The LQ–CONTROLLER SYNTHESIS PROBLEM for INFINITE–DIMENSIONAL SYSTEMS in FACTOR FORM

Piotr Grabowski

INSTITUTE OF AUTOMATICS

AGH UNIVERSITY OF SCIENCE AND TECHNOLOGY

CRACOW, POLAND

pgrab@ia.agh.edu.pl

AGH

CDPS 2011, 18–22 July, University of Wuppertal

Last modification: July 17, 2011

■ The general lq–problem with infinite time horizon for well–posed infinite–dimensional systems has been investigated in (Weiss and Weiss, 1997), and in (Staffans, 1997, 1999), (Mikkola and Staffans, 2004).

■ Our aim is to present a solution of a general lq - optimal controller synthesis problem for infinite–dimensional systems in factor form. The systems in factor form are an alternative to additive models, of the theory of well–posed systems and enable us to lead the analysis exclusively within the basic state space. As a result of applying the simplified analysis in terms of the factor systems, we obtain an equivalent, but, astonishingly not the same formulae expressing the optimal controller in the time–domain and a complement to the method of spectral factorization.

■ The results are illustrated by two examples of construction of the optimal control/controller for standard lq–problems met in literature: (Chapelon and Xu, 2003), to which we give full solution and an example of improving a river water quality by artificial aeration (active control) (Żołopa and Grabowski, 2008).

1 Introduction

Consider a control system governed by the model in factor form

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & \mathscr{A}\left[x(t) + \mathscr{D}u(t)\right] \\ y(t) & = & \mathscr{C}x(t) \end{array} \right\} \tag{1.1}$$

where the *state operator* $\mathscr{A}$ generates an **EXS** semigroup $\{S(t)\}_{t\geq 0}$ on a Hilbert space H with scalar product $\langle\cdot,\cdot\rangle_{\mathrm{H}}$, , i.e., there exit $M \geq 1$ and $\alpha > 0$ such that

$$\|S(t)x_0\|_{\mathrm{H}} \leq Me^{-\alpha t}\|x_0\|_{\mathrm{H}} \qquad \forall t \geq 0, \quad \forall x_0 \in \mathrm{H} \;\; ; \tag{1.2}$$

$\mathscr{C} : (D(\mathscr{C}) \subset \mathrm{H}) \longrightarrow \mathrm{Y}$, $\mathscr{C}\mathscr{A}^{-1} \in \mathbf{L}(\mathrm{H},\mathrm{Y})$, $\mathscr{D} \in \mathbf{L}(\mathrm{U},\mathrm{H})$ with $R(\mathscr{D}) \subset D(\mathscr{C})$, $\mathscr{C}\mathscr{D} \in \mathbf{L}(\mathrm{U},\mathrm{Y})$ and $u \in \mathrm{L}^2(0,\infty;\mathrm{U})$. Here Y and U are Hilbert spaces with scalar products $\langle\cdot,\cdot\rangle_{\mathrm{Y}}$ and $\langle\cdot,\cdot\rangle_{\mathrm{U}}$, respectively.

The LQ–optimal control problem with infinite time horizon is to minimize the quadratic integral performance index

$$J(x_0,u) = \int_0^\infty \begin{bmatrix} y(t) \\ u(t) \end{bmatrix}^* \begin{bmatrix} Q & N \\ N^* & R \end{bmatrix} \begin{bmatrix} y(t) \\ u(t) \end{bmatrix} \mathrm{d}t, \qquad (1.3)$$

where $Q = Q^* \in \mathbf{L}(\mathrm{Y})$, $N \in \mathbf{L}(\mathrm{U},\mathrm{Y})$ and $R = R^* \in \mathbf{L}(\mathrm{U})$, on trajectories of (1.1).

To solve this problem we shall assume that:

(A1) $\mathscr{C}$ is an admissible *observation operator*, i.e., $\mathscr{R}(\mathscr{Z}) \subset D(\mathscr{L}_\mathrm{Y})$, where

$$\mathscr{Z} \in \mathbf{L}(\mathrm{H}, \mathrm{L}^2(0,\infty;\mathrm{Y})),\ (\mathscr{Z}x_0)(t) := \mathscr{C}\mathscr{A}^{-1}S(t)x_0;$$
$$\mathscr{L}_\mathrm{Y} f = f',\ D(\mathscr{L}_\mathrm{Y}) = \mathrm{W}^{1,2}([0,\infty);\mathrm{Y}) \ .$$

Since $\mathscr{L}_\mathrm{Y}$ generates the *semigroup of left–shifts* on $\mathrm{L}^2(0,\infty;\mathrm{Y})$ then, by the closed–graph theorem, the admissibility of $\mathscr{C}$ holds iff

$$\Psi = \mathscr{L}_{\mathrm{Y}} \mathscr{Z} \in \mathbf{L}(\mathrm{H}, \mathrm{L}^2(0,\infty;\mathrm{Y})) \quad ,$$

and Ψ is called the system *observability map*.

(A2) $\mathscr{D}$ is an admissible *factor control operator*, i.e., $\mathscr{R}(\mathscr{W}) \subset D(\mathscr{A})$, where

$$\mathscr{W} \in \mathbf{L}(\mathrm{L}^2(0,\infty;\mathrm{U}),\mathrm{H}), \qquad \mathscr{W} f := \int_0^\infty S(t)\mathscr{D} f(t)\mathrm{d}t \quad .$$

By the closed–graph theorem, the admissibility of $\mathscr{D}$ holds iff

$$\Phi = \mathscr{A}\mathscr{W} \in \mathbf{L}(\mathrm{L}^2(0,\infty;\mathrm{U}),\mathrm{H}) \quad ,$$

and Φ is the system *reachability map*.

(A3) The system *transfer function* $\hat{G}(s) := s\mathscr{C}(sI-\mathscr{A})^{-1}\mathscr{D} - \mathscr{C}\mathscr{D}$ satisfies

$$\hat{G} \in \mathrm{H}^\infty(\mathbb{C}^+, \mathbf{L}(\mathrm{U},\mathrm{Y}))$$

(recall that $\hat{G} \in \mathrm{H}^\infty(\mathbb{C}^+, \mathrm{Z})$, for some Banach space Z, if

$\hat{G} : \mathbb{C}^+ \ni s \longmapsto \hat{G}(s) \in \mathrm{Z}$ is holomorphic and $\left\|\hat{G}\right\|_{\mathrm{H}^\infty(\mathbb{C}^+,\mathrm{Z})} = \sup_{s\in\mathbb{C}^+} \left\|\hat{G}(s)\right\|_{\mathrm{Z}} < \infty$; this definition applies as $\mathrm{Z} = \mathbf{L}(\mathrm{U},\mathrm{Y})$ is a Banach space). If the latter is met then the *input–output operator*, given by

$$(\mathbb{F}u)(t) := \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t (\Psi[\mathscr{D}u(\tau)])(t-\tau)\mathrm{d}\tau - (\mathscr{C}\mathscr{D})u(t) \quad .$$

satisfies $\mathbb{F} \in \mathbf{L}(\mathrm{L}^2(0,\infty;\mathrm{U}),\mathrm{L}^2(0,\infty;\mathrm{Y}))$. This follows from the Paley–Wiener theorem (Arendt et al, 2001, Theorem 1.8.3, p. 48; this version of the Paley–Wiener theorem does not require *separability* of a Hilbert space) upon taking the Laplace transforms: $(\widehat{\mathbb{F}u})(s) = \hat{G}(s)\hat{u}(s)$, $s \in \mathbb{C}^+$.

Let us remark that since $\hat{G}(s) = s^2\left(\mathscr{C}\mathscr{A}^{-1}\right)(sI - \mathscr{A})^{-1}\mathscr{D} - s\left(\mathscr{C}\mathscr{A}^{-1}\right)\mathscr{D} - \mathscr{C}\mathscr{D}$ then, by **EXS**, $\hat{G}$ is analytic on a set containing $\overline{\mathbb{C}^+}$, which jointly with (**A3**) yields $\left\|\hat{G}(j\omega)\right\|_{\mathbf{L}(\mathrm{U},\mathrm{Y})} \leq \left\|\hat{G}\right\|_{\mathrm{H}^\infty(\mathbb{C}^+,\mathbf{L}(\mathrm{U},\mathrm{Y}))}$ for every $\omega \in \mathbb{R}$.

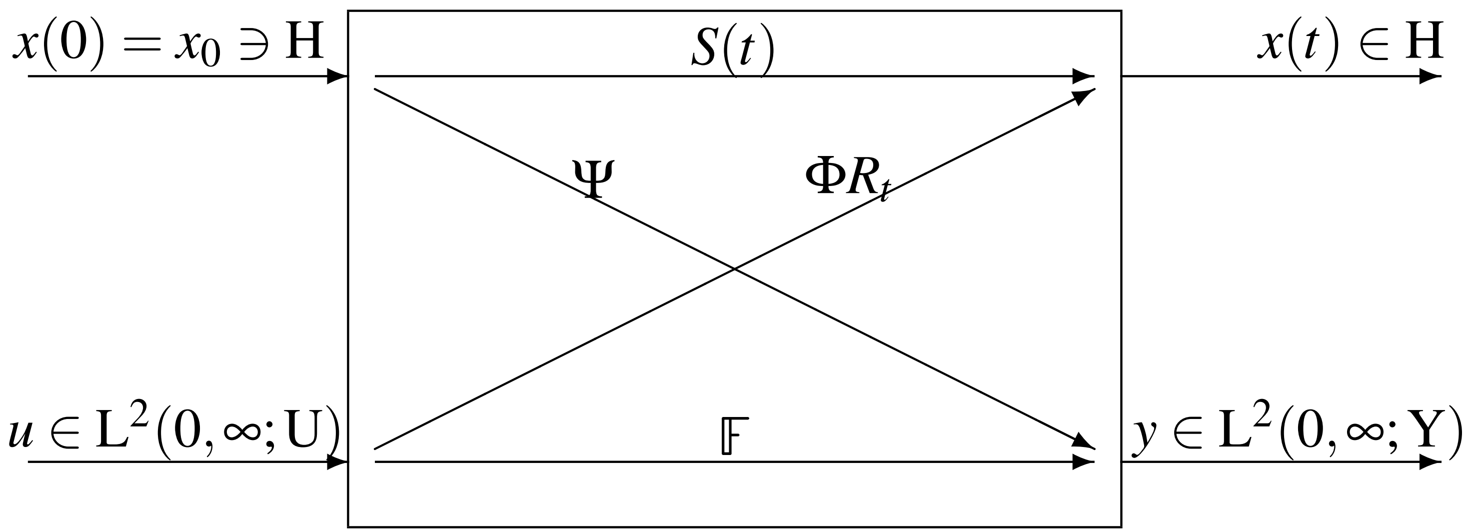

Figure 1.1: Basic control–theoretic operators and their action.

Remark 1.1. If $\mathscr{C}$ is not admissible, the operator $\Psi = \mathscr{L}_{\mathrm{Y}}\mathscr{Z}$ with natural domain $D(\Psi) = \{x \in \mathrm{H} : \mathscr{Z}x \in D(\mathscr{L}_{\mathrm{Y}})\}$ is *closed* and *densely defined*, with $\Psi|_{D(\mathscr{A})} = \mathscr{Z}\mathscr{A}$ (for $x_0 \in D(\mathscr{A})$, Ψx_0 is *homogeneous part* of the system output), and therefore it has *closed* and *densely defined* adjoint operator $\Psi^* = \mathscr{A}^*\mathscr{Z}^*$ with natural domain
$D(\Psi^*) = \{y \in \mathrm{L}^2(0,\infty;\mathrm{Y}) : \mathscr{Z}^*y \in D(\mathscr{A}^*)\}$, with $\Psi^*|_{D(\mathscr{R}_{\mathrm{Y}})} = \mathscr{Z}^*\mathscr{R}_{\mathrm{Y}}$,

$\mathscr{R}_{\mathrm{Y}} = \mathscr{L}_{\mathrm{Y}}^*$.

Similarly, if $\mathscr{D}$ is not admissible, the operator $\Phi = \mathscr{A}\mathscr{W}$ with natural domain $D(\Phi) = \{u \in \mathrm{L}^2(0,\infty;\mathrm{U}) : \mathscr{W}u \in D(\mathscr{A})\}$ is *closed* and *densely defined*, with $\Phi|_{D(\mathscr{R}_{\mathrm{U}})} = \mathscr{W}\mathscr{R}_{\mathrm{U}}$, $\mathscr{R}_{\mathrm{U}} = \mathscr{L}_{\mathrm{U}}^*$, and therefore it has *closed* and *densely defined* adjoint operator $\Phi^* = \mathscr{L}_{\mathrm{U}}\mathscr{W}^*$ with natural domain $D(\Phi^*) = \{x \in \mathrm{H} : \mathscr{W}^* x \in D(\mathscr{L}_{\mathrm{U}})\}$, with $\Phi^*|_{D(\mathscr{A}^*)} = \mathscr{W}^*\mathscr{A}^*$.

2 Time–domain considerations

Lemma 2.1. By (**A2**), for every $x_0 \in \mathrm{H}$ and $u \in \mathrm{L}^2(0,\infty;\mathrm{U})$

$$x(t) = S(t)x_0 + \underbrace{\Phi}_{=\mathscr{A}\mathscr{W}} R_t u,\ \left(R_t u\right)(\tau) := \left\{ \begin{array}{ccc} u(t-\tau) & \text{if} & \tau \le t \\ 0 & \text{if} & \tau < t \end{array} \right\}, \quad (2.1)$$

is a *weak* solution of (1.1), and $R_t \in \mathbf{L}(\mathrm{L}^2(0,\infty;\mathrm{U}))$ is called the *operator of reflection* at t.

