_{y\in\overline{B^*}\setminus\Omega_2}\text{dist}\left(\overline{B^*}\setminus\Omega_1,y\right)\right\}$$

$$\Omega_n\xrightarrow{\rho}\Omega,\ \text{ if } \rho(\Omega_n,\Omega)\to 0 \text{ as } n\to\infty.$$

$$\delta(K_1,K_2)=\max\left\{\sup_{x\in K_1}\text{dist}(x,K_2),\ \sup_{y\in K_2}\text{dist}(y,K_1)\right\},K_1,K_2 \text{ are compact}$$

$$\Omega_n\xrightarrow{\rho}\Omega\Longleftrightarrow\overline{B^*}\setminus\Omega_n\xrightarrow{\delta}\overline{B^*}\setminus\Omega.$$

Principle 1 [Compactness]$\{\Omega_m\}_{m=1}^{\infty}\subset\mathcal{C}\Rightarrow\Omega_{n_k}\xrightarrow{\rho}\Omega\in\mathcal{C}$;

Principle 2 [Continuity]: $\Omega_n\xrightarrow{\rho}\Omega\Rightarrow u_{\Omega_n}\to u_{\Omega}$!

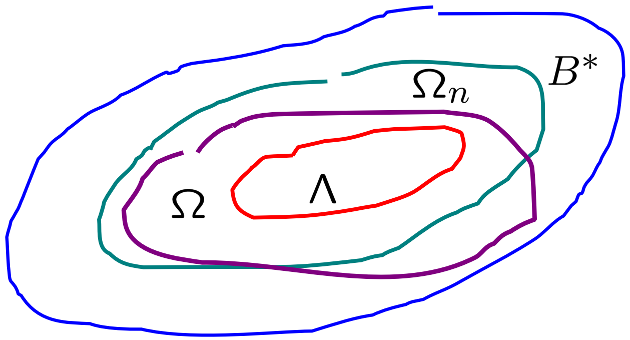
$\Longrightarrow$ Existence

Convex open set class (cannot be too small) meets the principles!

Hausdorff distance: known facts

Lemma 1: [Γ --property for open sets]

$\Omega_n \xrightarrow{\rho} \Omega \Rightarrow \forall$ open subset $\Lambda, \overline{\Lambda} \subset \Omega, \overline{\Lambda} \subset \Omega_n$ sufficiently large n.



Lemma 2:

$(\mathcal{O}, \delta)$ is a compact metric space, $\mathcal{O} = \{K \subset \overline{B^*} \mid K$ is compact$\}$:
$\Omega_n \subset B^*, \exists \Omega_{n_k} \xrightarrow{\rho} \Omega \in B^*$.

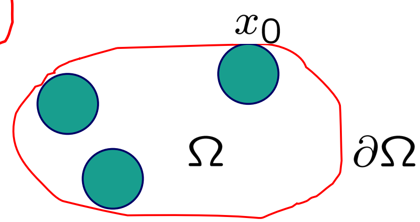
The class of all open sets of B^* is compact under Hausdorff distance!

But it is too large to have continuity!

Open set: interior ball condition

Definition 1: [interior ball condition]

$x_0 \in \partial\Omega$. Ω is said to satisfy the *interior ball condition* at x_0, if $\exists\ U(y(x_0), r(x_0)) \subset \Omega$ with $x_0 \in \partial U(y(x_0), r(x_0))$.

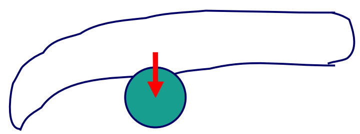


Ω is said to satisfy *the uniformly interior ball condition* , if $r(x_0) \geq r_\Omega > 0, \forall\ x_0 \in \partial\Omega$.

Ω cannot be too small!

Remark: This is first introduced by us, independent of convex!

Motivation: Smooth surface has interior ball property, like circle of curvature!



