Bounded real balanced truncation for strictly bounded real well-posed systems.

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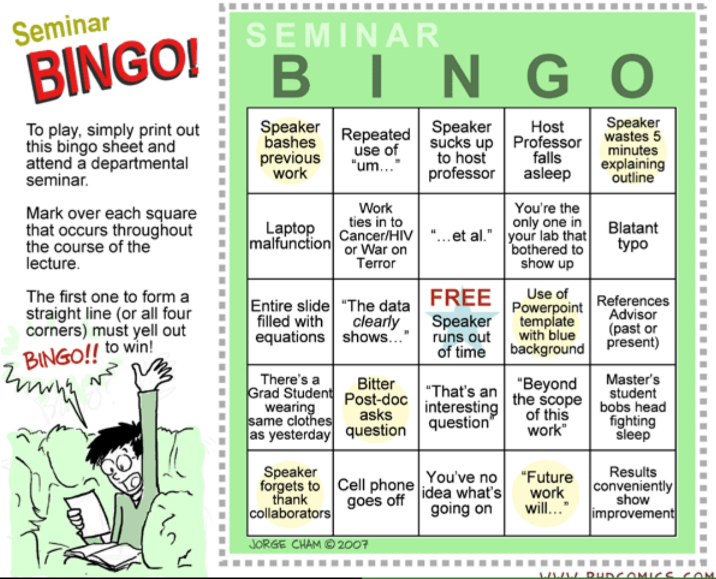
Seminar BINGO!
To play, simply print out this bingo sheet and attend a departmental seminar.
Mark over each square that occurs throughout the course of the lecture.
The first one to form a straight line (or all four corners) must yell out BINGO!! to win!
SEMINAR BINGO
Speaker bashes previous work
Repeated use of "um..."
Speaker sucks up to host professor
Host Professor falls asleep
Speaker wastes 5 minutes explaining outline
Laptop malfunction
Work ties in to Cancer/HIV or War on Terror
"...et al."
You're the only one in your lab that bothered to show up
Blatant typo
Entire slide filled with equations
"The data clearly shows..."
FREE Speaker runs out of time
Use of Powerpoint template with blue background
References Advisor (past or present)
There's a Grad Student wearing same clothes as yesterday
Bitter Post-doc asks question
"That's an interesting question"
"Beyond the scope of this work"
Master's student bobs head fighting sleep
Speaker forgets to thank collaborators
Cell phone goes off
You've no idea what's going on
"Future work will..."
Results conveniently show improvement
JORGE CHAM © 2007

Theorem

Let $G \in H^\infty(\mathbb{C}_0^+; B(\mathscr{U}, \mathscr{Y}))$ denote a strictly bounded real transfer function with summable bounded real singular values and where $\mathscr{U}$ and $\mathscr{Y}$ are finite dimensional. Then for each integer n there exists a rational transfer function denoted G_n such that

$$\|G - G_n\|_{H^\infty} \leq 2 \sum_{k \geq n+1} \sigma_k,$$

where σ_k are the bounded real singular values. The function G_n is bounded real.

- G_n is called the reduced order transfer function obtained by bounded real balanced truncation.

Bounded real transfer functions.

- For $D \subseteq \mathbb{C}$, $G : D \to B(\mathscr{U}, \mathscr{Y})$ is bounded real (or Schur) if

$$\|G\|_{H^\infty} \leq 1.$$

- Necessarily G bounded real implies $G \in H^\infty(\mathbb{C}_0^+; B(\mathscr{U}, \mathscr{Y}))$.
- G is strictly bounded real if

$$\|G\|_{H^\infty} < 1,$$

which is equivalent to $G \in H^\infty$ and

$$\exists\, \varepsilon > 0 \ : \ I - [G(s)]^* G(s) \geq \varepsilon I, \quad a.a.\ s \in \mathrm{i}\mathbb{R}.$$

Bounded real singular values.

- The bounded real singular values are some quantities associated with the system.
- They will be defined later.
- Note they are *not* the Hankel singular values (used in Lyapunov balancing).
- We will consider later when they are summable (form an ℓ^1 sequence).

For finite-dimensional systems (equivalently rational transfer functions)

- Bounded real balanced truncation (BRBT) first proposed by Opdenacker & Jonckheere [1988].
- Based on the model reduction scheme suggested by Moore [1981], now called Lyapunov balancing.
- Lyapunov balanced truncation is a model reduction scheme with error bounds.