Lemma 2.2. If in addition to (**A2**), the semigroup $\{S(t)\}_{t\geq 0}$ is **EXS** then the weak solution (2.1) is for every $x_0 \in \mathrm{H}$ and $u \in \mathrm{L}^2(0,\infty;\mathrm{U})$ in $\mathrm{BUC}_0([0,\infty),\mathrm{H})$, and $t \longmapsto \langle z, x(t)\rangle_{\mathrm{H}}$ is in $\mathrm{L}^2(0,\infty)$ for every $z \in \mathrm{H}$, $x_0 \in \mathrm{H}$ and $u \in \mathrm{L}^2(0,\infty;\mathrm{U})$.

Lemma 2.3. If (**A2**) holds then for every $u \in \mathrm{W}^{1,2}([0,\infty);\mathrm{U})$ and $x_0 \in \mathrm{H}$ such that $x_0 + \mathscr{D}u(0) \in D(\mathscr{A})$, (2.1) is a *classical* solution of (1.1).

The *output equation*

$$y(t) = \mathscr{C}x(t) = \mathscr{C}[x(t) + \mathscr{D}u(t)] - \mathscr{C}\mathscr{D}u(t) \tag{2.2}$$

is well–posed and is a continuous function of t. If, in addition (**A1**) holds, then

$$y(t) = (\Psi x_0)(t) + \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t (\Psi[\mathscr{D}u(\tau)])(t-\tau)\mathrm{d}\tau - \mathscr{C}\mathscr{D}u(t) \ . \tag{2.3}$$

Finally, if all assumptions (**A1**), (**A2**) and (**A3**) are met then for every

$x_0 \in \mathrm{H}$ and $u \in \mathrm{L}^2(0,\infty;\mathrm{U})$:

$$y = \Psi x_0 + \mathbb{F}u \quad . \tag{2.4}$$

Now we are in position to present the main result of this section.

Theorem 2.1. Let $\mathscr{A}$ generates an **EXS** semigroup on H and the assumptions (**A1**), (**A2**) and (**A3**) hold. If the operator

$$\mathscr{R} := R + N^*\mathbb{F} + \mathbb{F}^*Q\mathbb{F} + \mathbb{F}^*N = \mathscr{R}^* \in \mathbf{L}(\mathrm{L}^2(0,\infty;\mathrm{U}))$$

is *coercive* then there exists a unique optimal control, given by

$$u_{\mathrm{opt}} = \mathfrak{M}x_0, \quad \mathfrak{M} := -\mathscr{R}^{-1}(\mathbb{F}^*Q + N^*)\Psi \in \mathbf{L}(\mathrm{H}, \mathrm{L}^2(0,\infty;\mathrm{U})) \quad , \tag{2.5}$$

on which the performance index J achieves its minimum. The minimal value is $J(x_0) = \langle x_0, \mathscr{H}_{\mathrm{opt}} x_0 \rangle_{\mathrm{H}}$, where

$$\mathscr{H}_{\mathrm{opt}} := \Psi^*Q\Psi - \Psi^*(Q\mathbb{F} + N)\mathscr{R}^{-1}(\mathbb{F}^*Q + N^*)\Psi = \mathscr{H}_{\mathrm{opt}}^* \in \mathbf{L}(\mathrm{U}) \quad . \tag{2.6}$$

Next, define

$$N_- := N - Q(\mathscr{C}\mathscr{D}),\ R_- := R - (\mathscr{C}\mathscr{D})^* N - N^*(\mathscr{C}\mathscr{D}) + (\mathscr{C}\mathscr{D})^* Q(\mathscr{C}\mathscr{D}) = R_-^*$$

and assume, in addition, that R_- is *coercive*. Assume that $\mathscr{H} \in \mathbf{L}(\mathrm{H})$, $\mathscr{H} = \mathscr{H}^*$ solves the *Riccati operator equation*

$$\begin{aligned} &\langle \mathscr{A} z, \mathscr{H} z \rangle_{\mathrm{H}} + \langle z, \mathscr{H}\mathscr{A} z \rangle_{\mathrm{H}} + \langle Q\mathscr{C} z, \mathscr{C} z \rangle_{\mathrm{Y}} = \\ &= \left\langle -\mathscr{D}^* \mathscr{H}\mathscr{A} z + N_-^* \mathscr{C} z, R_-^{-1} \left(-\mathscr{D}^* \mathscr{H}\mathscr{A} z + N_-^* \mathscr{C} z \right) \right\rangle_{\mathrm{U}}, \quad z \in D(\mathscr{A}). \end{aligned} \tag{2.7}$$

Define

$$\mathscr{G} z := -\mathscr{D}^* \mathscr{H}\mathscr{A} z + N_-^* \mathscr{C} z, \qquad z \in D(\mathscr{A}) \tag{2.8}$$

and consider the feedback control law

$$u(t) = -R_-^{-1} \frac{\mathrm{d}}{\mathrm{d}t} \left[\mathscr{G}\mathscr{A}^{-1} x(t) \right] \ , \tag{2.9}$$

resulting in the closed–loop system

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathscr{A}^{-1}x\right] = x - \mathscr{D}R_-^{-1}\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathscr{G}\mathscr{A}^{-1}x(t)\right] \Leftrightarrow \frac{\mathrm{d}}{\mathrm{d}t}\left[\mathscr{A}^{-1}x + \mathscr{D}R_-^{-1}\mathscr{G}\mathscr{A}^{-1}x\right] = x \tag{2.10}$$

(I) If $u \in \mathrm{L}^2(0,\infty;\mathrm{U})$ then $u = u_{\mathrm{opt}}$, $\mathscr{H} = \mathscr{H}_{\mathrm{opt}}$ (in particular, this means that $\mathscr{H}_{\mathrm{opt}}$ solves (2.7)) , $\mathscr{G} = \mathscr{G}_{\mathrm{opt}}$, $s \longmapsto R_- + s\mathscr{G}_{\mathrm{opt}}(sI - \mathscr{A})^{-1}\mathscr{D}$ is in $\mathrm{H}^\infty(\mathbb{C}^+, \mathbf{L}(\mathrm{U}))$ and the solution x_{opt} of (2.10) with initial condition x_0, corresponding to u_{opt} reads as $x_{\mathrm{opt}}(t) = S_{\mathrm{opt}}(t)x_0 = [S(t) + \Phi R_t \mathfrak{M}]x_0$, and $\{S_{\mathrm{opt}}(t)\}_{t\geq 0}$ is an **EXS** semigroup on H.

(II) If a solution $\mathscr{H} = \mathscr{H}^* \in \mathbf{L}(\mathrm{H})$ to the Riccati operator equation (2.7) is such that for the corresponding $\mathscr{G}$, defined by (2.8), the operator–valued function $s \longmapsto [R_- + s\mathscr{G}(sI - \mathscr{A})^{-1}\mathscr{D}]$ is in $\mathrm{H}^\infty(\mathbb{C}^+, \mathbf{L}(\mathrm{U}))$ jointly with its $\mathbf{L}(\mathrm{U})$–inverse $s \longmapsto [R_- + s\mathscr{G}(sI - \mathscr{A})^{-1}\mathscr{D}]^{-1}$, then the implicitly defined feedback control (2.9) is in $\mathrm{L}^2(0,\infty;\mathrm{U})$ and therefore it is optimal, i.e., $u = u_{\mathrm{opt}}$, $\mathscr{H} = \mathscr{H}_{\mathrm{opt}}$ and $\mathscr{G} = \mathscr{G}_{\mathrm{opt}}$.

Remark 2.1. If $\mathscr{G}_{\text{opt}}$, originally defined on $D(\mathscr{A})$, extends to an operator $\mathscr{G}_{\Lambda}$ with domain $D(\mathscr{G}_{\Lambda})$ such that: (**i**) $R(\mathscr{D}) \subset D(\mathscr{G}_{\Lambda})$, (**ii**) $(R_{-} + \mathscr{G}_{\Lambda}\mathscr{D})$, $(R_{-} + \mathscr{G}_{\Lambda}\mathscr{D})^{-1} \in \mathbf{L}(\mathrm{U})$ then the equation $z + \mathscr{D}R_{-}^{-1}\mathscr{G}_{\text{opt}}z = x$, in definition of $D(\mathscr{A}_{\text{opt}})$, can be explicitly solved:

$$z + \mathscr{D}R_{-}^{-1}\mathscr{G}_{\text{opt}}z = x \Rightarrow \mathscr{G}_{\Lambda}z + \mathscr{G}_{\Lambda}\mathscr{D}R_{-}^{-1}\mathscr{G}_{\Lambda}z = (R_{-} + \mathscr{G}_{\Lambda}\mathscr{D})\,R_{-}^{-1}\mathscr{G}_{\Lambda}z = \mathscr{G}_{\Lambda}x$$
$$\Longrightarrow R_{-}^{-1}\mathscr{G}_{\Lambda}z = (R_{-} + \mathscr{G}_{\Lambda}\mathscr{D})^{-1}\,\mathscr{G}_{\Lambda}x \Longrightarrow z = x - \mathscr{D}\,(R_{-} + \mathscr{G}_{\Lambda}\mathscr{D})^{-1}\,\mathscr{G}_{\Lambda}x.$$

Consequently, the *closed–loop state operator* can be rewritten as

$$\begin{aligned} \mathscr{A}_{\text{opt}}x &= \mathscr{A}\left[x - \mathscr{D}\,(R_{-} + \mathscr{G}_{\Lambda}\mathscr{D})^{-1}\,\mathscr{G}_{\Lambda}x\right] \\ D(\mathscr{A}_{\text{opt}}) &= \left\{x \in D(\mathscr{G}_{\Lambda}) :\; x - \mathscr{D}\,(R_{-} + \mathscr{G}_{\Lambda}\mathscr{D})^{-1}\,\mathscr{G}_{\Lambda}x \in D(\mathscr{A})\right\} \;. \end{aligned}$$

This form of $\mathscr{A}_{\text{opt}}x$ suggests that the optimal feedback reads as

$$u = -\,(R_{-} + \mathscr{G}_{\Lambda}\mathscr{D})^{-1}\,\mathscr{G}_{\Lambda}x, \qquad x \in D(\mathscr{G}_{\Lambda}) \;, \tag{2.11}$$

what can easily be confirmed by the Laplace transformation.

A part of a proof of the Hille–Phillips–Yosida generation theorem is to show that the operator $\mathscr{A}_s \in \mathbf{L}(\mathrm{H})$, $\mathscr{A}_s f := s\mathscr{A}(sI-\mathscr{A})^{-1} f$ satisfies $\lim_{s\to\infty, s\in\mathbb{R}} \mathscr{A}_s f = \mathscr{A} f$ for every $f \in D(\mathscr{A})$ (Pazy, 1983, Lemma 3.3, p. 10). Therefore $\mathscr{A}_s$ has been called the *Yosida approximation* of $\mathscr{A}$. Since $\mathscr{G}\mathscr{A}^{-1} \in \mathbf{L}(\mathrm{H},\mathrm{U})$ the limit $\lim_{s\to\infty, s\in\mathbb{R}} s\mathscr{G}(sI-\mathscr{A})^{-1} z$ exists for $z \in D(\mathscr{A})$ and it is well–known that it may exist on some domain larger than $D(\mathscr{A})$. Thus the *Yosida approximation* of $\mathscr{G}_{\mathrm{opt}}$,

$$\begin{aligned} \mathscr{G}_\Lambda z &:= \lim_{s\to\infty, s\in\mathbb{R}} s\mathscr{G}_{\mathrm{opt}}(sI-\mathscr{A})^{-1} z, \\ D(\mathscr{G}_\Lambda) &= \{z \in \mathrm{H} : \exists \lim_{s\to\infty, s\in\mathbb{R}} s\mathscr{G}_{\mathrm{opt}}(sI-\mathscr{A})^{-1} z\} \end{aligned} \quad ,$$

or even its *restriction* to $R(\mathscr{D})$, may serve as the needed extension of $\mathscr{G}_{\mathrm{opt}}$, provided that the limit

$$(R_- + \mathscr{G}_\Lambda \mathscr{D})u = \lim_{s\to\infty, s\in\mathbb{R}} (R_- u + s\mathscr{G}_{\mathrm{opt}}(sI-\mathscr{A})^{-1}\mathscr{D}u), \quad u \in \mathrm{U}$$

defines a Banach isomorphism on U.

3 The frequency–domain approach

By the Paley–Wiener theorem (Arendt et al, 2001, Theorem 1.8.3, p. 48)

$$J(u,x_0) = J(\hat{u},x_0) = \langle \hat{u}, \Pi\hat{u}\rangle_{\mathrm{L}^2(j\mathbb{R},\mathrm{U})} + \langle \hat{u}, \left[\hat{G}^* Q + N^*\right] \widehat{\Psi x_0}\rangle_{\mathrm{L}^2(j\mathbb{R},\mathrm{U})} +$$

$$+ \langle \widehat{\Psi x_0}, \left[Q\hat{G} + N\right] \hat{u}\rangle_{\mathrm{L}^2(j\mathbb{R},\mathrm{Y})} + \langle \widehat{\Psi x_0}, Q\widehat{\Psi x_0}\rangle_{\mathrm{L}^2(j\mathbb{R},\mathrm{Y})},\ \hat{u} \in \mathrm{L}^2(j\mathbb{R},\mathrm{U}), x_0 \in \mathrm{H}$$

where Π stands for the *Popov spectral function*,

$$\Pi(j\omega) := R + 2\,\mathrm{Re}[N^*\hat{G}(j\omega)] + \hat{G}^*(j\omega)Q\hat{G}(j\omega) = \Pi^*(j\omega)\ , \qquad (3.1)$$

which, thanks to the continuity and boundedness of $\hat{G}$ on $j\mathbb{R}$, is $\mathbf{L}(\mathrm{U})$–valued bounded and continuous on $j\mathbb{R}$. Here we use the notation $2\,\mathrm{Re}\,Z := Z + Z^*$, $Z \in \mathbf{L}(\mathrm{U})$.

Proposition 3.1. Assume that the assumptions (**A1**), (**A2**) and (**A3**) hold, and $\mathscr{A}$ generates an **EXS** semigroup. Let Π be *coercive*. Then $\mathscr{R}$ is coercive and, by Theorem 2.1, the LQ–problem has a unique

$L^2(0,\infty;U)$–minimizer, whence, by the Paley–Wiener theorem, a unique $H^2(\mathbb{C}^+;U)$–minimizer.