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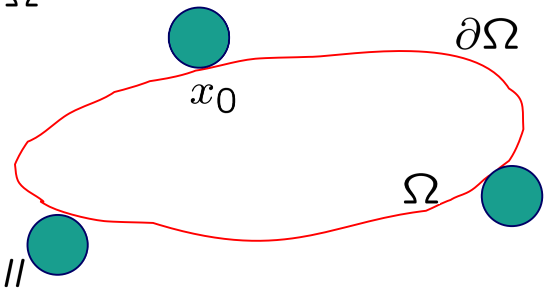

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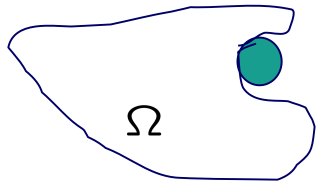
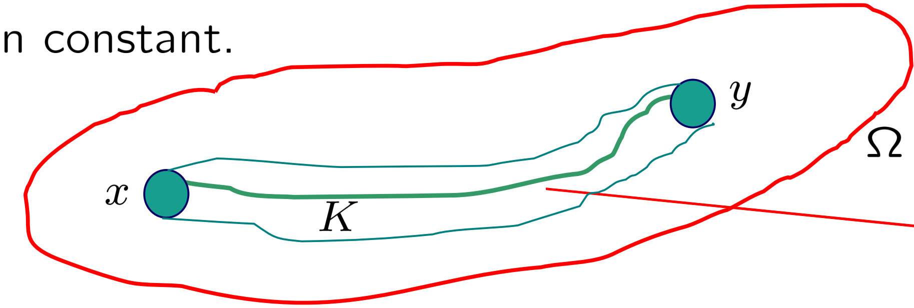


Remark: This is completely motivated from interior ball condition!

Open set: Property (C_M)

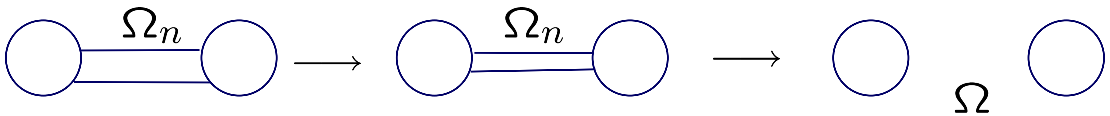
Definition 3: [Property (C_M)]

$\Omega \subset \mathbb{R}^N$ is said to have property (C_M), if for any $x, y \in \Omega$, $\exists$ compact set K with $x, y \in K$, such that $K \subset \Omega$ and $\cup_{z\in K} U(z, \frac{d^*}{M}) \subset \Omega$, $d^* = \min\{dist(x, \partial\Omega), dist(y, \partial\Omega)\}$, and $M > 1$ is a given constant.



Connected compact

Motivation: connected domains converge to connected doamin!



disconnected!

Three open sets class:

$$\begin{cases} \mathcal{C}_1 = & \{\Omega \subset B(0,R_0) \subset B^* | \\ & \Omega \text{ satisfies the uniformly interior ball condition and } r_\Omega \geq r_0\}. \\ \mathcal{C}_2 = & \{\Omega \subset B(0,R_0) \subset B^* | \, U(x_\Omega, R) \subset \Omega, \\ & \Omega \text{ satisfies the uniformly exterior ball condition and } r_\Omega \geq r_0\}. \\ \mathcal{C}_3 = & \{\Omega \subset B(0,R_0) \subset B^* | \, U(x_\Omega, R) \subset \Omega, \\ & \Omega \text{ is a open set, and has the property } (C_M)\}, \\ & \text{where } R > 0 \text{ is a given constant.} \end{cases}$$

Remark:

$\mathcal{C}_1 : r_\Omega \geq r_0 > 0;$

$\mathcal{C}_2 : U(x_\Omega, R) \subset \Omega, r_\Omega \geq r_0 > 0;$

$\mathcal{C}_3 : U(x_\Omega, R) \subset \Omega.$

Ω cannot be too small!

Advantage of $\mathcal{C}_3$: Any open set in $\mathcal{C}_3$ is connected!

First main result: compactness

Theorem 1 [compactness of open set class]:

For every given $i \in \{1,2,3\}$, if $\{\Omega_m\}_{m=1}^{\infty} \subset \mathcal{C}_i$, then there exist a subsequence $\{\Omega_{m_k}\}_{k=1}^{\infty}$ of $\{\Omega_m\}_{m=1}^{\infty}$, and $\Omega \in \mathcal{C}_i$ such that

$$\Omega_{m_k} \xrightarrow{\rho} \Omega \text{ as } k \to \infty.$$

In other words, each $(\mathcal{C}_i, \rho)$ is a compact metric space. Moreover, for any $i, j = 1, 2, 3$, $\left(\mathcal{C}_i \cap \mathcal{C}_j, \rho\right)$ is also a compact metric space.