- Since bounded real systems are stable systems, Lyapunov balancing is applicable.
- Natural question to ask is, why bounded real balancing?
- Bounded real systems occur frequently in physical examples.
- BRBT preserves bounded realness (contractivity) of the reduced order transfer function G_n, which Lyapunov balancing does not necessarily.
- There are error bounds for BRBT.

- Positive real balanced truncation (PRBT), sometimes also called stochastic balanced truncation in early literature, is very similar in principle to BRBT. PRBT was derived by Desai & Pal [1984].
- PRBT retains positive realness (passivity) of the reduced order transfer function G_n.
- Not the same H^∞ error bound as BRBT.

- We are aiming to extend BRBT and PRBT to the infinite dimensional case.
- This is still work in progress.
- Bounded real and positive real systems are closely related via the Cayley (diagonal) transform.
- As such bounded real results imply positive real results.
- Note positive real systems must be "square", $\mathscr{U} = \mathscr{Y}$.

- The model reduction schemes mentioned so far (Lyapunov, BRBT, PRBT) use certain ("balanced") realisations.
- In balanced realisations, certain functions of the state are equal or balanced.
- Model reduction by balanced truncation is a truncation method to create an approximate or reduced order system by truncating the state space.
- BRBT is based on Lyapunov balanced truncation.

- Given $G \in H^\infty(\mathbb{C}^{p\times m})$ rational we can find a minimal realisation denoted by $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ such that the system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \\ x(0) &= x_0, \end{aligned} \tag{1}$$

  with state-space $\mathbb{C}^n$, has transfer function G.
- Here A is stable and

$$G(s) = C(sI - A)^{-1}B + D,$$

  which is certainly defined for $s \in \mathbb{C}$ with $\operatorname{Re} s > 0$.
- If $T \in \mathbb{C}^{n\times n}$ is invertible then $\left[\begin{smallmatrix} T^{-1}AT & T^{-1}B \\ CT & D \end{smallmatrix}\right]$ is another realisation for G.

- Recall the controllability $\mathcal{Q}$ and observability $\mathcal{O}$ Gramians,

$$\mathcal{Q} = \Phi\Phi^*, \quad \mathcal{O} = \Psi^*\Psi,$$

  which are bounded operators $\mathbb{C}^n \to \mathbb{C}^n$.
- Note $\mathcal{Q}$ and $\mathcal{O}$ depend on the realisation.

Definition

The realisation $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is Lyapunov balanced if $\mathcal{Q} = \mathcal{O} =: \Sigma$ with Σ diagonal. The diagonal entries are the singular values of the Hankel operator $H = \Psi\Phi$, ordered in decreasing magnitude.

- The Hankel singular values are similarity invariants- so do not depend on the realisation.

- For Lyapunov balanced truncation, partition a Lyapunov balanced realisation $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ by

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & C_2 \end{bmatrix},$$

with $A_{11} \in \mathbb{C}^{r \times r}$, $r < n$ and B_1, C_1 conformly sized.

- States that correspond to larger singular values are kept, and the states corresponding to smaller singular values are omitted.
- Really

$$A_{11} = P_{\mathscr{X}_n} A|_{\mathscr{X}_n}, \quad B_1 = P_{\mathscr{X}_n} B, \quad C_1 = C|_{\mathscr{X}_n},$$

with $\mathscr{X}_n \subset \mathscr{X} = \mathbb{C}^n$.

- The reduced order system is defined by its realisation $\begin{bmatrix} A_{11} & B_1 \\ C_1 & D \end{bmatrix}$, so that

$$G_n(s) = C_1(sI - A_{11})^{-1}B_1 + D.$$

- It can be proven that $\begin{bmatrix} A_{11} & B_1 \\ C_1 & D \end{bmatrix}$ is minimal, A_{11} is stable and the error bound

$$\|G - G_n\|_{H^\infty} \leq 2 \sum_{k=n+1}^{r} \sigma_k,$$

holds. σ_k are the Hankel singular values.

- Let $G \in H^\infty(\mathbb{C}^{p\times m})$ be rational, proper and bounded real with minimal realisation $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$. Then from the Bounded Real Lemma there exists $P = P^* \geq 0$, K, W such that

$$\begin{aligned} A^*P + PA + C^*C &= -K^*K, \\ PB + C^*D &= -K^*W, \\ I - D^*D &= W^*W. \end{aligned} \tag{2}$$

- There is minimal, non-negative, self-adjoint solution to (2) P_m which satisfies

$$-\langle x_0, P_m x_0\rangle = \inf_u \int_{\mathbb{R}^+} \|u(s)\|^2 - \|y(s)\|^2 \, ds, \tag{3}$$

subject to (1).
- When W is invertible, $(W^*W)^{-1}K$ is the optimal feedback operator for the optimal control problem (3).