There exists a spectral factorization

$$\Pi(j\omega) = \Xi^*(j\omega)\Xi(j\omega) \ , \tag{3.2}$$

where $\Xi \in H^\infty(\mathbb{C}^+, \mathbf{L}(U))$ jointly with $\mathbb{C}^+ \ni s \longmapsto \Xi^{-1}(s) \in \mathbf{L}(U)$. This spectral factorization is uniquely determined up to a constant, i.e., independent of s, unitary operator multiplier which belongs to $\mathbf{L}(U)$.

Let P_+ stand for the projection from $L^2(j\mathbb{R};U)$ onto its closed subspace $H^2(\mathbb{C}^+;U)$. Then the $H^2(\mathbb{C}^+;U)$–minimizer is given by

$$\hat{u}(s) = -\Xi^{-1}(s)P_+\left\{\Xi^{-*}(j\omega)\left[\hat{G}^*(j\omega)Q + N^*\right]\widehat{(\Psi x_0)}(j\omega)\right\} \ . \tag{3.3}$$

Proposition 3.2. $\mathscr{A}$ generates an **EXS** semigroup. Assume that the assumptions (**A1**), (**A2**) and (**A3**) hold. Let $\Pi(j\omega)$ be coercive. Then $R_- = \Pi(0) = \Xi^*(0)\Xi(0)$ is coercive, so we can discuss the operator Riccati equation (2.7). To each its solution $\mathscr{H}$, or to each $\mathscr{G}$ given by

(2.8), there corresponds a spectral factorization (3.2), where

$$\Xi(s) := V + V^{-*}\mathscr{G}s(sI-\mathscr{A})^{-1}\mathscr{D} \in \mathbf{L}(\mathrm{U}) \tag{3.4}$$

and $s \longmapsto \Xi(s) \in \mathrm{H}^\infty(\mathbb{C}^+, \mathbf{L}(\mathrm{U}))$. Furthermore, $V^{-*}\mathscr{G}$ is admissible.

If $\mathbf{L}(\mathrm{U})$–inverse of Ξ is in $\mathrm{H}^\infty(\mathbb{C}^+, \mathbf{L}(\mathrm{U}))$ then the implicit formula (2.9) defines optimal feedback controller.

Finally,
$\exists \lim_{s\to\infty, s\in\mathbb{R}} s\mathscr{G}(sI-\mathscr{A})^{-1}\mathscr{D}u := \mathscr{G}_\Lambda \mathscr{D}u \iff \exists \lim_{s\to\infty, s\in\mathbb{R}} \Xi(s)u := Du$ and then $V^{-*}(R_- + \mathscr{G}_\Lambda\mathscr{D}) = D$. Thus $R_- + \mathscr{G}_\Lambda\mathscr{D}$ is invertible iff so is D, a fact important for verification whether the explicit formula for the optimal feedback controller (2.11) holds true.

4 The method of spectral factorization

Let us treat (3.4) not as a definition of a spectral factor but an equation determining $\mathscr{G}$. Such the equation is said to be the *realization identity* or *equation*. Then, by (2.8) and (3.4) a unique spectral factor corresponds to the optimal cost, thus this spectral factor is necessarily in $\mathrm{H}^{\infty}(\mathbb{C}^{+}, \mathbf{L}(\mathrm{U}))$ jointly with its inverse and is determined up to a unitary operator which is hidden in V. Thus if the LHS of (3.4) is a spectral factor in $\mathrm{H}^{\infty}(\mathbb{C}^{+}, \mathbf{L}(\mathrm{U}))$ jointly with its inverse then the realization identity must be satisfied, out of uniqueness, by $\mathscr{G}$, corresponding to the optimal control/controller.

It should be emphasised that the realization equation is generally not uniquely solvable. Nevertheless, if the system is *approximately controllable*, i.e., if $\overline{\mathscr{R}(\Phi)} = \mathrm{H} \iff \ker \Phi^{*} = \{0\}$, then the realization identity cannot have more then one solution, so it determines uniquely the optimal controller (in its implicit form), provided that the LHS of the realization identity is a spectral factor belonging to $\mathrm{H}^{\infty}(\mathbb{C}^{+}, \mathbf{L}(\mathrm{U}))$ jointly with its inverse.

Thus if, in addition, the system is *approximately controllable*, then $\mathscr{G}_\Lambda$ or $D^{-1}V^{-*}\mathscr{G}_\Lambda$ are *uniquely* determined by the following equivalent *realization equations*

$$\Xi^*(0)\Xi(s) = R_- + \mathscr{G}_\Lambda\mathscr{D} + \hat{G}_{\mathscr{G}}(s) \iff \Xi^*(0)\left[\Xi(s) - D\right] = \hat{G}_{\mathscr{G}}(s) \iff$$
$$\iff \Xi(s) = D\left[I + D^{-1}V^{-*}\mathscr{G}_\Lambda\mathscr{A}(sI-\mathscr{A})^{-1}\mathscr{D}\right] \quad , \tag{4.1}$$

where $\hat{G}_{\mathscr{G}}(s) := \mathscr{G}_\Lambda\mathscr{A}(sI-\mathscr{A})^{-1}\mathscr{D}$ and the second line arises by acting with the operator $D^{-1}V^{-*}$ on both sides of the last identity in the first line.

Comment 4.1. If τ is the *operator of boundary control* then, since $D(\mathscr{A}) \subset \ker\tau$, $\tau\mathscr{D} = -I$, one has

$$\tau\mathscr{A}(sI-\mathscr{A})^{-1}\mathscr{D} = s\tau(sI-\mathscr{A})^{-1}\mathscr{D} - \tau\mathscr{D} = I$$

and (4.1) can also be written as

$$\Xi(s) = (D\tau + V^{-*}\mathscr{G}_\Lambda)\mathscr{A}(sI-\mathscr{A})^{-1}\mathscr{D} \quad .$$

5 Comparison with earlier works

Consider the *tower* (or *scale*) of Hilbert spaces

$$\mathrm{H}_1 \hookrightarrow \mathrm{H}(=\mathrm{H}^*) \hookrightarrow \mathrm{H}_{-1} \quad ,$$

with continuous dense embeddings, where $\mathrm{H}_1 = (D(\mathscr{A}), \|\cdot\|_{\mathscr{A}})$, $\|x\|_{\mathscr{A}} := \|\mathscr{A}x\|_{\mathrm{H}}$ whilst H_{-1} stands for the completion of H under the norm $\|x\|_{\mathrm{H}_{-1}} := \|\mathscr{A}^{-1}x\|_{\mathrm{H}}$; the latter arises by taking the limits of all sequences of H, which are Cauchy sequences with respect to $\|x\|_{\mathrm{H}_{-1}}$.

Parallely, consider also the *tower* of Hilbert spaces

$$\mathrm{Z}_{-1} \hookleftarrow \mathrm{H}(=\mathrm{H}^*) \hookleftarrow \mathrm{Z}_1 \quad ,$$

with continuous dense embeddings, where $\mathrm{Z}_1 = (D(\mathscr{A}^*), \|\cdot\|_{\mathscr{A}^*})$, $\|x\|_{\mathscr{A}^*} := \|\mathscr{A}^*x\|_{\mathrm{H}}$ whilst Z_{-1} stands for the completion of H under the norm $\|x\|_{\mathrm{Z}_{-1}} := \|\mathscr{A}^{-*}x\|_{\mathrm{H}}$; the latter arises by taking the limits of all sequences of H, which are Cauchy sequences with respect to $\|x\|_{\mathrm{Z}_{-1}}$.

The bilinear form

$$\langle x, z\rangle_{H_{-1}\times Z_1} := \left\langle \mathscr{A}_e x, \mathscr{A}^{-*} z\right\rangle_{H\times H} \quad ,$$

where $\mathscr{A}_e \in \mathbf{L}(H, H_{-1})$ denotes the extension of $\mathscr{A} \in \mathbf{L}(H_1, H)$, an isometry from H_1, onto H, defines duality pairing between H_{-1} and Z_1. Here H_{-1} is isomorphic with $[D(\mathscr{A}^*)]^*$ whilst and Z_{-1} is isomorphic with $[D(\mathscr{A})]^*$.

It is proved in (Weiss and Weiss, 1997) that if Π has the spectral factorization $\Pi(j\omega) = [\Xi(j\omega)]^*\Xi(j\omega)$, where Ξ, $\Xi^{-1} \in H^\infty(\mathbb{C}^+, \mathbf{L}(U))$ and $\Xi(s) \longrightarrow D$ as $s \to \infty$, $s \in \mathbb{R}$ with D and $D^{-1} \in \mathbf{L}(U)$ (*regular spectral function*), then the optimal cost operator X solves the operator Riccati (Weiss and Weiss, 1997, Theorem 12.8, p. 322, especially formula (12.7)) and (Staffans, 1997, Corollary 45, p. 3712); see also (Mikkola and Staffans, 2004, Theorem 3, especially formula (6))

$$\mathscr{A}^*X + X\mathscr{A} + C^*QC = (B^*_{\Lambda_w}X + NC)^*(D^*D)^{-1}(B^*_{\Lambda_w}X + NC) \quad , \qquad (5.1)$$

where all terms are in $\mathbf{L}(\mathrm{H}_1, \mathrm{Z}_{-1})$ and, actually, X maps $D(\mathscr{A})$ into $D(B^*_{\Lambda_w})$. Here $B \in \mathbf{L}(\mathrm{U}, \mathrm{H}_{-1}) \iff B^* \in \mathbf{L}(\mathrm{Z}_1, \mathrm{U})$, $C \in \mathbf{L}(\mathrm{H}_1, \mathrm{Y}) \iff C^* \in \mathbf{L}(\mathrm{Y}, \mathrm{Z}_{-1})$, $B^*_{\Lambda_w}$ (B^*_{Λ}) denotes weak (strong) extension of B^*, defined as the weak (strong) limit of $sB^*(sI - \mathscr{A})^{-1}x$ as $s \to \infty$, $s \in \mathbb{R}$ and $D(B^*_{\Lambda_w})$ consists of those $x \in \mathrm{H}$ for which the weak limit exists ($D(B_{\Lambda^*})$ consists of those $x \in \mathrm{H}$ for which the strong limit exists). The optimal controller is given on $D(\mathscr{A})$ as

$$Fx = -(D^*D)^{-1}(B^*_{\Lambda_w}X + NC)x, \qquad x \in D(\mathscr{A}) \ .$$

The spectral factor Ξ can be realized as a transfer function of the system with the state operator $\mathscr{A}$, control operator B, observation operator $-DF_\Lambda$ and the feedtrough operator D (Weiss and Weiss, 1997, p. 329, Formula (12.5)), i.e.,

$$\Xi(s) = D - DF_\Lambda(sI - \mathscr{A})^{-1}B = D\left[I - F_\Lambda(sI - \mathscr{A})^{-1}B\right] \ . \tag{5.2}$$

Finally, the state operator of the optimal closed–loop system reads as

$$\mathscr{A}_{\text{opt}} = \mathscr{A} + BF_\Lambda, \qquad D(\mathscr{A}_{\text{opt}}) = \{x_0 \in D(F_\Lambda) : \ (\mathscr{A} + BF_\Lambda)x_0 \in \mathrm{H}\}$$

so the optimal controller is $u = F_{\text{opt}}x_0$, where $F_{\text{opt}}x_0 = F_\Lambda x_0$ for $x_0 \in D(\mathscr{A}_{\text{opt}})$.

Our Riccati operator equation (2.7) slightly differs from (5.1) as:

(**a**) it does not employ the feedthrough operator D,

(**b**) it is stated in a weak sense within the state space H,

(**c**) even if we identify X with $\mathscr{H}$ (both operators express the minimal cost), C with $\mathscr{C}$ and notify that $B^*_{\Lambda_w}$ is an extension of $\mathscr{D}^*\mathscr{A}^*$ then the ordering of operators defining $\mathscr{G}$ and F is not the same and in (2.7) the operator N_- appears instead of N in (5.1). Thus our Riccati equation (2.7) is astonishingly not the same as (5.1).

Next, **EXS** of $\{S_{\text{opt}}(t)\}_{t\geq 0}$ is not shown in (Weiss and Weiss, 1997), though we still do not know whether it decays with the same rate or faster than $\{S(t)\}_{t\geq 0}$.