The proof is quite elementary!

Special attention for $\mathcal{C} = \mathcal{C}_1 \cap \mathcal{C}_2$

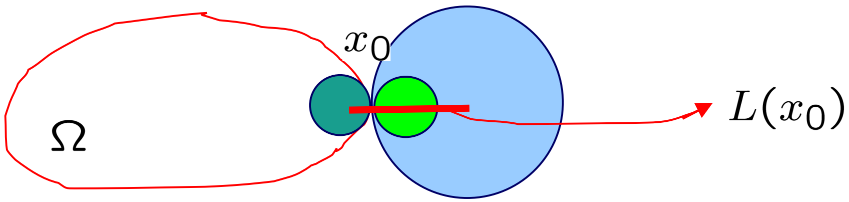
Let $\mathcal{C} = \mathcal{C}_1 \cap \mathcal{C}_2$. $\mathcal{C}$ has both uniformly interior and exterior ball property

Expect: $\mathcal{C}$ is more smooth:

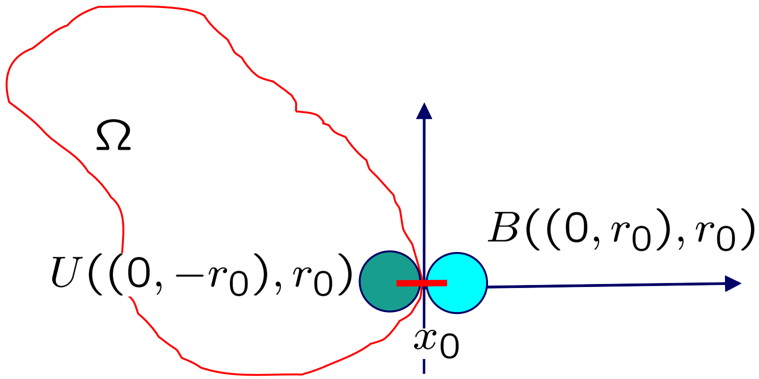
Remind: any smooth surface has interior property!

The smoothness of $\mathcal{C}$ is the inverse of above property!

Lemma 3: Let $\Omega \in \mathcal{C}_1 \cap \mathcal{C}_2$. Then for every $x_0 \in \partial\Omega$, there exists a straight line $L(x_0)$ passing through the point x_0, such that all centers of the exterior and interior balls at x_0 lying in this line. Moreover, $L(x_0)$ is unique.



Special attention for $\mathcal{C} = \mathcal{C}_1 \cap \mathcal{C}_2$

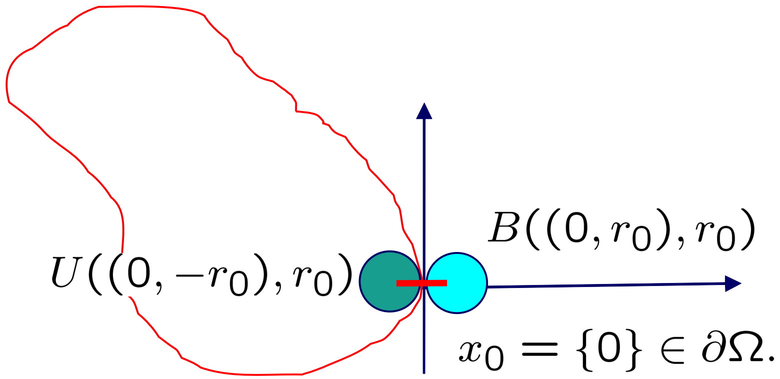


By rotation and translation, we may suppose that $x_0 = \{0\} \in \partial\Omega$.

Lemma 4: For any $x' \in U\left(0, \frac{r_0}{16}\right) \subset \mathbb{R}^{N-1}$, the line $L_{x'} = \{(x', s); s \in \mathbb{R}\}$ intersects with the set $[U(0, \frac{r_0}{16}) \times (-\frac{r_0}{4}, \frac{r_0}{4})] \cap \partial\Omega$ only one point.

From Lemma 4, we see that there exists a function $f : U(0, \frac{r_0}{16})(\subset \mathbb{R}^{N-1}) \to \mathbb{R}, x' \mapsto f(x')$ such that $f(0) = 0$ and $\{(x', f(x'))|\ x' \in U(0, \frac{r_0}{16})\} = [U(0, \frac{r_0}{16}) \times (-\frac{r_0}{4}, \frac{r_0}{4})] \cap \partial\Omega$.

