Algebraic Properties of Riccati equations

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- **The noncommutative case.**
- **Conclusions.**

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Matrix Riccati equation result

Let $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{m\times n}$, $C \in \mathbb{C}^{n\times p}$. If (A, B) is stabilizable and (A, C) is detectable, then $\exists$ a unique stabilizing solution $P \in \mathbb{C}^{n\times n}$ of

$$A^*P + PA - PBB^*P + C^*C = 0.$$

Stabilizing solution: $P = P^* \geq 0$ and $A - BB^*P$ is stable.

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Fundamental question

Let $\mathfrak{A}$ be a complex Banach algebra, with an involution $\cdot^\dagger$. When will the following LQR Riccati equation have a unique exponentially stabilizing solution $P \in \mathfrak{A}^{n\times n}$?

$$A^\dagger P + PA - PBB^\dagger P + C^\dagger C = 0.$$

Stabilizing solution: $P = P^\dagger$ and $A - BB^\dagger P$ is exponentially stable, i.e., the semigroup e^{At} on $\mathfrak{A}^{n\times n}$ is exponentially stable $\iff$

$$\sup\{\operatorname{Re} \lambda : \lambda \in \sigma(A)\} < 0.$$

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LQR Riccati equation

Suppose that Z is a Hilbert space and $A,\ B,\ C \in \mathcal{L}(Z)$. If (A, B) is exponentially stabilizable and (A, C) is exponentially detectable, then there exists a unique nonnegative solution $P \in \mathcal{L}(Z)$, $P = P^* \geq 0$ of the LQR control Riccati equation

$$A^*P + PA - PBB^*P + C^*C = 0.$$

There also exists a unique nonnegative solution $Q \in \mathcal{L}(Z)$, $Q = Q^* \geq 0$ of the LQR filter Riccati equation

$$AQ + QA^* - QC^*CQ + BB^* = 0.$$

Moreover, $A - BB^*P$ and $A - QCC^*$ generate exponentially stable semigroups.

Complex Banach algebra $\mathfrak{A}$ with an involution

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A complex Banach space with a *multiplication operation*:

- $a, b \in \mathfrak{A} \implies ab \in \mathfrak{A}$
- $a(bc) = (ab)c$,
- $\|ab\| \leq \|a\|\|b\|$.

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Involution is a map from $\mathfrak{A}$ to itself: $a \to a^\dagger$ with the properties:

- $(a+b)^\dagger = a^\dagger + b^\dagger$,
- $(\alpha b)^\dagger = \overline{\alpha} b^\dagger$,
- $(ab)^\dagger = a^\dagger b^\dagger$,
- $(a^\dagger)^\dagger = a$.

Commutative complex Banach algebra:

$$ab = ba \quad \text{for all} \quad a, b \in \mathfrak{A}.$$

Example (Even-Weighted Wiener algebra $W_\alpha(\mathbb{T})$)

Let $\alpha = (\alpha_n)_{n\in\mathbb{Z}}$ be any sequence of positive real numbers satisfying

$$\alpha_{n+m} \leq \alpha_n \alpha_m, \quad \alpha_{-n} = \alpha_n \quad \text{(even)}.$$

$$W_\alpha(\mathbb{T}) := \Big\{ f : f(z) = \sum_{n\in\mathbb{Z}} f_n z^n \,, z \in \mathbb{T} = \text{ unit circle} \\ \text{and } \|f\|_{W_\alpha(\mathbb{T})} := \sum_{n\in\mathbb{Z}} \alpha_n |f_n| < \infty \Big\},$$

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$$f^\dagger(z) := \overline{f(\overline{z})} = \sum_{n\in\mathbb{Z}} \overline{f_n} z^n,$$
$$\text{and } f^\sim(z) := f(\overline{1/z})^* = \sum_{n\in\mathbb{Z}} \overline{f_n} z^{-n}.$$

Motivation: Spatially Invariant Systems

Consider the subalgebra of bounded *convolution* operators $T : \ell_2 \to \ell_2$ given by

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Spatially invariant systems: $\Sigma(A, B, C)$ with state space $\ell_2(\mathbb{Z}; \mathbb{C}^n)$. A, B, C are matrices whose entries are convolution operators and $A, B, C \in \mathcal{L}(\ell_2(\mathbb{Z}; \mathbb{C}^n))$, a Banach algebra with the involution the adjoint operation $\cdot^*$.

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Under the Fourier transform $\mathfrak{F} : \ell_2(\mathbb{Z}; \mathbb{C}^n) \to \mathbf{L}_2(\mathbb{T}; \mathbb{C}^n)$. So the study of spatially invariant systems $\Sigma(A, B, C)$ is transformed to the study of the isometrically isomorphic systems $\Sigma(\mathfrak{F}A\mathfrak{F}^{-1}, \mathfrak{F}B\mathfrak{F}^{-1}, \mathfrak{F}C\mathfrak{F}^{-1}) := \Sigma(\hat{A}, \hat{B}, \hat{C})$ on $\mathbf{L}_2(\mathbb{T}; \mathbb{C}^n)$. Since $\hat{A}, \hat{B}, \hat{C}$ are multiplication operators on $\mathbf{L}_\infty(\mathbb{T}; \mathbb{C}^n)$ they are much easier to handle mathematically.