On the other side our results and those of (Weiss and Weiss, 1997) are very close in the frequency–domain aspects as:

(d) the idea of Remark 2.1 coincides with the concept of a *regular spectral factor*,

(e) comparing the second line of (4.1) with (5.2) we get a relationship between $\mathcal{G}_\Lambda$ and F_Λ,

$$F_\Lambda = -D^{-1}V^{-*}\mathcal{G}_\Lambda \quad . \tag{5.3}$$

6 Solution of the example by Chapelon and Xu

The state operator $\mathscr{A}$ acts in $\mathrm{H} = \mathrm{L}^2(0,1) \oplus \mathrm{L}^2(0,1)$,

$$\mathscr{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -m_1 x_1' \\ m_2 x_2' \end{bmatrix}, \quad m_1 > 0,\ m_2 > 0;$$

$$D(\mathscr{A}) = \left\{ x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathrm{W}^{1,2}(0,1) \oplus \mathrm{W}^{1,2}(0,1) : \begin{array}{c} x_1(0) = \alpha x_2(0) \\ x_2(1) = \beta x_1(1) \end{array} \right\}$$

and generates an **EXS** C_0–semigroup, provided that $\alpha^2\beta^2 < 1$. This fact is not explicitly proved in (Chapelon and Xu, 2003), where the authors recall an older result due to D. Russell (Chapelon and Xu, 2003, Proposition 3.1, p. 592), however we are able to give a separate Lyapunov–type proof. For that, define the following matrix operators of multiplication $\mathscr{E}_1 = \mathscr{E}_1^*$, $\mathscr{E}_2 = \mathscr{E}_2^*$ and $\mathscr{E}_3 = \mathscr{E}_3^* \in \mathbf{L}(\mathrm{H})$, $\mathscr{E}_3^* \geq 0$:

$$(\mathscr{E}_1 x)(\theta) := \frac{1}{1-\alpha^2\beta^2}\,\mathrm{diag}\left\{\frac{1}{m_1}, 0\right\} x(\theta),$$
$$(\mathscr{E}_2 x)(\theta) := \frac{1}{1-\alpha^2\beta^2}\,\mathrm{diag}\left\{0, \frac{1}{m_2}\right\} x(\theta)$$
$$(\mathscr{E}_3 x)(\theta) = \mathrm{diag}\left\{\frac{1-\theta}{m_1}, \frac{\theta}{m_2}\right\} x(\theta), \qquad x \in \mathrm{H} \quad .$$

Notice that its linear combination $k_1\mathscr{E}_1 + k_2\mathscr{E}_2 + \mathscr{E}_3$ satisfies

$$\langle \mathscr{A}x, (k_1\mathscr{E}_1 + k_2\mathscr{E}_2 + \mathscr{E}_3)x\rangle_{\mathrm{H}} + \langle x, (k_1\mathscr{E}_1 + k_2\mathscr{E}_2 + \mathscr{E}_3)\mathscr{A}x\rangle_{\mathrm{H}} =$$
$$= -\|x\|_{\mathrm{H}}^2 + \left\{\beta^2 - \frac{k_1}{1-\alpha^2\beta^2} + \frac{k_2\beta^2}{1-\alpha^2\beta^2}\right\} x_1^2(1) + \tag{6.1}$$
$$+\left\{\alpha^2 + \frac{k_1\alpha^2}{1-\alpha^2\beta^2} - \frac{k_2}{1-\alpha^2\beta^2}\right\} x_2^2(0), \qquad x \in D(\mathscr{A}) \quad .$$

Solving an appropriate linear system of equations determining k_1, k_2 we establish that $\mathscr{E} := (\alpha^2+1)\beta^2\mathscr{E}_1 + (\beta^2+1)\alpha^2\mathscr{E}_2 + \mathscr{E}_3$ satisfies the

Lyapunov operator equation

$$\langle \mathscr{A} x, \mathscr{E} x \rangle_{\mathrm{H}} + \langle x, \mathscr{E}\mathscr{A} x \rangle_{\mathrm{H}} = - \|x\|^2_{\mathrm{H}}, \qquad x \in D(\mathscr{A}) \ .$$

Now **EXS** for $\alpha^2\beta^2 < 1 \iff \mathscr{E} \geq 0$ follows from Datko's theorem. If the latter holds then $\mathscr{C} = I$ is admissible (**A1**).

(Chapelon and Xu, 2003) have used the framework of well–posed systems rather than (1.1), so it is worth to note that the *operator of boundary control* equals

$$\tau \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1(0) - \alpha x_2(0) \\ x_2(1) - \beta x_1(1) \end{bmatrix}, \ D(\tau) \subset \mathrm{W}^{1,2}(0,1) \oplus \mathrm{W}^{1,2}(0,1). \quad (6.2)$$

$\mathrm{U} = \mathbb{R}^2$, and the factor control operator $\mathscr{D}$ is given by

$$\mathscr{D} u = \mathbf{D} u, \qquad \mathbf{D} = \frac{1}{\alpha\beta - 1} \begin{bmatrix} \mathbf{1} & \alpha \mathbf{1} \\ \beta \mathbf{1} & \mathbf{1} \end{bmatrix} \ .$$

The adjoint operator of $\mathscr{A}$ is

$$\mathscr{A}^* \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} m_1 v_1' \\ -m_2 v_2' \end{bmatrix}, \quad D(\mathscr{A}^*) =$$

$$= \left\{ v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathrm{W}^{1,2}(0,1) \oplus \mathrm{W}^{1,2}(0,1) : \begin{array}{c} m_1 v_1(1) = \beta m_2 v_2(1) \\ \alpha m_1 v_1(0) = m_2 v_2(0) \end{array} \right\} .$$

Hence

$$\mathscr{D}^* \mathscr{A}^* v = \mathbf{D}^T \int_0^1 (\mathscr{A}^* v)(\theta) \mathrm{d}\theta = \begin{bmatrix} m_1 v_1(0) \\ m_2 v_2(1) \end{bmatrix}$$

and this observation operator is admissible (**A2**). Indeed, the operator

$$(\mathscr{H}_\Phi v)(\theta) = \frac{1}{1 - \alpha^2 \beta^2} \operatorname{diag}\left\{ (\alpha^2 + 1) m_1, (\beta^2 + 1) m_2 \right\} v(\theta), \qquad v \in \mathrm{H}$$

is the system controllability gramian, because it solves the Lyapunov operator equation, i.e., for $v \in D(\mathscr{A}^*)$ there holds:

$$\langle \mathscr{A}^* v, \mathscr{H}_\Phi v\rangle_{\mathrm{H}} + \langle v, \mathscr{H}_\Phi \mathscr{A}^* v\rangle_{\mathrm{H}} = -m_1^2 v_1^2(0) - m_2 v_2^2(1) = -\left\| \mathscr{D}^* \mathscr{A}^* v\right\|_{\mathrm{U}}^2,$$

The system is infinite-time exactly controllable as $\mathscr{H}_\Phi$ is a coercive operator, whence the method of spectral factorization is applicable because for standard lq–problems the Popov function is always coercive.

Next,

$$(\mathscr{A}(sI-\mathscr{A})^{-1}z)(\theta) = \begin{bmatrix} se^{-\frac{s\theta}{m_1}}\mathfrak{c} - z_1(\theta) + \frac{s}{m_1}\int_0^\theta e^{-\frac{s(\theta-\tau)}{m_1}} z_1(\tau)\mathrm{d}\tau \\ \frac{1}{\alpha} s e^{\frac{s\theta}{m_2}}\mathfrak{c} - z_2(\theta) - \frac{s}{m_2}\int_0^\theta e^{\frac{s(\theta-\tau)}{m_2}} z_2(\tau)\mathrm{d}\tau \end{bmatrix},$$

where

$$\mathfrak{c} = \frac{1}{\frac{1}{\alpha}e^{\frac{s}{m_2}} - \beta e^{-\frac{s}{m_1}}}\left[\frac{\beta}{m_1}\int_0^1 e^{-\frac{s(1-\tau)}{m_1}} z_1(\tau)\mathrm{d}\tau + \frac{1}{m_2}\int_0^1 e^{\frac{s(1-\tau)}{m_2}} z_2(\tau)\mathrm{d}\tau\right].$$

Hence, taking $z = \mathbf{D}u$, we get

$$\hat{G}(s) = \frac{1}{1-\alpha\beta e^{-s\left(\frac{1}{m_1}+\frac{1}{m_2}\right)}} \begin{bmatrix} e^{-\frac{s\theta}{m_1}} & \alpha e^{-\frac{s}{m_2}} e^{-\frac{s\theta}{m_1}} \\ \beta e^{-\frac{s}{m_1}} e^{-\frac{s(1-\theta)}{m_2}} & e^{-\frac{s(1-\theta)}{m_2}} \end{bmatrix} . \quad (6.3)$$

with $\hat{G} \in \mathrm{H}^\infty(\mathbb{C}^+, \mathbf{L}(\mathrm{U}))$, i.e., (**A3**) is met.

The transfer function can be represented in the *right coprime* form $\hat{G}(s) = \mathbf{U}(s)\mathbf{M}^{-1}(s)$ with

$$\mathbf{U}(s) = \begin{bmatrix} e^{-\frac{s\theta}{m_1}} & 0 \\ 0 & e^{-\frac{s(1-\theta)}{m_2}} \end{bmatrix}, \quad \mathbf{M}(s) = \begin{bmatrix} 1 & -\alpha e^{-\frac{s}{m_2}} \\ -\beta e^{-\frac{s}{m_1}} & 1 \end{bmatrix} .$$

Denoting by $\mathbf{Z}_*(s) := \mathbf{Z}^T(-s)$ the *para–Hermitian adjoint* of $\mathbf{Z}(s)$, we see that $\mathbf{U}_*(s) = \mathbf{U}^T(-s) = \mathbf{U}(-s) = \mathbf{U}^{-1}(s)$, so $\mathbf{U}(s)$ is *para–unitary*. Now

$$\Pi(s) = I + \hat{G}_*(s)\hat{G}(s) = I + \mathbf{M}^{-T}(-s)\mathbf{M}^{-1}(s)$$

which facilitates finding a spectral factor of $\Pi(j\omega) \geq I$ by reducing the problem to finding a spectral factor of an entire matrix–valued function

$$I+\mathbf{M}_*(s)\mathbf{M}(s) = \begin{bmatrix} 2+\beta^2 & -\alpha e^{-\frac{s}{m_2}} - \beta e^{\frac{s}{m_1}} \\ -\alpha e^{\frac{s}{m_2}} - \beta e^{-\frac{s}{m_1}} & 2+\alpha^2 \end{bmatrix}.$$

We shall seek for a factorization $I+\mathbf{M}_*(s)\mathbf{M}(s) = \mathbf{X}_*(s)\mathbf{X}(s)$ with

$$\mathbf{X}(s) = \begin{bmatrix} \mathfrak{m} & -\mathfrak{n}e^{-\frac{s}{m_2}} \\ -\mathfrak{p}e^{-\frac{s}{m_1}} & \mathfrak{q} \end{bmatrix} .$$

This leads to the system of equations:

$$\mathfrak{m}^2+\mathfrak{p}^2 = 2+\beta^2 \tag{6.4}$$

$$\mathfrak{n}\mathfrak{m} = \alpha \tag{6.5}$$

$$\mathfrak{p}\mathfrak{q} = \beta \tag{6.6}$$

$$\mathfrak{n}^2+\mathfrak{q}^2 = 2+\alpha^2 \ . \tag{6.7}$$

Eliminating $\mathfrak{n}$, $\mathfrak{p}$ from (6.7)/(6.4) with the aid of (6.5)/(6.6), we get

$$\mathfrak{n} = \frac{\alpha}{\mathfrak{m}}, \quad \mathfrak{p} = \frac{\beta}{\mathfrak{q}}, \quad \mathfrak{q}^2 = 2 + \alpha^2 - \frac{\alpha^2}{\mathfrak{m}^2} \Longrightarrow \mathfrak{p}^2 = \frac{\beta^2 \mathfrak{m}^2}{(2+\alpha^2)\mathfrak{m}^2 - \alpha^2}$$

and a biquadratic equation determining $\mathfrak{m}$:

$$(2+\alpha^2)\mathfrak{m}^4 - \left[\alpha^2 - \beta^2 + (2+\alpha^2)(2+\beta^2)\right]\mathfrak{m}^2 + \alpha^2(2+\beta^2) = 0 \ . \quad (6.8)$$

Observe that the LHS of (6.8) at $\mathfrak{m}^2 = 0$ equals: $\alpha^2(2+\beta^2) \geq 0$. Let

$$\mu := \alpha^2 - \beta^2 + (2+\alpha^2)(2+\beta^2) =$$
$$(2+\beta^2) + (2+\alpha^2) + \alpha^2(2+\beta^2) \geq 4 + 2\alpha^2 > 0$$

and observe that the determinant of (6.8) satisfies

$$\mu^2 \geq \Delta = \mu^2 - 4\alpha^2(2+\alpha^2)(2+\beta^2) >$$
$$> \left[(2+\alpha^2) + \alpha^2(2+\beta^2)\right]^2 - 4\alpha^2(2+\alpha^2)(2+\beta^2) =$$
$$= \left[(2+\alpha^2) - \alpha^2(2+\beta^2)\right]^2 \geq 0.$$

Hence (6.8) has four real roots $\mathfrak{m}_B > \mathfrak{m}_A \geq 0 \geq -\mathfrak{m}_A > -\mathfrak{m}_B$ with equality signs iff $\alpha = 0$. Furthermore, the LHS of (6.8) at $\mathfrak{m}^2 = 2$ equals: $-\beta^2(2+\alpha^2) \leq 0$, so $\mathfrak{m}_B \geq \sqrt{2}$ $(= \iff \beta = 0)$ and $\mathfrak{q}_B \geq \sqrt{2}$ $(= \iff \alpha = 0)$. Take the solution

$$\mathfrak{m} = \mathfrak{m}_B := \sqrt{\frac{\mu + \sqrt{\Delta}}{2(2+\alpha^2)}},\ \mathfrak{n} = \frac{\alpha}{\mathfrak{m}_B},$$

$$\mathfrak{p} = \frac{\beta \mathfrak{m}_B}{\sqrt{(2+\alpha^2)\mathfrak{m}_B^2 - \alpha^2}},\ \mathfrak{q} = \frac{\sqrt{(2+\alpha^2)\mathfrak{m}_B^2 - \alpha^2}}{\mathfrak{m}_B}.$$

Since

$$\mathbf{X}^{-1}(s) = \frac{1}{\mathfrak{m}\mathfrak{q}\left[1 - \frac{\mathfrak{n}\mathfrak{p}}{\mathfrak{m}\mathfrak{q}} e^{-(\frac{s}{m_1} + \frac{s}{m_2})}\right]} \begin{bmatrix} \mathfrak{q} & \mathfrak{n} e^{-\frac{s}{m_2}} \\ \mathfrak{p} e^{-\frac{s}{m_1}} & \mathfrak{m} \end{bmatrix}.$$

then $s \longmapsto \mathbf{X}(s) \in \mathrm{H}^{\infty}(\mathbb{C}^{+}, \mathbf{L}(\mathbb{C}^{2}))$ jointly with $s \longmapsto \mathbf{X}^{-1}(s)$ iff

$$1 > \frac{\mathfrak{n}^2\mathfrak{p}^2}{\mathfrak{m}^2\mathfrak{q}^2} = \frac{\alpha^2\beta^2}{\left[(2+\alpha^2)\mathfrak{m}_B^2 - \alpha^2\right]^2} \Leftrightarrow \left[(2+\alpha^2)\mathfrak{m}_B^2 - \alpha^2\right]^2 > \alpha^2\beta^2, \quad (6.9)$$

but the last inequality holds as $\alpha^2\beta^2 < 1$ and $\left[(2+\alpha^2)\mathfrak{m}_B^2 - \alpha^2\right]^2 \geq 4$.