Second main result: $\mathcal{C} \in C^{1,1}$.



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The property of f :

a). $\frac{\partial f}{\partial x_i}(0) = 0$ for every $i \in \{1, \cdots, N-1\}$.

b). $\frac{\partial f}{\partial \xi_j} : U(0, \frac{r_0}{64N})(\subset \mathbb{R}^{N-1}) \to \mathbb{R}$ is Lipschitz continuous for every $j \in \{1, \cdots, N-1\}$.

No more!

$\Rightarrow \mathcal{C} \in C^{1,1}$.

Second main result: $\mathcal{C} \in C^{1,1}$.

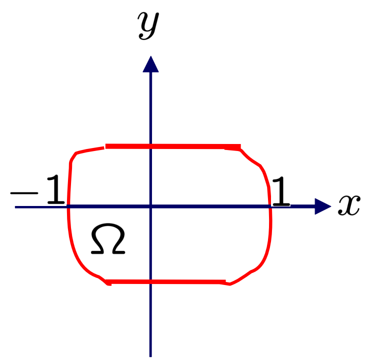
Let $\Omega \in \mathbb{R}^2$ be the interior domain surrounded by the following four curves $\Gamma_i, i = 1, 2, 3, 4.$

Γ_1: $\{(x, y) \in \mathbb{R}^2 | x \in [-1, 1], y = 1\}$;

Γ_2: $\{(x, y) \in \mathbb{R}^2 | x \in [-1, 1], y = -1\}$;

Γ_3: $\{(x, y) \in \mathbb{R}^2 | x = 1 + \sqrt{1 - y^2}, y \in [-1, 1]\}$;

Γ_4: $\{(x, y) \in \mathbb{R}^2 | x = -1 - \sqrt{1 - y^2}, y \in [-1, 1]\}$.



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$$f(x) = \begin{cases} 1, & x \in (0, 1]; \\ \sqrt{1 - (x - 1)^2}, & x \in [1, 2). \end{cases}$$

Then $\Omega \in C^{1,1}$ but $\Omega \notin C^2$.

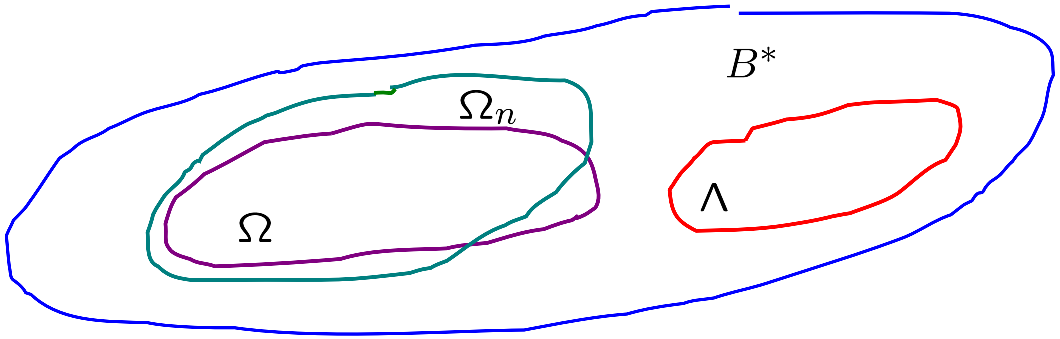
$f''_-(1) = 0, \; f''_+(1) = -1.$

Exterior $\Gamma-$ Property of $\mathcal{C}=\mathcal{C}_1\cap\mathcal{C}_2$

Theorem 2: $\mathcal{C}\in C^{1,1}$.