Example

Take $\hat{A} = 0$, $\hat{B}(z) = 10 - z - 1/z$, $\hat{C} = 1$. The LQR Riccati equation on $\mathbf{L}_2(\mathbb{T})$ is

$$\hat{A}(z)^*\hat{P}(z) + \hat{P}(z)\hat{A}(z) - \hat{P}(z)\hat{B}(z)\hat{B}(z)^*\hat{P}(z) + \hat{C}(z)^*\hat{C}(z) = 0.$$

This can be solved algebraically for each $z \in \mathbb{T}$ to obtain the unique positive solution

$$\begin{aligned} \hat{P}(z) &= \frac{1}{10 - z - 1/z} \\ &= \frac{1}{4\sqrt{6}} \sum_{k \in \mathbb{Z}} \delta^{-|k|} z^k, \end{aligned}$$

where $\delta = 5 + \sqrt{24}$.

Control of Platoon-type spatially invariant systems

Discrete spatially invariant operators are in $\mathcal{L}(\ell_2(\mathbb{Z};\mathbb{C}^n)) \simeq \mathcal{L}(\mathbf{L}_2(\mathbb{T};\mathbb{C}^n)) = \mathbf{L}_\infty(\mathbb{T};\mathbb{C}^{n\times n})$.

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Existence of Riccati solutions

If $(\hat{A}(z), \hat{B}(z), \hat{C}(z))$ is stabilizable and detectable for all $z \in \mathbb{T}$ the Riccati equation has a unique s.a stabilizing solution.

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$\sum_{r\in\mathbb{T}} \alpha_r p_r < \infty$ for some (α_r) or $\hat{P}$ must be in a Wiener algebra $W_\alpha(\mathbb{T}) \subset \mathbf{L}_\infty(\mathbb{T})$.

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Moreover, the larger the weights α_r, the better the approximation will be; for example, an exponential weight $\alpha_r = e^{|r|}$ would be perfect.

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Every commutative Banach algebra is isomorphic to an algebra of continuous functions on its maximal ideal space $M(\mathfrak{A})$ (a compact Hausdorff space, equipped with the weak * topology). The *Gelfand transform* is a map $\check{\cdot} : \mathfrak{A} \to \mathbb{C}$ given by

$$\check{a}(\varphi) = \varphi(a), \quad \forall \varphi \in M(\mathfrak{A}), \quad \forall a \in \mathfrak{A}.$$

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Example (The commutative Banach algebra $W_\alpha(\mathbb{T})$)

This has the maximal ideal space which is isomorphic to the annulus around $\mathbb{T}$:

$$\mathbb{A}(\rho) = \{z \in \mathbb{C} : 1/\rho \leq |z| \leq \rho\}, \quad \rho = \inf_{n>0} \sqrt[n]{\alpha_n} = \lim_{n\to\infty} \sqrt[n]{\alpha_n}.$$

For $f(z) = \sum_{k\in\mathbb{Z}} f_k z^k$ for $z \in \mathbb{T}$ the Gelfand transform is

$$\check{f}(z) = \sum_{k\in\mathbb{Z}} f_k z^k \quad \text{for } z \in \mathbb{A}(\rho), \quad \text{the whole annulus}.$$

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The idea was to note that the Gelfand transforms $(\check{A}(\varphi), \check{B}(\varphi)), \check{C}(\varphi))$ are just complex matrices and so consider the isomorphic finite-dimensional Riccati equation for each φ

$$\check{A}(\varphi)^*\check{P}(\varphi)+\check{P}(\varphi)\check{A}(\varphi)-\check{P}(\varphi))\check{B}(\varphi)\check{B}(\varphi)^*\check{P}(\varphi))+\check{C}(\varphi))^*\check{C}(\varphi)) = 0.$$

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Theorem

Suppose that $\mathfrak{A}$ be a commutative, unital, complex Banach algebra, with an involution $\cdot^\dagger$. Denote by $\check{A}$ the Gelfand transform.
Let $A \in \mathfrak{A}^{n\times n}$, $B \in \mathfrak{A}^{n\times m}$, $C \in \mathfrak{A}^{p\times n}$ be such that for all $\varphi \in M(\mathfrak{A})$, $(\check{A}(\varphi), \check{B}(\varphi))$ is controllable and $(\check{A}(\varphi), \check{C}(\varphi))$ is observable. Then there exists a solution $P \in \mathfrak{A}^{n\times n}$ such that

$$A^\dagger P + PA - PBB^\dagger P + C^\dagger C = 0, \tag{1}$$

and $A - BB^\dagger P$ is asymptotically stable.

FALSE: The involution $\cdot^\dagger$ needs to match the complex conjugate.