Consequently the spectral factor $\Xi(s)$ of Π reads as

$$\Xi(s) = \mathbf{X}(s)\mathbf{M}^{-1}(s) =$$

$$= \frac{1}{1-\alpha\beta e^{-\left(\frac{s}{m_1}+\frac{s}{m_2}\right)}} \begin{bmatrix} \mathfrak{m} - \mathfrak{n}\beta e^{-\left(\frac{s}{m_1}+\frac{s}{m_2}\right)} & (\mathfrak{m}\alpha - \mathfrak{n})e^{-\frac{s}{m_2}} \\ (\mathfrak{q}\beta - \mathfrak{p})e^{-\frac{s}{m_1}} & \mathfrak{q} - \mathfrak{p}\alpha e^{-\left(\frac{s}{m_1}+\frac{s}{m_2}\right)} \end{bmatrix}.$$

and belongs to $\mathrm{H}^{\infty}(\mathbb{C}^{+}, \mathbf{L}(\mathbb{C}^{2}))$ jointly with $\Xi^{-1}(s)$,

$$\Xi^{-1}(s) = \frac{1}{\mathfrak{m}\mathfrak{q} - \mathfrak{n}\mathfrak{p}e^{-\left(\frac{s}{m_1}+\frac{s}{m_2}\right)}} \begin{bmatrix} \mathfrak{q} - \mathfrak{p}\alpha e^{-\left(\frac{s}{m_1}+\frac{s}{m_2}\right)} & (\mathfrak{n} - \mathfrak{m}\alpha)e^{-\frac{s}{m_2}} \\ (\mathfrak{p} - \mathfrak{q}\beta)e^{-\frac{s}{m_1}} & \mathfrak{m} - \mathfrak{n}\beta e^{-\left(\frac{s}{m_1}+\frac{s}{m_2}\right)} \end{bmatrix}.$$

For obtaining the optimal controller we get

$$D = \lim_{s\to\infty, s\in\mathbb{R}} \Xi(s) = \begin{bmatrix} \mathfrak{m} & 0 \\ 0 & \mathfrak{q} \end{bmatrix}, \quad \Xi(0) = \frac{1}{1-\alpha\beta} \begin{bmatrix} \mathfrak{m}-\mathfrak{n}\beta & \mathfrak{m}\alpha-\mathfrak{n} \\ \mathfrak{q}\beta-\mathfrak{p} & \mathfrak{q}-\mathfrak{p}\alpha \end{bmatrix}$$

and, since $\mathfrak{m}\mathfrak{q} = \sqrt{(2+\alpha)^2\mathfrak{m}_B^2-\alpha^2} \geq \sqrt{2}$,

$$D^{-1} = \begin{bmatrix} \frac{1}{\mathfrak{m}} & 0 \\ 0 & \frac{1}{\mathfrak{q}} \end{bmatrix}, \quad \Xi^{-1}(0) = \frac{1}{\mathfrak{m}\mathfrak{q}-\mathfrak{n}\mathfrak{p}} \begin{bmatrix} \mathfrak{q}-\mathfrak{p}\alpha & \mathfrak{n}-\mathfrak{m}\alpha \\ \mathfrak{p}-\mathfrak{q}\beta & \mathfrak{m}-\mathfrak{n}\beta \end{bmatrix}.$$

From the realization identity (4.1), which here takes the form:

$$I - D^{-1} \underbrace{\Xi(s)}_{=\mathbf{X}(s)\mathbf{M}^{-1}(s)} = \overbrace{-D^{-1} \underbrace{V^{-*}}_{=\Xi^{-*}(0)} \mathscr{G}_\Lambda}^{=F_\Lambda} \overbrace{\mathscr{A}(sI-\mathscr{A})^{-1}\mathscr{D}}^{=\hat{G}(s)} = F_\Lambda \underbrace{\hat{G}(s)}_{=\mathbf{U}(s)\mathbf{M}^{-1}(s)}$$

$$\iff \mathbf{M}(s) - D^{-1}\mathbf{X}(s) = F_\Lambda \mathbf{U}(s)$$

and (5.3) or (2.11), we determine the optimal controller

$$u = F_\Lambda x = -(\underbrace{R_-}_{=\Pi(0)} + \mathscr{G}_\Lambda \mathscr{D})^{-1} \mathscr{G}_\Lambda x = \begin{bmatrix} (\frac{\mathfrak{n}}{\mathfrak{m}} - \alpha) x_2(0) \\ (\frac{\mathfrak{p}}{\mathfrak{q}} - \beta) x_1(1) \end{bmatrix}$$

$$\mathscr{G}_\Lambda x = \frac{1}{1-\alpha\beta} \begin{bmatrix} (\mathfrak{m} - \mathfrak{n}\beta)(\alpha\mathfrak{m} - \mathfrak{n}) x_2(0) + (\mathfrak{q}\beta - \mathfrak{p})^2 x_1(1) \\ (\alpha\mathfrak{m} - \mathfrak{n})^2 x_2(0) + (\mathfrak{q} - \mathfrak{p}\alpha)(\mathfrak{q}\beta - \mathfrak{p}) x_1(1) \end{bmatrix},$$

$$D(F_\Lambda) = D(\mathscr{G}_\Lambda) \supset \mathrm{W}^{1,2}(0,1) \oplus \mathrm{W}^{1,2}(0,1) \supset R(\mathscr{D}) \ .$$

The Riccati operator equation (2.7) takes the form

$$\begin{aligned} &\langle \mathscr{A} x, \mathscr{H} x \rangle_{\mathrm{H}} + \langle x, \mathscr{H} \mathscr{A} x \rangle_{\mathrm{H}} = -\|x\|_{\mathrm{H}}^2 + (\alpha\mathfrak{m} - \mathfrak{n})^2 x_2^2(0) + \\ &+ (\beta\mathfrak{q} - \mathfrak{p})^2 x_1^2(1) = -\|\mathscr{C} x\|_{\mathrm{H}}^2 + \|V^{-*} \mathscr{G} x\|_{\mathrm{U}}^2, \quad x \in D(\mathscr{A}) \ , \end{aligned} \tag{6.10}$$

It is not difficult to see that its solution $\mathscr{H} \in \mathbf{L}(\mathrm{H})$, $\mathscr{H} = \mathscr{H}^*$ equals

$$\mathscr{H} = (\mathfrak{m}^2 - 2)(1 - \alpha^2\beta^2)\mathscr{E}_1 + (\mathfrak{q}^2 - 2)(1 - \alpha^2\beta^2)\mathscr{E}_2 + \mathscr{E}_3 \geq 0$$

and it was found using (6.1), where k_1, k_2 have been determined from the linear system of equations

$$\begin{bmatrix} -1 & \beta^2 \\ \alpha^2 & -1 \end{bmatrix} \begin{bmatrix} k_1(1-\alpha^2\beta^2)^{-1} \\ k_2(1-\alpha^2\beta^2)^{-1} \end{bmatrix} = \begin{bmatrix} (\beta\mathfrak{q}-\mathfrak{p})^2-\beta^2 \\ (\alpha\mathfrak{m}-\mathfrak{n})^2-\alpha^2 \end{bmatrix} .$$

Using Theorem 2.1/(**II**) we can confirm optimality of the above $\mathscr{G}_\Lambda$

$$\mathscr{G}x = -\mathscr{D}^*\mathscr{H}\mathscr{A}x + N_-^*\mathscr{C}x = -\mathscr{D}^*[\mathscr{H}\mathscr{A}x + x] =$$
$$= \frac{1}{1-\alpha\beta}\begin{bmatrix} (2-\mathfrak{m}^2+\beta^2\mathfrak{q}^2-\beta^2)x_1(1)+(2\beta-\mathfrak{q}^2\beta+\alpha\mathfrak{m}^2-\alpha)x_2(0) \\ (2\alpha-\mathfrak{m}^2\alpha+\beta\mathfrak{q}^2-\beta)x_1(1)+(2-\mathfrak{q}^2+\alpha^2\mathfrak{m}^2-\alpha^2)x_2(0) \end{bmatrix}$$
$$= \mathscr{G}_\Lambda x, \qquad x \in D(\mathscr{A}) .$$

Finally, we determine the closed-loop system state operator. Let

$$✐ := \frac{1}{1-\alpha\beta}\frac{\alpha\mathfrak{m}-\mathfrak{n}}{\mathfrak{m}}x_2(0)\mathbf{1} + \frac{1}{1-\alpha\beta}\frac{\alpha(\mathfrak{q}\beta-\mathfrak{p})}{\mathfrak{q}}x_1(1)\mathbf{1} ,$$

$$\text{✎} := \frac{1}{1-\alpha\beta}\frac{\beta(\alpha\mathfrak{m}-\mathfrak{n})}{\mathfrak{m}}x_2(0)\mathbf{1} + \frac{1}{1-\alpha\beta}\frac{\mathfrak{q}\beta-\mathfrak{p}}{\mathfrak{q}}x_1(1)\mathbf{1} \; .$$

Since for $x \in \mathrm{W}^{1,2}(0,1) \oplus \mathrm{W}^{1,2}(0,1)$ one has:

$$x - \mathscr{D}(R_- + \mathscr{G}_\Lambda \mathscr{D})^{-1}\mathscr{G}_\Lambda x = \begin{bmatrix} x_1 + \text{✐} \\ x_2 + \text{✎} \end{bmatrix} \in D(\mathscr{A}) \iff$$

$$\iff x_1(0) = \frac{\mathfrak{n}}{\mathfrak{m}}x_2(0),\; x_1(1) = \frac{\mathfrak{p}}{\mathfrak{q}}x_2(1)$$

then

$$\mathscr{A}_{\mathrm{opt}}x = \mathscr{A}[x - \mathscr{D}(R_- + \mathscr{G}_\Lambda \mathscr{D})^{-1}\mathscr{G}_\Lambda x] = \mathscr{A}x = \begin{bmatrix} -m_1 x_1' \\ m_2 x_2' \end{bmatrix},\;\; D(\mathscr{A}_{\mathrm{opt}}) =$$

$$= \left\{ x \in \mathrm{W}^{1,2}(0,1) \oplus \mathrm{W}^{1,2}(0,1) :\; x_1(0) = \frac{\mathfrak{n}}{\mathfrak{m}}x_2(0),\; x_1(1) = \frac{\mathfrak{p}}{\mathfrak{q}}x_2(1) \right\} \; .$$

$\mathscr{A}_{\mathrm{opt}}$ has the same structure as $\mathscr{A}$ with α and β replaced by $\frac{\mathfrak{n}}{\mathfrak{m}}$ and $\frac{\mathfrak{p}}{\mathfrak{q}}$, respectively. Hence the result concerning **EXS** of the semigroup

$\{S(t)\}_{t\geq 0}$ applies to $\{S_{\mathrm{opt}}(t)\}_{t\geq 0}$, i.e., $\{S_{\mathrm{opt}}(t)\}_{t\geq 0}$ is **EXS** iff $\left(\frac{\mathfrak{n}}{\mathfrak{m}}\frac{\mathfrak{p}}{\mathfrak{q}}\right)^2 < 1$. However, the last inequality was shown to be true – see (6.9), confirming the general **EXS** result of Theorem 2.1.

Observe that $F_\Lambda|_{D(\mathscr{A}_{\mathrm{opt}})} = \tau|_{D(\mathscr{A}_{\mathrm{opt}})}$, where τ is the operator of boundary control given by (6.2). This fact is basic for establishing the structure of optimal control closed–loop system depicted in Figure 6.1,

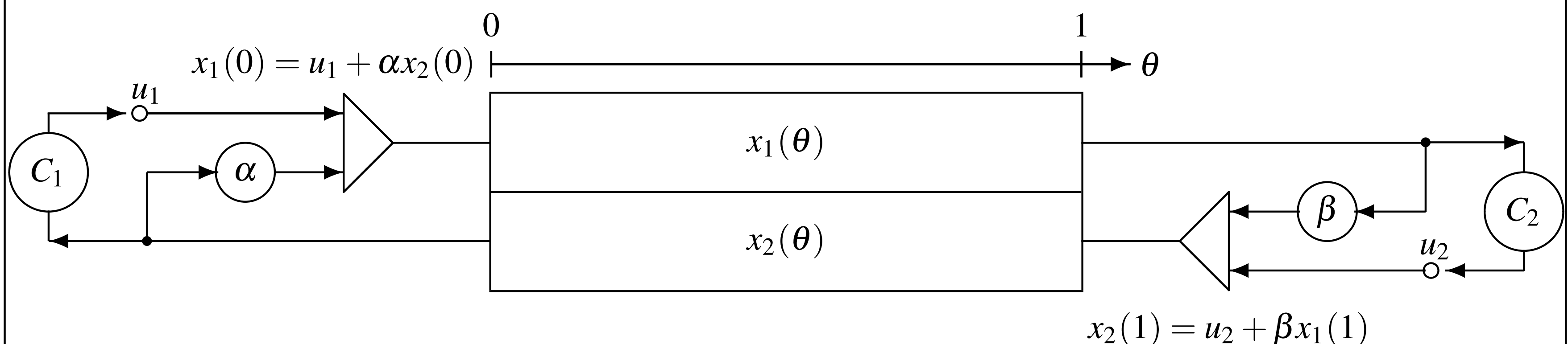

Figure 6.1: Open/closed–loop control system for the Chapelon–Xu example; $C_1 := \frac{\mathfrak{n}}{\mathfrak{m}} - \alpha$, $C_2 := \frac{\mathfrak{p}}{\mathfrak{q}} - \beta$.

where the external connections realize the optimal feedback control $u = F_\Lambda x = -(R_- + \mathscr{G}_\Lambda \mathscr{D})^{-1}\mathscr{G}_\Lambda x$.