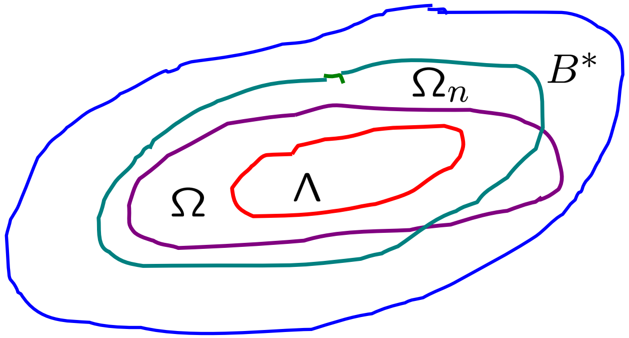
Theorem 3: [Exterior $\Gamma-$ property of $\mathcal{C}\in C^{1,1}$]

If $\{\Omega_n\}_{n=1}^{\infty}\subset\mathcal{C}_1\cap\mathcal{C}_2$ and $\Omega_n\xrightarrow{\rho}\Omega$. Then for each open subset Λ satisfying $\overline{\Lambda}\subset B^*\setminus\overline{\Omega}$, there exists a positive integer n_Λ depending on Λ such that $\overline{\Lambda}\subset B^*\setminus\overline{\Omega_n}$ for all $n\geq n_\Lambda$.



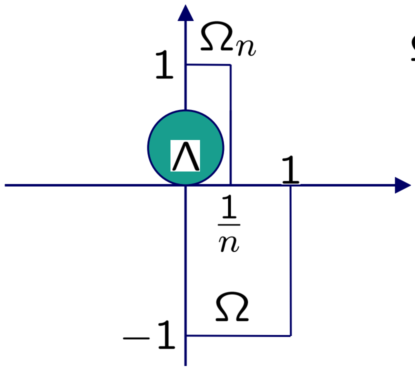
Exterior Γ- Property of $\mathcal{C}=\mathcal{C}_1\cap\mathcal{C}_2$

Remind: interior Γ- property is always true if only $\Omega_n \xrightarrow{\rho} \Omega$



However, the exterior Γ - property can not be deduced from $\Omega_n \xrightarrow{\rho} \Omega$

Example:



$$\Omega_n = \{(0,1)\times(-1,0)\}\cup\{(0,1-\tfrac{1}{n})\times[0,1)\}$$

$$\downarrow$$

$$\Omega = (0,1)\times(-1,0)$$

Continuity conditions

$\{\Omega_n\}_{n=1}^{\infty} \subset B^*$, let u_n be the solution of Equation:

$$-\mathcal{A}u_n = f \text{ in } \Omega_n,\ u_n \in H_0^1(\Omega_n).$$

Assume that

(i). $\Omega_n \xrightarrow{\rho} \Omega$. a.e., $\overline{B^*} \setminus \Omega_n \xrightarrow{\delta} \overline{B^*} \setminus \Omega$.

Needs special property of class

(ii). $\chi_{\overline{B^*}\setminus\Omega_n} \to l$ in L^∞ weak star topology, $l > 0$ a.e. in $\overline{B^*} \setminus \Omega$.

(iii). If $w \in H^1(B^*), w\chi_{\overline{B^*}\setminus\Omega} = 0$, then $w|_\Omega \in H_0^1(\Omega)$, where χ_Ω denotes the characteristic function of Ω.

smoothness of $\partial\Omega$

Then $u_n \to u$ in $H^1(\Omega)$, where u is the solution of Equation below

$$-\mathcal{A}u = f \text{ in } \Omega, \quad u \in H_0^1(\Omega).$$

Existence for class $\mathcal{C} = \mathcal{C}_1 \cap \mathcal{C}_2$

Lemma 5: If $\Omega_n \in \mathcal{C} = \mathcal{C}_1 \cap \mathcal{C}_2$, Ω is a nonempty open set, and

$\Omega_n \xrightarrow{\rho} \Omega$, then

It needs exterior $\Gamma-$property

$$\chi_{\Omega_n} \to \chi_\Omega \text{ in } L^\infty.$$

So condition (ii) is satisfied!

Condition (iii) is the direct consequence of fact $\mathcal{C} \in C^{1,1}$.

Theorem 4: [existence of optimal shape in the class $\mathcal{C} = \mathcal{C}_1 \cap \mathcal{C}_2$]

The shape optimization problem

$$J(\Omega^*) = \inf_{\Omega \in \mathcal{C}} J(\Omega) = \inf_{\Omega \in \mathcal{C}} \frac{1}{2} \int_{B^*} |u_\Omega - g|^2 dx$$

admits at least one solution Ω for the open sets class $\mathcal{C} = \mathcal{C}_1 \cap \mathcal{C}_2$. In particular, if $f \in L^\infty(B^*)$, then $u_\Omega \in C^{1,1}(\Omega)$.