New result by Amol Sasane (special case of SIAM 2011 paper)

Let $\mathfrak{A}$ be a commutative, unital, complex, semisimple Banach algebra. Suppose that $A \in \mathfrak{A}^{n\times n}$, $B \in \mathfrak{A}^{n\times m}$, $C \in \mathfrak{A}^{p\times n}$ satisfy the following for all $\varphi \in M(\mathfrak{A})$,

(A1) $(\check{A^\dagger})(\varphi) = (\check{A}(\varphi))^*$, $(\check{B^\dagger})(\varphi) = (\check{B}(\varphi))^*$, $(\check{C^\dagger})(\varphi) = (\check{C}(\varphi))^*$.

(A2) $(\check{A}(\varphi), \check{B}(\varphi))$ is stabilizable,

(A3) $(\check{A}(\varphi), \check{C}(\varphi))$ is detectable.

Then there exists a $P \in \mathfrak{A}^{n\times n}$ such that

- $A^\dagger P + PA - PBB^\dagger P + C^\dagger C = 0$,
- $A - BB^\dagger P$ is exponentially stable, and $P^\dagger = P$.

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- $A^\dagger P + PA - PBB^\dagger P + C^\dagger C = 0$,
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Note: condition (A1) on the involution: it needs to match complex conjugation: In general

$f^\dagger(z) := \overline{f(\overline{z})h} \neq f(z)^*$, $\qquad f^\sim(z) := f(\overline{1/z})^* \neq f(z)^*$.

Banach algebras satisfying (A1) = *symmetric Banach algebras*.

Example (Symmetric even-weighted Wiener algebras)

Gelfand-Raikov-Shilov condition on the weights α_n:

$$\rho = \inf_{n>0} \sqrt[n]{\alpha_n} = \lim_{n\to\infty} \sqrt[n]{\alpha_n} = 1.$$

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When the Gelfand-Raikov-Shilov condition is satisfied, the annulus $\mathbb{A}(\rho)$ degenerates to the circle $\mathbb{T}$, and for the Banach algebra $W_\alpha(\mathbb{T})$ the involution $\cdot^\sim$ reduces to

$$f^\sim(z) := f(z)^* \;\; (z \in \mathbb{T}), \text{for} f \in W_\alpha(\mathbb{T}).$$

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With this involution $W_\alpha(\mathbb{T})$ is a *symmetric* Banach algebra.
Thus for matrices A, B, C with entries from $W_\alpha(\mathbb{T})$ their involution is the usual Hermitian adjoint operation: $A^\sim(z) = A(z)^*$ and assumption (A1) is always satisfied.

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They considered LQR control of a general class of spatially distributed systems. These include the platoon type spatially invariant systems whose operators have components in the even-weighted algebras $W_\alpha(\mathbb{T})$. In particular, $W_\tau(\mathbb{T})$ with the exponential weights $\alpha_n = e^{\tau|n|}$.

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Theorem: spatially invariant scalar case

If $A, B, C \in W_\tau(\mathbb{T})$ and the LQR Riccati equation

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has a unique positive definite solution $P \in \mathcal{L}(\mathbf{L}_2(\mathbb{T}))$, then $P \in W_\tau(\mathbb{T})$.

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Counterexample: Ruth Curtain, IEEE 2008.

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Recent result (Curtain 2011)

Suppose that $\mathfrak{A}$ is a unital symmetric Banach algebra and

- $\mathfrak{A}$ is a Banach *-subalgebra of $\mathcal{L}(Z)$, where Z is a Hilbert space;
- $\mathfrak{A}$ has the inverse-closed property:
  $D \in \mathfrak{A}, \quad D^{-1} \in \mathcal{L}(Z) \Longrightarrow D^{-1} \in \mathfrak{A}$;
- $\|A\|_{\mathfrak{A}} \geq M\|A\|_{\mathcal{L}(Z)}$.

If $A, B, C \in \mathfrak{A}$ and (A, B, C) is exponentially stabilizable and detectable wrt Z, then $P \in \mathfrak{A}$, where $P \in \mathcal{L}(Z)$ is the unique nonnegative solution to the control Riccati equations:

$$A^*P + PA - PBB^*P + C^*C = 0.$$

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With recent results by Gröchenig & Leinert (2006) this result covers the spatially distributed LQR problem posed by Motee & Jadbabaie.

Curious

An essential step in the proof uses the following result from Curtain & Opmeer MCSS, 2006:

Let P, Q be the self-adjoint solutions to the control and filter Riccati equations

$$A^*P + PA - PBB^*P + C^*C = 0$$

$$QA + QA^* - QC^*CQ + B^*B = 0.$$

Then the solution to the following Lyapunov equation

$$(A - BB^*P)X + X(A - BB^*P)^* = -BB^*$$

is $X = P(I + PQ)^{-1}$.

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- **The results have direct applications to spatially distributed systems.**
- **Generalization to the algebraic properties of the LQR Riccati equation when A is an unbounded operator on $\mathfrak{A}$ but it generates a semigroup on $\mathfrak{A}$.**

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