7 LQ–problem of Żołopa and Grabowski

7.1 The SISO case

In (Żołopa and Grabowski, 2008) the LQ–problem has been formulated for the dynamical system modelling propagation of pollutants in a river. In this section we solve the standard LQ–problem for a controllable part of this model arising from a general one (Żołopa and Grabowski, 2008, p. 174) by extracting its second component. This is possible as the second component is affected by the first component but not conversely and the control does not excite the first component, which therefore remains uncontrolled.

Suppose also that we consider the SISO case, i.e., the case of a one point control (one aerator) located at $\theta = \eta > 0$ and one output (one sensor measuring dissolved oxygen) located at $\theta = \gamma > \eta$ as depicted in Figure 7.1.

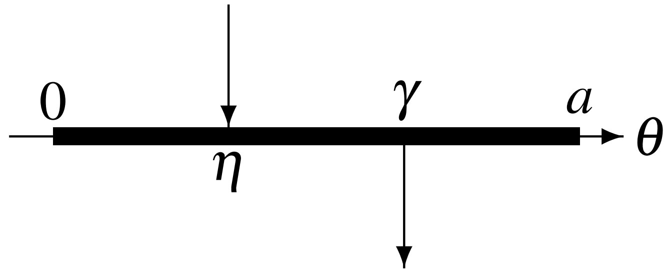

Figure 7.1: Configuration of measurement and control in the SISO case.

The state space is $\mathrm{H} = \mathrm{L}^2(0,a)$, $a > 0$, whilst $\mathrm{U} = \mathrm{Y} = \mathbb{R}$. The state operator is

$$\mathscr{A}x = -vx' - K_2 x, \qquad D(\mathscr{A}) = \mathrm{W}_0^{1,2}(0,a), \qquad K_2 > 0$$

and it generates an **EXS** or even decaying in a finite–time semigroup on H. The observation functional is given by

$$\mathscr{C}x = x(\gamma), \qquad D(\mathscr{C}) = \{x \in \mathrm{H} :\ x \text{ is continuous at } \theta = \gamma\} \ .$$

Finally, the factor control vector $d \in \mathrm{H}$ takes the form

$$d(\theta) = -\frac{1}{v} e^{-\frac{K_2}{v}(\theta-\eta)} \mathbb{1}(\theta - \eta) = -\frac{1}{v} e^{-\frac{K_2}{v}(\theta-\eta)} \chi_{[\eta,a]}(\theta), \quad \theta \in [0,a] \ .$$

$\mathscr{C}$ is admissible as the operator $\mathscr{H}_\Psi \in \mathbf{L}(\mathrm{H})$, $\mathscr{H}_\Psi = \mathscr{H}_\Psi^* \geq 0$, defined as

$$(\mathscr{H}_\Psi x)(\theta) := \frac{1}{v} e^{-\frac{2K_2}{v}(\gamma-\theta)} \chi_{[0,\gamma]}(\theta) x(\theta), \quad x \in \mathrm{H} \ , \tag{7.1}$$

is the *observability gramian*:

$$\langle \mathscr{A} x, \mathscr{H}_\Psi x \rangle_\mathrm{H} + \langle x, \mathscr{H}_\Psi \mathscr{A} x \rangle_\mathrm{H} = -x^2(\gamma) = -|\mathscr{C} x|^2, \qquad x \in D(\mathscr{A}) \ .$$

Next, d is an admissible factor control vector. Indeed, by duality, its is enough to show that the observation functional $d^* \mathscr{A}^*$ is admissible with respect to the adjoint semigroup. Here

$$\mathscr{A}^* w = v w' - K_2 w, \qquad D(\mathscr{A}^*) = \left\{ w \in \mathrm{W}^{1,2}(0,a) : \ w(a) = 0 \right\} \tag{7.2}$$

and the admissibility of $d^* \mathscr{A}^*$ follows from Lyapunov characterization of admissibility as $\mathscr{H}_\Phi \in \mathbf{L}(\mathrm{H})$, $\mathscr{H}_\Phi = \mathscr{H}_\Phi^* \geq 0$

$$(\mathscr{H}_\Phi x)(\theta) := \frac{1}{v} e^{\frac{2K_2}{v}(\eta-\theta)} \chi_{[\eta,a]}(\theta) x(\theta), \quad x \in \mathrm{H} \ ,$$

solves the Lyapunov operator equation ($\mathscr{H}_\Phi$ is the *controllability* *gramian*)

$$\langle \mathscr{A}^* w, \mathscr{H}_\Phi w\rangle_{\mathrm{H}} + \langle w, \mathscr{H}_\Phi \mathscr{A}^* w\rangle_{\mathrm{H}} = -w^2(\eta) = -\left|d^* \mathscr{A}^* w\right|^2, \quad w \in D(\mathscr{A}^*).$$

$\ker \mathscr{H}_\Psi$ and $\ker \mathscr{H}_\Phi$ are both nontrivial, whence the system is neither (infinite–time) approximately observable nor approximately controllable, so the method of spectral factorization is not applicable.

Since

$$\begin{aligned} \left((sI-\mathscr{A})^{-1}x\right)(\theta) &= \tfrac{1}{v}\int_0^\theta e^{-\frac{s+K_2}{v}(\theta-\xi)}x(\xi)\mathrm{d}\xi, \\ \left(\mathscr{A}(sI-\mathscr{A})^{-1}x\right)(\theta) &= -x(\theta)+\tfrac{s}{v}\int_0^\theta e^{-\frac{s+K_2}{v}(\theta-\xi)}x(\xi)\mathrm{d}\xi \end{aligned} \tag{7.3}$$

then with $\delta := \frac{K_2}{v}(\gamma-\eta) > 0$

$$\hat{G}(s) = \mathscr{C}\mathscr{A}(sI-\mathscr{A})^{-1}d = \frac{1}{v}e^{-\frac{s}{v}(\gamma-\eta)}e^{-\delta}, \qquad \hat{G} \in \mathrm{H}^\infty(\mathbb{C}^+) \ .$$

Recalling that the resolvent of $\mathscr{A}$ is Laplace transform of the semigroup generated by $\mathscr{A}$ and substituting $t = \frac{\theta - \xi}{v}$ in (7.3) we get

$$(S(t)X)(\theta) = e^{-K_2 t} \left\{ \begin{array}{ccc} X(\theta - vt) & \text{if} & a \geq \theta \geq vt \\ 0 & \text{if} & \theta < vt \end{array} \right\}, \quad t \geq 0,\ \theta \in [0, a]. \tag{7.4}$$

7.1.1 Direct solution using Theorem 2.1/(I)

In the case of standard LQ–problem $Q = R = 1$, $N = 0$ and thus

$$N_-^* = -\mathscr{C} d = \frac{1}{v} e^{-\delta}, \qquad R_- := 1 + \frac{1}{v^2} e^{-2\delta} \quad .$$

Here the Riccati operator equation (2.7) reduces to

$$\langle \mathscr{A} z, \mathscr{H} z \rangle_{\mathrm{H}} + \langle z, \mathscr{H} \mathscr{A} z \rangle_{\mathrm{H}} + (\mathscr{C} z)^2 = R_-^{-1} \left[\langle \mathscr{A} z, \mathscr{H} d \rangle_{\mathrm{H}} + (\mathscr{C} d)^* \mathscr{C} z \right]^2,$$

where $z \in D(\mathscr{A})$.

As a result of seeking its solution in the form $\mathscr{H} = \mathfrak{a}\mathscr{H}_\Psi + \mathfrak{b}\mathscr{H}_1$, where

$$(\mathscr{H}_1 x)(\theta) := \frac{1}{v} e^{-\frac{2K_2}{v}(\eta-\theta)} \chi_{[\eta,\gamma]}(\theta) x(\theta) = \frac{e^{2\delta}}{v} e^{-\frac{2K_2}{v}(\gamma-\theta)} \chi_{[\eta,\gamma]}(\theta) x(\theta)$$

one obtains

$$\mathfrak{a} = \frac{v^2}{v^2 + e^{-2\delta}}, \qquad \mathfrak{b} = (1-\mathfrak{a})e^{-2\delta} = \frac{e^{-4\delta}}{v^2 + e^{-2\delta}} \ ,$$

which yields

$$(\mathscr{H} x)(\theta) = \frac{1}{v} e^{-\frac{2K_2}{v}(\gamma-\theta)} \left\{ \begin{array}{lll} \mathfrak{a} & \text{on} & [0,\eta) \\ 1 & \text{on} & [\eta,\gamma] \\ 0 & \text{on} & (\gamma,a] \end{array} \right\} x(\theta) \ , \tag{7.5}$$

$$\mathscr{G} z := [\langle \mathscr{A} z, -\mathscr{H} d \rangle_{\mathrm{H}} - (\mathscr{C} d)^* \mathscr{C} z] = \frac{e^{-2\delta}}{v}\, z(\eta), \qquad z \in D(\mathscr{A}) \ .$$

If $z \in \mathrm{Reg}[0,a]$ – the space of *regulated functions*, i.e., functions having one–sided (finite) limits at every point $\theta \in [0,a]$ then, by the Lebesgue dominated convergence theorem:

$$\lim_{s\to\infty, s\in\mathbb{R}} s\mathscr{G}(sI-\mathscr{A})^{-1}z = \frac{e^{-2\delta}}{v^2}\lim_{s\to\infty, s\in\mathbb{R}} s\int_0^\eta e^{-\frac{s+K_2}{v}(\eta-\xi)}z(\xi)\mathrm{d}\xi =$$

$$= \tfrac{e^{-2\delta}}{v}\lim_{s\to\infty, s\in\mathbb{R}} \frac{s}{s+K_2}\int_0^\infty e^{-t}z\left(\eta-\frac{v}{s+K_2}t\right)\mathbb{1}\left(\frac{s+K_2}{v}\eta-t\right)\mathrm{d}t =$$

$$= \frac{e^{-2\delta}}{v}\, z(\eta-) \Longrightarrow \quad \mathscr{G}_\Lambda z := \frac{e^{-2\delta}}{v}\, z(\eta-), \qquad D(\mathscr{G}_\Lambda) = \mathrm{Reg}[0,a] \ .$$

Hence, $d\in D(\mathscr{G}_\Lambda)$ and (2.11) gives

$$u = \frac{-ve^{-2\delta}}{v^2+e^{-2\delta}}\, z(\eta-), \qquad z\in D(\mathscr{G}_\Lambda) = \mathrm{Reg}[0,a] \ .$$

Consequently, the closed–loop operator reads as

$$\mathscr{A}_{\mathrm{opt}}x = -vz' - K_2 z, \quad z(\theta) := x(\theta) + \tfrac{e^{-2\delta}}{v^2+e^{-2\delta}}x(\eta-)e^{-\frac{K_2}{v}(\theta-\eta)}\chi_{[\eta,a]} \ ;$$

$$D(\mathscr{A}_{\mathrm{opt}}) = \left\{x\in \mathrm{H}:\ z\in \mathrm{W}_0^{1,2}[0,a],\ x(\eta+) = \tfrac{v^2}{v^2+e^{-2\delta}}x(\eta-)\right\} \ ,$$

whence (on θ–intervals $[0,\eta]$, $[\eta,a]$ there holds $\mathscr{A}_{\rm opt}x = -vx' - k_2x$)

$$x \in D(\mathscr{A}_{\rm opt}) \Longrightarrow \quad \langle x, \mathscr{A}_{\rm opt}x\rangle_{\rm H} + \langle \mathscr{A}_{\rm opt}x, x\rangle_{\rm H} = \langle x, \mathscr{A} z\rangle_{\rm H} + \langle \mathscr{A} z, x\rangle_{\rm H} =$$

$$= -vx^2(\eta-) - vx^2(a) + vx^2(\eta+) - 2K_2\, \|x\|_{\rm H}^2 =$$

$$= -\tfrac{vc^2(c^2+2)}{(1+c^2)^2}x^2(\eta-) - vx^2(a) - 2K_2\, \|x\|_{\rm H}^2,\ c = \tfrac{1}{v}e^{-\delta},$$

and

$$x(\theta) = \left\{ \begin{array}{rl} \frac{1}{v}\int_0^\theta e^{-\frac{(K_2+\lambda)(\theta-\xi)}{v}}X(\xi)\mathrm{d}\xi & \text{if} \quad 0 \le \theta < \eta \\ \blacklozenge + \frac{1}{v}\int_\eta^\theta e^{-\frac{(K_2+\lambda)(\theta-\xi)}{v}}X(\xi)\mathrm{d}\xi & \text{if} \quad \eta \le \theta \le a \end{array} \right\}, \quad (7.6)$$

where

$$\blacklozenge := \frac{1}{v(1+c^2)}\int_0^\eta e^{-\frac{(K_2+\lambda)(\theta-\xi)}{v}}X(\xi)\mathrm{d}\xi \ ,$$

solves the resolvent equation $\lambda x - \mathscr{A}_{\rm opt}x = X$. By the Lummer–Phillips theorem, $\mathscr{A}_{\text{opt}}$ generates an **EXS** semigroup on H. Moreover, since

$$x \in D(\mathscr{A}_{\text{opt}}) \Longrightarrow \langle x, \mathscr{A}_{\text{opt}} x \rangle_{\text{H}} + \langle \mathscr{A}_{\text{opt}} x, x \rangle_{\text{H}} \leq -\frac{vc^2(c^2+2)}{(1+c^2)^2} x^2(\eta-)$$

then, by Lyapunov characterization of admissibility, the functional $x \longmapsto x(\eta-)$ is admissible with respect to the semigroup generated by $\mathscr{A}_{\text{opt}}$, which confirms that the control is in $\text{L}^2(0,\infty)$, so its is optimal. Actually, we have more. Since (7.6) defines the resolvent of $\mathscr{A}_{\text{opt}}$ then, substituting $t = \frac{\theta-\xi}{v}$ in (7.6) and applying the Laplace transformation, we obtain

$$\left(S_{\text{opt}}(t)X\right)(\theta) = e^{-K_2 t} \left\{ \begin{array}{lll} X(\theta - vt) & \text{if} & 0 \leq t \leq \frac{\theta}{v}, 0 \leq \theta < \eta \\ X(\theta - vt) & \text{if} & 0 \leq t \leq \frac{\theta-\eta}{v}, \eta \leq \theta < a \\ \frac{1}{1+c^2} X(\theta - vt) & \text{if} & \frac{\theta-\eta}{v} \leq t \leq \frac{\theta}{v}, \eta \leq \theta \leq a \\ 0 & & \text{elsewhere} \end{array} \right\}$$

from which we deduce that the semigroup $\{S_{\text{opt}}(t)\}_{t \geq 0}$ decays to 0 in a natural finite time a/v. The rate of decaying of $\{S_{\text{opt}}(t)\}_{t\geq 0}$ is for $\theta \geq \eta$ faster than that of $\{S(t)\}_{t\geq 0}$ given by (7.4).