Boundary optimization problem in $\mathcal{C} = \mathcal{C}_1 \cap \mathcal{C}_2$

- Consider boundary shape optimization problem:

$$J(\Omega^*) = \min_{\Omega \in \mathcal{C}} J(\Omega) = \min_{\Omega \in \mathcal{C}} \int_{\partial\Omega} f(x, \nu(x)) d\mathcal{H}^{N-1}$$

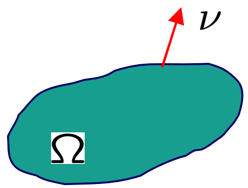
where

$\mathcal{C} = \mathcal{C}_1 \cap \mathcal{C}_2$

$\mathcal{H}^{N-1}$ is the N-1-dimensional Hausdorff measure on $\partial\Omega$.

f is a nonnegative function

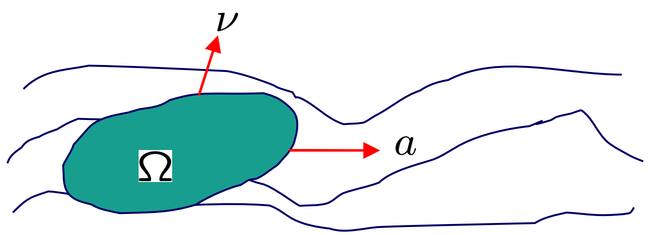
ν is the normal unit vector exterior to Ω



Newtonian resistance in N-dimensional body Ω:

$$f(x, \nu) = \left((a \cdot \nu)^+\right)^3$$

$a \rightarrow$ direction of the motion



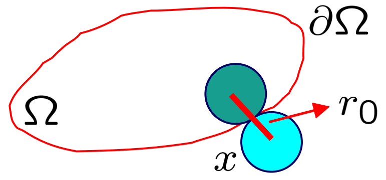
Boundary optimization problem in $\mathcal{C} = \mathcal{C}_1 \cap \mathcal{C}_2$

Write $x \in \mathbb{R}^N$ as $x = (x^1, \cdots, x^{N-1}, x^N) = (x', x^N)$.

In the proof of Theorem 2: $\mathcal{C} \in C^{1,1}$:

$\Omega \in \mathcal{C}, x \in \partial\Omega$, $\exists$ a $C^{1,1}$ function f_x:

$f_x : U'(0, 3a_0) \to \mathbb{R}$ with $f_x(0) = 0$, $a_0 = \frac{1}{3} \cdot \frac{r_0}{256N}$.



After rotation and translation: we have a $C^{1,1}$ map:

$$\Psi_x : (\xi', \xi^N) \in U(x, 3a_0) \longmapsto (\zeta', \zeta^N - f_x(\zeta'))$$

(i). $\Psi_x(U(x, 3a_0) \cap \Omega) \subset \mathbb{R}^N_+$; (ii). $\Psi_x(U(x, 3a_0) \cap \partial\Omega) \subset \partial\mathbb{R}^N_+$;

(iii). $\Psi_x \in C^{1,1}(U(x, 3a_0))$, $\Psi_x^{-1} \in C^{1,1}(D)$, $D \equiv \Psi_x(U(x, 3a_0))$

$\{\Psi_x\} \to$ coordinate charts of $\partial\Omega$

Computation of Hausdorff measure

In order to show the existence of boundary shape optimization we need

$$\Omega_n \xrightarrow{\rho} \Omega \Rightarrow \mathcal{H}^{N-1}(\partial\Omega_{n_k}) \to \mathcal{H}^{N-1}(\partial\Omega).$$

To do so, we have to find how to compute measure?

$$\Downarrow$$

Find coordinate charts of $\partial\Omega$

$$\Downarrow$$

Find coordinate representation

$$\Uparrow$$

Riemannian manifold

Find coordinate charts of Ω

Write $x \in \mathbb{R}^N$ as $x = (x^1, \cdots, x^{N-1}, x^N) = (x', x^N)$.