- Similarly $Q_m = Q_m^* \geq 0$ solving the optimal control problem

$$-\langle x_0, Q_m x_0 \rangle = \inf_{u_d} \int_{\mathbb{R}^+} \|u_d(s)\|^2 - \|y_d(s)\|^2 \, ds, \qquad (4)$$

subject to the dual system of (1), is the minimal, self-adjoint solution of the dual bounded real equations

$$\begin{aligned} AQ + QA^* + BB^* &= -LL^*, \\ QC^* + BD^* &= -LV^*, \\ I - DD^* &= VV^*. \end{aligned} \qquad (5)$$

Definition

The realisation $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is bounded real balanced if $P_m = Q_m =: \Sigma$ with Σ diagonal. The diagonal entries are the bounded real singular values, which are the squareroots of the eigenvalues of $P_m Q_m$, ordered in decreasing magnitude.

- If $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is bounded real balanced with $P_m = Q_m =: \Sigma$ then from the bounded real equations (2) and (5)

$$\begin{aligned} A^*\Sigma + \Sigma A + \left[\begin{smallmatrix} C^* & K^* \end{smallmatrix}\right]\left[\begin{smallmatrix} C \\ K \end{smallmatrix}\right] &= 0, \\ A\Sigma + \Sigma A^* + \left[\begin{smallmatrix} B & L \end{smallmatrix}\right]\left[\begin{smallmatrix} B^* \\ L^* \end{smallmatrix}\right] &= 0. \end{aligned} \tag{6}$$

- $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ bounded real balanced implies $\begin{bmatrix} A & \left[\begin{smallmatrix} B & L \end{smallmatrix}\right] \\ \left[\begin{smallmatrix} C \\ K \end{smallmatrix}\right] & - \end{bmatrix}$ is Lyapunov balanced.
- Bounded real singular values are the Hankel singular values of the extended system.
- Error bound now follows from Lyapunov balanced case.

- Now make some remarks on the proof of the main result.

Theorem

Transfer function G strictly bounded real, summable bounded real singular values $(\sigma_k)_{k\in\mathbb{N}}$ then there exists rational G_n such that

$$\|G - G_n\|_{H^\infty} \leq 2 \sum_{k\geq n+1} \sigma_k.$$

- Argument is similar to finite-dimensional case:
  - Construct extended system.
  - Apply Lyapunov balanced truncation to extended system.

- In the finite dimensional case the Bounded Real Lemma gives the extended system.
- Bounded Real Lemma harder for infinite dimensional case.
- Can still make sense of the optimal control problems.
- We use the Weiss & Weiss [1997] optimal control paper.

- Start with a stable well-posed linear realisation $\left[\begin{smallmatrix} \mathbb{T} & \Phi \\ \Psi & \mathbb{F} \end{smallmatrix}\right]$ of G.
- The strict bounded realness assumption implies existence of invertible spectral factors $\theta \in H^\infty(B(\mathscr{U}))$, $\xi \in H^\infty(B(\mathscr{Y}))$ such that

$$I - G^*G = \theta^*\theta, \quad I - GG^* = \xi^*\xi.$$

- The factors θ and ξ have input-output maps $\mathbb{F}_\theta$ and $\mathbb{F}_\xi$ and we define

$$\Psi_\theta = -\mathbb{F}_\theta^{-1}\mathbb{F}^*\Psi.$$

- Ψ_θ is an output map for $\mathbb{T}$.

- We obtain first extended system

$$\begin{bmatrix} \mathbb{T} & \Phi \\ \Psi & \mathbb{F} \end{bmatrix} \rightarrow \begin{bmatrix} \mathbb{T} & \Phi \\ \begin{bmatrix} \Psi \\ \Psi_\theta \end{bmatrix} & \begin{bmatrix} \mathbb{F} \\ \mathbb{F}_\theta \end{bmatrix} \end{bmatrix},$$

which has observability Gramian P_m, solution of optimal control problem. "Extended output."

- Dual process gives input map Φ_ξ and second extended system

$$\begin{bmatrix} \mathbb{T} & \Phi \\ \Psi & \mathbb{F} \end{bmatrix} \rightarrow \begin{bmatrix} \mathbb{T} & [\Phi \ \Phi_\xi] \\ \Psi & [\mathbb{F} \ \mathbb{F}_\xi] \end{bmatrix},$$

which has controllability Gramian Q_m, solution of dual optimal control problem. "Extended input."