7.1.2 Operator–theoretic approach

Since $\mathscr{A}$ can be represented as $\mathscr{A} = v\mathscr{R}_F - K_2 I$, where $\mathscr{R}_F$, stands for the generator of the semigroup of right–shifts on H then the semigroup generated by $\mathscr{A}$ equals (7.4). Hence the homogeneous part of the output, due to initial condition $x_0 \in \mathrm{H} = \mathrm{L}^2(0,a)$ is

$$y_{\text{hom}}(t) = \left\{ \begin{array}{ccc} e^{-K_2 t} x_0(\gamma - vt) & \text{if} & \left(\frac{a}{v} \geq\right) \frac{\gamma}{v} \geq t \\ 0 & \text{if} & \frac{\gamma}{v} < t \end{array} \right\} = (\Psi x_0)(t) \quad (7.7)$$

for almost all $t \geq 0$, where $\Psi \in \mathbf{L}(\mathrm{H}, \mathrm{L}^2(0,\infty))$ denotes the observability map.

Recall that the system transfer function is $\hat{G}(s) = \dfrac{1}{v} e^{-\delta} e^{-s\frac{\delta}{K_2}}$,

$\hat{G} \in \mathrm{H}^\infty(\mathbb{C}^+)$, whence the nonhomogeneous part of the output, due to a control $u \in \mathrm{L}^2(0,\infty)$, takes the form

$$y_{\text{nonhom}}(t) = \left\{ \begin{array}{ccc} \frac{1}{v} e^{-\delta} u\left(t - \frac{\delta}{K_2}\right) & \text{if} & t \geq \frac{\delta}{K_2} \\ 0 & \text{if} & t < \frac{\delta}{K_2} \end{array} \right\}, \quad \text{for almost all } t \geq 0 \ , \tag{7.8}$$

and therefore the extended input–output map and its adjoint are

$$\mathbb{F} = \frac{1}{v} e^{-\delta} S_{\mathscr{R}}\left(\frac{\delta}{K_2}\right), \qquad \mathbb{F}^* = \frac{1}{v} e^{-\delta} S_{\mathscr{L}}\left(\frac{\delta}{K_2}\right)$$

and $\mathbb{F}, \mathbb{F}^* \in \mathbf{L}(\mathrm{L}^2(0,\infty))$, where $\{S_{\mathscr{R}}(t)\}_{t\geq 0}$ and $\{S_{\mathscr{L}}(t)\}_{t\geq 0}$ stand for the semigroups of right–, respectively, left–shifts on $\mathrm{L}^2(0,\infty)$.

By (2.5), the optimal control (to be more precise its time–domain form) is

$$u = -\left(\mathbb{F}^*\mathbb{F} + I\right)^{-1} \mathbb{F}^* \Psi x_0 \ . \tag{7.9}$$

But

$$\left(\mathbb{F}^*\Psi x_0\right)(t) = \frac{1}{v}e^{-\delta}\left(S_{\mathscr{L}}\left(\frac{\delta}{K_2}\right)\Psi x_0\right)(t) = $$
$$= \frac{1}{v}e^{-2\delta}\left\{\begin{array}{ccc} e^{-K_2 t}x_0(\eta - vt) & \text{if} & t \in [0, \frac{\eta}{v}] \\ 0 & \text{if} & t > \frac{\eta}{v} \end{array}\right\}.$$

Since $S_{\mathscr{L}}(t)S_{\mathscr{R}}(t) = I$, then $\left(\mathbb{F}^*\mathbb{F} + I\right)^{-1} = \frac{v^2}{v^2+e^{-2\delta}}I$ and from (7.9) one gets

$$u(t) = \left\{\begin{array}{cc} -\frac{ve^{-2\delta}}{v^2+e^{-2\delta}}\, e^{-K_2 t}x_0(\eta - vt) & \text{for almost all } t \in [0, \frac{\eta}{v}] \\ 0 & \text{for almost all } t > \frac{\eta}{v} \end{array}\right. \tag{7.10}$$

From (2.6) we get the optimal cost operator

$$\mathscr{H}x_0 = \mathscr{H}_\Psi x_0 - \Psi^*\mathbb{F}\left(\mathbb{F}^*\mathbb{F} + I\right)^{-1}\mathbb{F}^*\Psi x_0 \ .$$

Directly from definition of the adjoint operator we find

$$(\Psi^* f)(\theta) = \frac{1}{v} f\left(\frac{\gamma-\theta}{v}\right) e^{-\frac{K_2(\gamma-\theta)}{v}} \chi_{[0,\gamma]}(\theta), \qquad \theta \in [0,a] \ . \quad (7.11)$$

Since $S_{\mathscr{R}}(t) S_{\mathscr{L}}(t) = \chi_{[t,\infty)} I$ then

$$-\mathbb{F}\left(\mathbb{F}^*\mathbb{F}+I\right)^{-1}\mathbb{F}^* = -\frac{e^{-2\delta}}{v^2+e^{-2\delta}} \chi_{[\frac{\delta}{K_2},\infty)} I \ ,$$

whence

$$-\Psi^*\mathbb{F}\left(\mathbb{F}^*\mathbb{F}+I\right)^{-1}\mathbb{F}^*\Psi x_0 = -\frac{e^{-2\delta}}{v(v^2+e^{-2\delta})} e^{-\frac{K_2(\gamma-\theta)}{v}} \chi_{[0,\eta]}(\theta) x_0(\theta)$$

and finally

$$(\mathscr{H} x_0)(\theta) =$$
$$= \frac{1}{v} e^{-\frac{2K_2}{v}(\gamma-\theta)} \chi_{[0,\gamma]}(\theta) x_0(\theta) - \frac{e^{-2\delta}}{v(v^2+e^{-2\delta})} e^{-\frac{2K_2(\gamma-\theta)}{v}} \chi_{[0,\eta]}(\theta) x_0(\theta) \ .$$

The last formula coincides with (7.5) except for one point $\theta = \eta$.

Now, we show that the feedback realization of the optimal control (7.10) is correct. Indeed, an initial condition x_0 and a control u, not necessarily optimal, give rise to the full state $x(t) = S(t)x_0 + x_{\text{nonhom}}(t)$, $t \geq 0$. Since

$$\begin{aligned}\hat{x}_{\text{nonhom}}(s)(\theta) = &\quad (\mathscr{A}(sI - \mathscr{A})^{-1}d)(\theta)\hat{u}(s) = \frac{1}{v}\, e^{-\frac{s+K_2}{v}(\theta-\eta)}\mathbb{1}(\theta-\eta)\hat{u}(s) \\ = &\quad \frac{1}{v}\, e^{-\frac{K_2}{v}(\theta-\eta)}\chi_{[\eta,a]}(\theta)e^{-\frac{s}{v}(\theta-\eta)}\hat{u}(s)\ ,\end{aligned}$$

then

$$x_{\text{nonhom}}(t) = \left\{\begin{array}{cll} \frac{1}{v}e^{-\frac{K_2}{v}(\theta-\eta)}\chi_{[\eta,a]}(\theta)u\left(t - \frac{\theta-\eta}{v}\right) & \text{if} & t \geq \frac{\theta-\eta}{v} \\ 0 & \text{if} & t < \frac{\theta-\eta}{v} \end{array}\right\} .$$

Thus if $x_0 \in \text{Reg}[0,a]$ then one has $S(t)x_0 \in \text{Reg}[0,a]$ for every $t \geq 0$ and the optimal feedback controller equation yields

$$u(t) = -\frac{ve^{-2\delta}}{v^2+e^{-2\delta}} \lim_{\theta\to\eta-} x(t)(\theta) = -\frac{ve^{-2\delta}}{v^2+e^{-2\delta}} \lim_{\theta\to\eta-} (S(t)x_0)(\theta) =$$

$$= -\frac{ve^{-2\delta}}{v^2+e^{-2\delta}} \left\{ \begin{array}{ccc} e^{-K_2 t} x_0(\eta - vt-) & \text{if} & 0 \le t < \frac{\eta}{v} \\ 0 & \text{if} & t \ge \frac{\eta}{v} \end{array} \right\},$$

what agrees with (7.10).

7.2 The TISO case

Consider the TISO case, i.e., the case of two point controls (two aerators) located at $\theta = \eta_1 > 0$, $\theta = \eta_2 > \eta_1$ and one output (one sensor measuring dissolved oxygen) located at $\theta = \gamma > \eta_2$ as depicted in Figure 7.2; therefore still we have $\mathrm{Y} = \mathbb{R}$ but now $\mathrm{U} = \mathbb{R}^2$. The optimal controller can be found by mixing the results of Sections 7.1.1 and 7.1.2

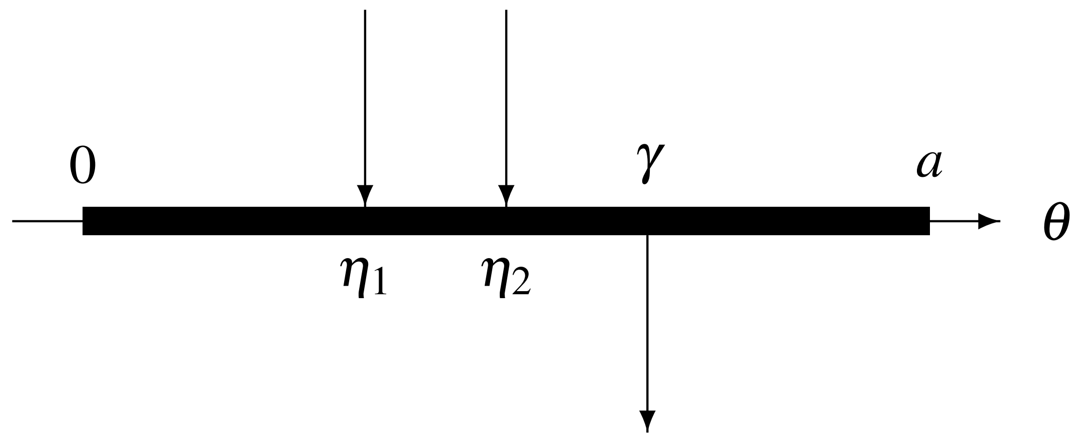

Figure 7.2: Configuration of measurement and controls in the TISO case.

The factor control operator modifies as follows

$$\mathscr{D} = \begin{bmatrix} d_1 & d_2 \end{bmatrix}, \quad d_i(\theta) = -\frac{1}{v} e^{-\frac{K_2}{v}(\theta - \eta_i)} \chi_{[\eta_i, a]}(\theta), \; \theta \in [0,a], \; i = 1,2$$

and consequently $\mathscr{D}^* = \begin{bmatrix} d_1^* \\ d_2^* \end{bmatrix}$. Similarly to the SISO case,

$\mathscr{H}_\Phi \in \mathbf{L}(\mathrm{H})$, $\mathscr{H}_\Phi = \mathscr{H}_\Phi^* \geq 0$,

$$(\mathscr{H}_\Phi x)(\theta) := \frac{1}{v} \left[e^{\frac{2K_2}{v}(\eta_1 - \theta)} \chi_{[\eta_1, a]}(\theta) + e^{\frac{2K_2}{v}(\eta_2 - \theta)} \chi_{[\eta_2, a]}(\theta) \right] x(\theta), \quad x \in \mathrm{H},$$

solves the Lyapunov operator equation ($\mathscr{H}_\Phi$ is the *controllability gramian*)

$$\langle \mathscr{A}^* w, \mathscr{H}_\Phi w \rangle_{\mathrm{H}} + \langle w, \mathscr{H}_\Phi \mathscr{A}^* w \rangle_{\mathrm{H}} =$$
$$= -w^2(\eta_1) - w^2(\eta_2) = -\|\mathscr{D}^* \mathscr{A}^* w\|^2_{\mathrm{U}}, \qquad w \in D(\mathscr{A}^*) \;\; ,$$

whence $\mathscr{D}$ is admissibility, though the system is not (infinite-time) approximately controllable as $\ker \mathscr{H}_\Phi \neq \{0\}$.

The observability map Ψ is still given by (7.7) whilst the input–output operator has components somewhat similar to the SISO case:

$$\mathbb{F}u = \begin{bmatrix} \mathbb{F}_1 & \mathbb{F}_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \; \mathbb{F}_i = c_i S_{\mathscr{R}} \left(\frac{\delta_i}{K_2} \right), \quad i = 1,2 \;\; ,$$

$$\mathbb{F}^* y = \begin{bmatrix} \mathbb{F}_1^* \\ \mathbb{F}_2^* \end{bmatrix} y, \qquad \mathbb{F}_i^* = c_i S_{\mathscr{L}} \left(\frac{\delta_i}{K_2} \right), \quad i = 1,2 \;\; ,$$

where $c_i := \frac{1}{v}e^{-\delta_i}$, $\delta_i := \frac{K_2(\gamma-\eta_i)}{v}$ ($\eta_1 < \eta_2 \iff \delta_1 > \delta_2$).