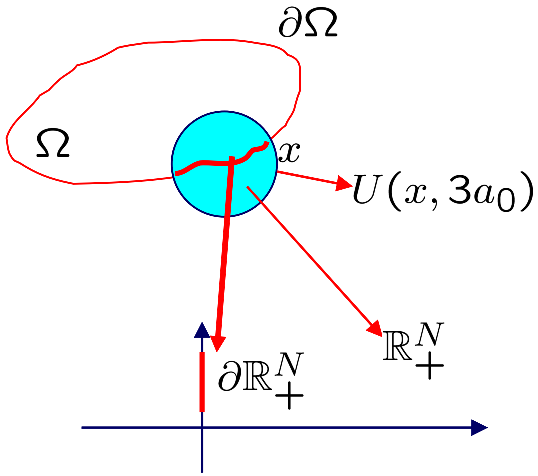
$$\Omega \in \mathcal{C}, x \in \partial\Omega, \ \exists \text{ a } C^{1,1} \text{ function } f_x:$$

$$f_x : U'(0, 3a_0) \to \mathbb{R} \text{ with } f_x(0) = 0, \ a_0 = \frac{1}{3} \cdot \frac{r_0}{256N}.$$

$$\Psi_x : (\xi', \xi^N) \in U(x, 3a_0) \longmapsto (\zeta', \zeta^N - f_x(\zeta'))$$

$$V_x \equiv \overline{\Omega} \cap U(x, 3a_0), \ \mathscr{A}^\Omega \equiv \{(V_x, \Psi_x)\}_{x \in \partial\Omega} \cup \{(\Omega, id_\Omega)\}$$

where id_Ω is the identity map of Ω.



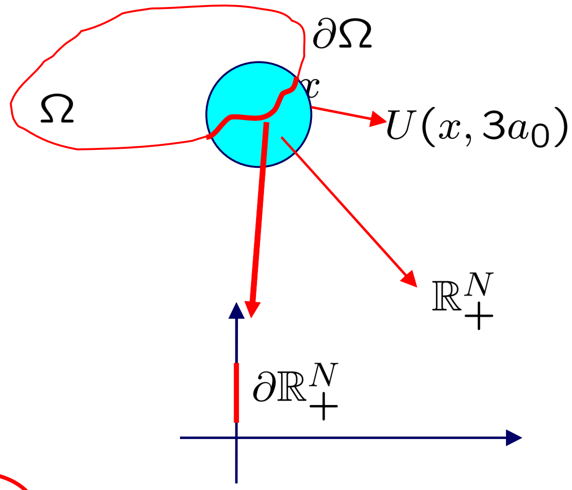
$\mathscr{A}^\Omega \to$ Coordinate charts of Ω

Lemma 6: $\overline{\Omega}$ is Riemannian manifold:

$\overline{\Omega}$ is an oriented C^1 Riemannian manifold determined by $\mathscr{A}^\Omega$ with boundary $\partial\Omega$, with Euclidean metric.

Find coordinate charts of $\partial\Omega$

Lemma 7: $\partial\overline{\Omega}$ is Riemannian manifold:



Let $(\overline{\Omega}, \mathscr{A}^{\Omega})$ be the Riemannian manifold with the Euclidean metric $\overline{g} = \langle \cdot, \cdot \rangle$ confirmed by Lemma 4. Then the including map $i : \partial\Omega \to \overline{\Omega}$ is an embedding. Moreover, $\partial\Omega$ is also an oriented C^1 Riemannian manifold determined by

$$\tilde{\mathscr{A}}^{\Omega} = \left\{ (\tilde{V}_x, \tilde{\Psi}_x);\ (V_x, \Psi_x) \in \mathscr{A}^{\Omega},\ \text{and } \tilde{V}_x = V_x \cap \partial\Omega, \tilde{\Psi}_x = \Psi_x|_{\tilde{V}_x} \right\}$$

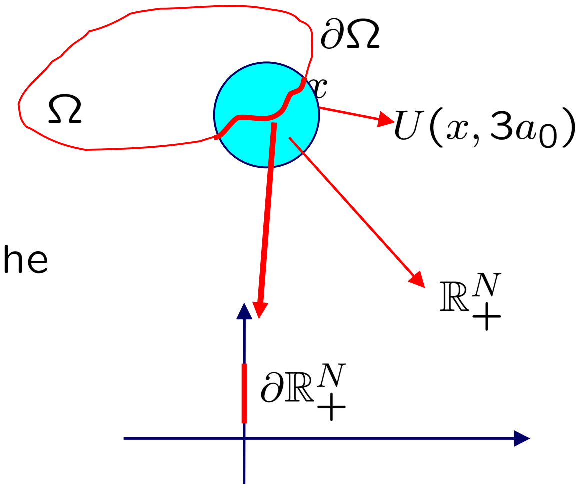
with induced metric $g = i^* \circ \overline{g}$.