- The extended system is defined by combining the two extended systems

$$\begin{bmatrix} \mathbb{T} & \Phi \\ \begin{bmatrix} \Psi \\ \Psi_\theta \end{bmatrix} & \begin{bmatrix} \mathbb{F} \\ \mathbb{F}_\theta \end{bmatrix} \end{bmatrix}, \begin{bmatrix} \mathbb{T} & [\Phi \ \Phi_\xi] \\ \Psi & [\mathbb{F} \ \mathbb{F}_\xi] \end{bmatrix} \to \begin{bmatrix} \mathbb{T} & [\Phi \ \Phi_\xi] \\ \begin{bmatrix} \Psi \\ \Psi_\theta \end{bmatrix} & \begin{bmatrix} \mathbb{F} & \mathbb{F}_\xi \\ \mathbb{F}_\theta & ? \end{bmatrix} \end{bmatrix} =: \begin{bmatrix} \mathbb{T} & \Phi_E \\ \Psi_E & \mathbb{F}_E \end{bmatrix}.$$

- Has transfer function $G_E = \begin{bmatrix} G & \xi \\ \theta & ? \end{bmatrix}$.
- Unclear presently how to finish defining the transfer function G_E and input-output map $\mathbb{F}_E$, but we can make sense of the Hankel operator $H_E = \Psi_E \Phi_E$.

- The bounded real singular values are the singular values of the product $P_m Q_m$.
- If the bounded real singular values are summable then the extended Hankel operator H_E is nuclear (or trace class).
- Nuclear Hankel operators have lots of nice properties. For example the transfer function is regular and the Hankel operator determines the transfer function up to a constant (the feedthrough).
- Can then make sense of the input-output map and transfer function of the extended system.

- Lyapunov balanced truncation has been extended to a class of infinite dimensional systems by Glover *et al.* [1988], and we make use of some of their ideas. Those results have recently been extended by Guiver, Opmeer [2011].
- We truncate the exactly observable shift realisation on L^1 of H_E

$$\begin{bmatrix} \mathbb{S} & H_E \\ I & \mathbb{F}_E \end{bmatrix},$$

by truncating the generators of the above realisation.
- Key is we do not truncate a balanced (or output-normal) realisation.

Which systems have summable bounded real singular values?

- Summable bounded real singular values corresponds to a nuclear Hankel operator of the extended system.
- Sufficient conditions for a Hankel operator to be nuclear have been investigated by others, Opmeer [2010], Curtain & Sasane [2001].
- If the semigroup is analytic and the control B and observation operators C are not too unbounded, i.e.

$$C : \mathscr{X}_\alpha \to \mathscr{Y}, \quad B : \mathscr{U} \to \mathscr{X}_\beta, \quad \alpha - \beta < 1,$$

  then the Hankel operator is nuclear.
- From Staffans [1997], for strictly bounded real systems, the extended operators are no more unbounded than the original operators.

- Transfer results to positive real case. In the finite-dimensional case the gap metric error bound

$$\delta(G, G_n) \leq 2 \sum_{k=n+1}^{r} \sigma_k,$$

holds, Guiver, Opmeer [2010] and Timo Reis.

- Investigate whether strict bounded realness is required. In the finite-dimensional theory it is not required for the error bound.
- Look at ways to compute G_n numerically etc.

- Under the assumptions of strict bounded realness, and summable bounded real singular values, bounded real balanced truncation has been extended to infinite dimensional systems.
  - BRBT truncation is Lyapunov balanced truncation of a certain extended system, related to the solution of two optimal control problems.
  - The extended system is constructed using spectral factorisations of the Popov functions $I - G^*G$ and $I - GG^*$, which uses strict bounded realness.
- Bounded real balanced truncation gives rise to an H^∞ error bound, analogous to that for finite dimensional bounded real balanced truncation.
  - Error bound follows from the error bound for Lyapunov balanced truncation.

- There are checkable conditions for summable bounded real singular values for strictly bounded real systems
  - Using BRSV are the Hankel singular values of the extended system.
  - Require analytic semigroup, B and C not too unbounded.
- Does not provide a constructive method of finding reduced order transfer function G_n.
- Under the Cayley transform bounded real balanced truncation (will probably) become positive real balanced truncation.

Thank you!