Using the operator attempt of Section 7.1.2 we find the minimal cost operator, $\mathscr{H} = \mathscr{H}_\Psi - \Psi^*\mathbb{F}(I+\mathbb{F}^*\mathbb{F})^{-1}\mathbb{F}^*\Psi$,

$$(\mathscr{H}x_0)(\theta) = \left\{ \begin{array}{ccr} \frac{1}{1+c_1^2+c_2^2} & \text{on} & [0,\eta_1] \\ \frac{1}{1+c_2^2} & \text{on} & (\eta_1,\eta_2] \\ 1 & \text{on} & (\eta_2,\gamma] \\ 0 & \text{on} & (\gamma,a] \end{array} \right\} x_0(\theta) \ .$$

Now we switch to the attempt of Section 7.1.1. From (2.8) we get

$$\mathscr{G}z = -\mathscr{D}^*\mathscr{H}\mathscr{A}z + N_-^*\mathscr{C}z = \left[\begin{array}{c} \frac{vc_1^2}{1+c_2^2}\, z(\eta_1) + \frac{vc_1c_2^3}{1+c_2^2}\, z(\eta_2) \\ vc_2^2\, z(\eta_2) \end{array} \right] \qquad (7.12)$$

for $z \in D(\mathscr{A}) = \mathrm{W}_0^{1,2}(0,a)$. It is enough to determine an extension of $\mathscr{G}$ onto $\mathrm{Reg}[0,a]$. Let $z \in \mathrm{Reg}[0,a]$. Then, by (7.3), (7.12) and the Lebesgue dominated convergence theorem:

$$\lim_{s\to\infty, s\in\mathbb{R}} s\mathscr{G}(sI-\mathscr{A})^{-1}z = \begin{bmatrix} \frac{vc_1^2}{1+c_2^2}z(\eta_1-) + \frac{vc_1c_2^3}{1+c_2^2}z(\eta_2-) \\ vc_2^2 z(\eta_2-) \end{bmatrix} := \mathscr{G}_\Lambda z \ , \tag{7.13}$$

whence, by (2.11),

$$u = -(R_- + \mathscr{G}_\Lambda \mathscr{D})^{-1}\mathscr{G}_\Lambda x = -\begin{bmatrix} \frac{vc_1^2}{1+c_1^2+c_2^2}\ x(\eta_1-) \\ \frac{vc_2^2}{1+c_2^2}\ x(\eta_2-) \end{bmatrix} \ . \tag{7.14}$$

Remark 7.1 (Limit passage from TISO to SISO case)**.** Taking $\eta_1 \to -\infty$, which implies $\delta_1 \to \infty$, $c_1 \to 0$ and fixing $\eta_2 = \eta$, $\delta_2 = \delta$, $c_2 = c = \frac{1}{v}e^{-\delta}$ we get $u_1 = 0$ and $u_2 = u$, where u is the optimal control or controller in the SISO case.

The controller (7.14) has astonishingly simple realization, depicted in Figure 7.3.

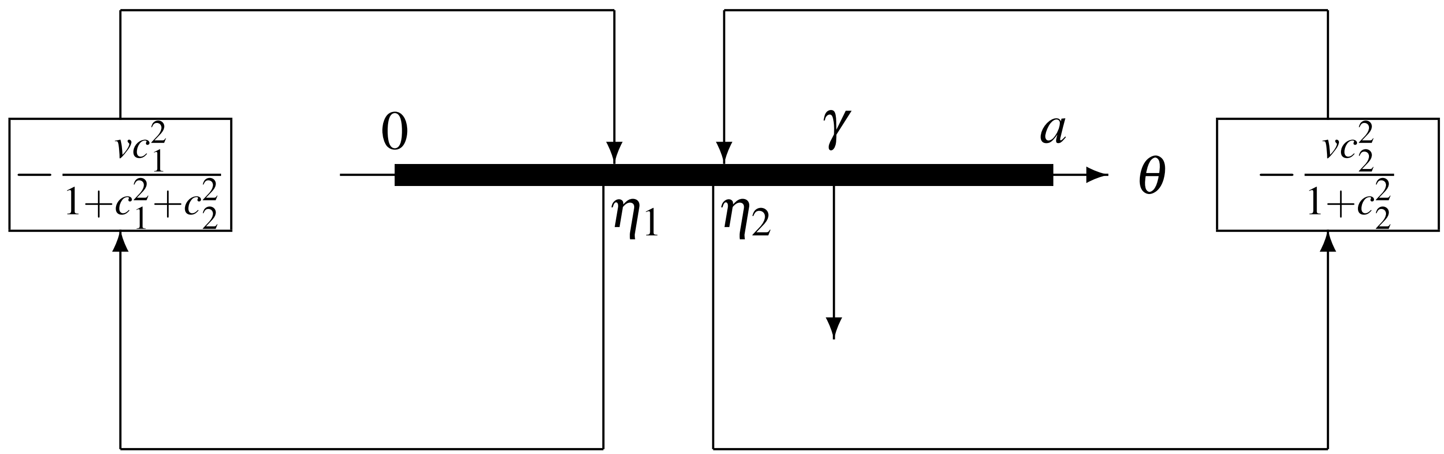

Figure 7.3: Optimal feedback controller realization in the TISO case.

Let $x \in \mathrm{Reg}[0,a]$ and u be the optimal controller (7.14). A discussion of conditions ensuring that $z = x + \mathscr{D}u \in D(\mathscr{A})$ (in particular, $z = x + \mathscr{D}u$ must be continuous on [0,a]), leads to the following form of the closed–loop state operator:

$$\mathscr{A}_{\mathrm{opt}}x = -vz' - K_2 z \ , \quad z(\theta) := x(\theta) + \frac{c_1^2}{1+c_1^2+c_2^2}x(\eta_1-)e^{-\frac{K_2}{v}(\theta-\eta_1)}\chi_{[\eta_1,a]} + \\ + \frac{c_2^2}{1+c_2^2}x(\eta_2-)e^{-\frac{K_2}{v}(\theta-\eta_2)}\chi_{[\eta_2,a]}; \quad D(\mathscr{A}_{\mathrm{opt}}) =$$

$$\left\{x \in \mathrm{H} : z \in \mathrm{W}_0^{1,2}[0,a], x(\eta_1+) = \tfrac{1+c_2^2}{1+c_1^2+c_2^2} x(\eta_1-), x(\eta_2+) = \tfrac{1}{1+c_2^2} x(\eta_2-)\right\}.$$

Hence (on θ-intervals $[0,\eta_1]$, $[\eta_1,\eta_2]$, $[\eta_2,a]$ there holds $\mathscr{A}_{\text{opt}} x = -vx' - k_2 x$)

$$x \in D(\mathscr{A}_{\text{opt}}) \Longrightarrow \langle x, \mathscr{A}_{\text{opt}} x\rangle_{\mathrm{H}} + \langle \mathscr{A}_{\text{opt}} x, x\rangle_{\mathrm{H}} = \langle x, \mathscr{A} z\rangle_{\mathrm{H}} + \langle \mathscr{A} z, x\rangle_{\mathrm{H}} =$$

$$= -vx^2(\eta_1-) - vx^2(\eta_2-) - vx^2(a) + vx^2(\eta_1+) + vx^2(\eta_2+) - 2K_2 \|x\|_{\mathrm{H}}^2 =$$

$$= -\tfrac{vc_2^2(c_2^2+2)}{(1+c_2^2)^2} x^2(\eta_1-) - \tfrac{vc_1^2(c_1^2+2c_2^2+2)}{(1+c_1^2+c_2^2)^2} x^2(\eta_2-) - vx^2(a) - 2K_2 \|x\|_{\mathrm{H}}^2 \ ,$$

and

$$x(\theta) = \left\{ \begin{array}{ccc} \dfrac{1}{v}\displaystyle\int_0^\theta e^{-\frac{(K_2+\lambda)(\theta-\xi)}{v}} X(\xi) \mathrm{d}\xi & \text{if} & 0 \le \theta < \eta_1 \\ \spadesuit & \text{if} & \eta_1 \le \theta < \eta_2 \\ \clubsuit & \text{if} & \eta_2 \le \theta \le a \end{array} \right\} , \quad (7.15)$$

where

$$\spadesuit = \frac{1+c_2^2}{v(1+c_1^2+c_2^2)}\int_0^{\eta_1} e^{-\frac{(K_2+\lambda)(\theta-\xi)}{v}} X(\xi)\mathrm{d}\xi + \frac{1}{v}\int_{\eta_1}^{\theta} e^{-\frac{(K_2+\lambda)(\theta-\xi)}{v}} X(\xi)\mathrm{d}\xi \quad ,$$

$$\clubsuit = \quad \frac{1}{v(1+c_1^2+c_2^2)}\int_0^{\eta_1} e^{-\frac{(K_2+\lambda)(\theta-\xi)}{v}} X(\xi)\mathrm{d}\xi +$$

$$\frac{1}{v(1+c_2^2)}\int_{\eta_1}^{\eta_2} e^{-\frac{(K_2+\lambda)(\theta-\xi)}{v}} X(\xi)\mathrm{d}\xi + \frac{1}{v}\int_{\eta_2}^{\theta} e^{-\frac{(K_2+\lambda)(\theta-\xi)}{v}} X(\xi)\mathrm{d}\xi \quad ,$$

solves the resolvent equation $\lambda x - \mathscr{A}_{\mathrm{opt}} x = X$ which, by the Lummer–Phillips theorem, implies that $\mathscr{A}_{\mathrm{opt}}$ generates an **EXS** semigroup on H. Moreover, since

$$x \in D(\mathscr{A}_{\mathrm{opt}}) \Longrightarrow \langle x, \mathscr{A}_{\mathrm{opt}} x\rangle_{\mathrm{H}} + \langle \mathscr{A}_{\mathrm{opt}} x, x\rangle_{\mathrm{H}} \leq$$

$$\leq -\frac{vc_2^2(c_2^2+2)}{(1+c_2^2)^2} x^2(\eta_1-) - \frac{vc_1^2(c_1^2+2c_2^2+2)}{(1+c_1^2+c_2^2)^2} x^2(\eta_2-)$$

then, by Lyapunov characterization of admissibility, the functionals

$x \longmapsto x(\eta_1-)$, $x \longmapsto x(\eta_2-)$ are admissible with respect to the semigroup genarated by $\mathscr{A}_{\text{opt}}$, which confirms that the optimal control is in $\mathrm{L}^2(0,\infty;\mathbb{R}^2)$. Now (7.15) defines the resolvent of $\mathscr{A}_{\text{opt}}$. Thus substituting $t = \frac{\theta-\xi}{v}$ in (7.15) and applying the definition of Laplace transformation, we obtain

$$\left(S_{\text{opt}}(t)X\right)(\theta) = e^{-K_2 t} \begin{cases} X(\theta - vt) & \text{if} \quad 0 \le t \le \frac{\theta}{v}, \quad 0 \le \theta < \eta_1 \\ \frac{1+c_2^2}{1+c_1^2+c_2^2} X(\theta - vt) & \text{if} \quad \frac{\theta-\eta_1}{v} \le t \le \frac{\theta}{v}, \quad \eta_1 \le \theta < \eta_2 \\ X(\theta - vt) & \text{if} \quad 0 \le t \le \frac{\theta-\eta_1}{v}, \quad \eta_1 \le \theta < \eta_2 \\ \frac{1}{(1+c_2^2)(1+c_1^2+c_2^2)} X(\theta - vt) & \text{if} \quad \frac{\theta-\eta_1}{v} \le t \le \frac{\theta}{v}, \quad \eta_2 \le \theta \le a \\ \frac{1}{1+c_2^2} X(\theta - vt) & \text{if} \quad \frac{\theta-\eta_2}{v} \le t \le \frac{\theta-\eta_1}{v}, \quad \eta_2 \le \theta \le a \\ X(\theta - vt) & \text{if} \quad 0 \le t \le \frac{\theta-\eta_2}{v}, \quad \eta_2 \le \theta \le a \\ 0 & \text{elsewhere} \end{cases}$$

from which we deduce that actually the semigroup $\{S_{\text{opt}}(t)\}_{t\ge 0}$ decays to 0 is a natural finite time a/v. The rate of decaying of $\{S_{\text{opt}}(t)\}_{t\ge 0}$ is for $\theta \ge \eta$ faster than that of $\{S(t)\}_{t\ge 0}$ given by (7.4).

References

ARENDT W., BATTY C.J.K., HIEBER M., NEUBRANDER F., *Vector–Valued Laplace Transforms and Cauchy Problems*. Basel: Birkhäuser, 2001.

CHAPELON A., XU C.-Z., *Boundary control of class of hyperbolic systems*, EUROPEAN JOURNAL of CONTROL, **9** (2003), pp. 589-604.

MIKKOLA K.M., STAFFANS O., *Riccati equations and optimal control for infinite–dimensional linear systems*, Proceedings of Sixteenth International Conference on Mathematical Networks and Systems (MTNS 2004), Katholike Universiteit Leuven, Belgium July 5-9, 2004. Available from `http://users.abo.fi/staffans/publ.html`

PAZY A., *Semigroups of Linear Operators and Applications to Partial Differential Equations*, New York: Springer Verlag, 1983.

STAFFANS O.J., *Quadratic optimal control of stable well–posed systems*, TRANSACTIONS of the AMERICAN MATHEMATICAL SOCIETY, **349** (1997), pp. 3679 - 3715.

STAFFANS O.J., *Quadratic optimal control through spectral and coprime factorization*, EUROPEAN JOURNAL of CONTROL, **5** (1999), pp. 167-179.

WEISS G., WEISS M., *Optimal control of stable weakly regular linear systems*, MATHEMATICS of CONTROL, SIGNALS and SYSTEMS, **10** (1997), pp. 287 - 330.

ŻOŁOPA E., GRABOWSKI P., *Abstract dynamical model of propagation of pollutants in a river*. AUTOMATYKA (AGH), **12** (2008), pp. 169 - 182. Available from: `http://journals.bg.agh.edu.pl/AUTOMATYKA/2008-02/Auto01.pdf`