Representation of coordinate of $\partial\Omega$

Lemma 8: Coordinate representation:

$$\overline{g} = \sum_{j=1}^{N} (d\xi^j)^2 \text{ coordinate of } \mathbb{R}^N$$

the immersion i has the following representation on the local chart $(\tilde{V}_x, \tilde{\Psi}_x)$ of $\partial\Omega$



$$\xi^j = \sum_{l=1}^{N-1} b_{lj}(x)\zeta^l + b_{Nj} f_x(\zeta') + x^j, \; j = 1, \cdots, N.$$

$$g|_{\tilde{V}_x} = \sum_{i,j=1}^{N-1} \left(\delta_{ij} + \frac{\partial f_x}{\partial \zeta^i}\frac{\partial f_x}{\partial \zeta^j} \right) d\zeta^i d\zeta^j,$$

$$dV_g|_{\tilde{V}_x} = \sqrt{1 + |Df_x|^2} d\zeta^1 \wedge \cdots \wedge d\zeta^{N-1}, Df_x = \left(\frac{\partial f_x}{\partial \zeta^1}, \cdots, \frac{\partial f_x}{\partial \zeta^{N-1}} \right).$$

Coordinate charts of $\partial\Omega_n$

Lemma 9:

Local coordinate chart of $\partial\Omega_n$ is $(\tilde{V}_{X_i,n}, \tilde{\Psi}_{X_i,n})$. $\tilde{\mathscr{A}}_n = \left\{(\tilde{V}_{X_i,n}, \tilde{\Psi}_{X_i,n});\ i = 1, \cdots, M\right\}$. $(\partial\Omega_n, \tilde{\mathscr{A}}_n)$ is an oriented C^1 Riemannian manifold with Riemannian metric $g_n = i_n^* \circ \overline{g}$, and

$$dV_{g_n}|_{\tilde{V}_{X_i,n}} = \sqrt{1 + |Df_{X_i,n}|^2} d\zeta^1 \wedge \cdots \wedge d\zeta^{N-1}.$$

Moreover, $\tilde{\mathscr{A}}^{\Omega_n}$ and $\tilde{\mathscr{A}}_n$ are C^1-compatible and coherently oriented.

Boundary convergence

Lemma 10:

For every $\xi' \in U'(0, 2a_0)$, one has

$$Df_{X_i,n}(\xi') \to Df(\xi').$$

$$\int_{U'(0,2a_0)} \sqrt{1 + |Df_{X_i,n}(\xi')|^2} d\xi' \to \int_{U'(0,2a_0)} \sqrt{1 + |Df_{X_i}(\xi')|^2} d\xi'.$$

Lemma 11:

$$\Omega_n \xrightarrow{\rho} \Omega \Longrightarrow \mathrm{Vol}(\partial\Omega_{n_k}) \to \mathrm{Vol}(\partial\Omega),\ \mathrm{Vol}(\partial\Omega_n) = \int_{\partial\Omega_n} dV_{g_n}.$$

Lemma 12: $\mathrm{Vol}(\partial\Omega) = \mathcal{H}^{N-1}(\partial\Omega)$ for any $\Omega \in \mathcal{C}$.

Theorem 5: $\mathcal{C} = \mathcal{C}_1 \cap \mathcal{C}_2$. $\Omega_n \xrightarrow{\rho} \Omega \Longrightarrow \mathcal{H}^{N-1}(\partial\Omega_{n_k}) \to \mathcal{H}^{N-1}(\partial\Omega)$.

Existence of boundary shape optimization

Theorem 6: $\mathcal{C} = \mathcal{C}_1 \cap \mathcal{C}_2$.

Let $f : \mathbb{R}^N \times \mathbb{S}^{N-1} \to \mathbb{R}^+$ be a lower semicontinuous function, where $\mathbb{S}^{N-1}$ denotes the $(N-1)$-dimensional unit sphere of $\mathbb{R}^N$. Then the minimum problem:

$$J(\Omega^*) = \min_{\Omega \in \mathcal{C}} J(\Omega) = \min_{\Omega \in \mathcal{C}} \int_{\partial\Omega} f(x, \nu(x)) d\mathcal{H}^{N-1}$$

admits at least one solution.

Proof: Measure theory + $\Omega \in C^{1,1}$.

---The end---

---The end---

Thank You！