Indirect damping for coupled systems

Piermarco Cannarsa

University of Rome “Tor Vergata” (Italy)
http://www.mat.uniroma2.it

CDPS 2011
University of Wuppertal July 18-22, 2011

Outline

Outline

Outline

Outline

Outline

a conservative system

$\Omega \subset \mathbb{R}^n$ bounded the wave equation

$$\begin{aligned} \partial_t^2 v - \Delta v = 0 \qquad & \text{in} \qquad \Omega \times \mathbb{R}\,, \\ v = 0 \qquad & \text{on} \qquad \partial\Omega \times \mathbb{R}\,, \end{aligned}$$

describes a conservative system: the energy of a solution

$$E(u(t)) = \frac{1}{2}\int_\Omega \left(|Du(t,x)|^2 + |\partial_t u(t,x)|^2\right) dx$$

is constant in t

a conservative system

$\Omega \subset \mathbb{R}^n$ bounded the wave equation

$$\begin{aligned} \partial_t^2 v - \Delta v = 0 \qquad & \text{in} \qquad \Omega \times \mathbb{R}\,, \\ v = 0 \qquad & \text{on} \qquad \partial\Omega \times \mathbb{R}\,, \end{aligned}$$

describes a conservative system: the energy of a solution

$$E(u(t)) = \frac{1}{2}\int_\Omega \Big(|Du(t,x)|^2 + |\partial_t u(t,x)|^2 \Big)\, dx$$

is constant in t

a dissipative system

the damped wave equation

$$\begin{aligned} \partial_t^2 u - \Delta u + \partial_t u = 0 \qquad & \text{in} \qquad \Omega \times \mathbb{R}\,, \\ u = 0 \qquad & \text{on} \qquad \partial\Omega \times \mathbb{R}\,, \end{aligned}$$

is exponentially stable as $t \to \infty$

$$E(u(t)) \le E(u(0))e^{c(1-t)} \qquad (c > 0)$$

a dissipative system

the damped wave equation

$$\begin{aligned} \partial_t^2 u - \Delta u + \partial_t u = 0 \qquad & \text{in} \qquad \Omega \times \mathbb{R}\,, \\ u = 0 \qquad & \text{on} \qquad \partial\Omega \times \mathbb{R}\,, \end{aligned}$$

is exponentially stable as $t \to \infty$

$$E(u(t)) \leq E(u(0))e^{c(1-t)} \qquad (c > 0)$$

a weakly coupled system

consider the coupling through zero order terms

$$\begin{cases} \partial_t^2 u - \Delta u + \partial_t u + \alpha v = 0 \\ \partial_t^2 v - \Delta v + \alpha u = 0 \end{cases} \quad \text{in} \quad \Omega \times \mathbb{R}$$

$$u = 0 = v \quad \text{on} \quad \partial\Omega \times \mathbb{R}$$

any kind of stability for $\alpha \neq 0$?

a weakly coupled system

consider the coupling through zero order terms

$$\begin{cases} \partial_t^2 u - \Delta u + \partial_t u + \alpha v = 0 \\ \partial_t^2 v - \Delta v + \alpha u = 0 \end{cases} \quad \text{in} \quad \Omega \times \mathbb{R}$$

$$u = 0 = v \quad \text{on} \quad \partial\Omega \times \mathbb{R}$$

any kind of stability for $\alpha \neq 0$?

lack of exponential stability

recast

$$\begin{cases} \partial_t^2 u - \Delta u + \partial_t u + \alpha v = 0 \\ \partial_t^2 v - \Delta v + \alpha u = 0 \end{cases} \quad \text{in} \quad \Omega \times \mathbb{R}$$

as an evolution equation in $\mathcal{H} = [H_0^1(\Omega) \times L^2(\Omega)]^2$

$$\begin{pmatrix} u \\ u' \\ v \\ v' \end{pmatrix}' = \begin{pmatrix} L_1 & K \\ K & L_2 \end{pmatrix} \begin{pmatrix} u \\ u' \\ v \\ v' \end{pmatrix} =: \mathcal{L} \begin{pmatrix} u \\ u' \\ v \\ v' \end{pmatrix},$$

- L_1, L_2 generators of $\mathcal{C}_0$-semigroups on $H_0^1(\Omega) \times L^2(\Omega)$
- K compact operator in $H_0^1(\Omega) \times L^2(\Omega)$

lack of exponential stability

recast

$$\begin{cases} \partial_t^2 u - \Delta u + \partial_t u + \alpha v = 0 \\ \partial_t^2 v - \Delta v + \alpha u = 0 \end{cases} \quad \text{in} \quad \Omega \times \mathbb{R}$$

as an evolution equation in $\mathcal{H} = [H_0^1(\Omega) \times L^2(\Omega)]^2$

$$\begin{pmatrix} u \\ u' \\ v \\ v' \end{pmatrix}' = \begin{pmatrix} L_1 & K \\ K & L_2 \end{pmatrix} \begin{pmatrix} u \\ u' \\ v \\ v' \end{pmatrix} =: \mathcal{L} \begin{pmatrix} u \\ u' \\ v \\ v' \end{pmatrix},$$

- L_1, L_2 generators of $\mathcal{C}_0$-semigroups on $H_0^1(\Omega) \times L^2(\Omega)$
- K compact operator in $H_0^1(\Omega) \times L^2(\Omega)$

lack of exponential stability

recast

$$\begin{cases} \partial_t^2 u - \Delta u + \partial_t u + \alpha v = 0 \\ \partial_t^2 v - \Delta v + \alpha u = 0 \end{cases} \qquad \text{in} \quad \Omega \times \mathbb{R}$$

as an evolution equation in $\mathcal{H} = [H_0^1(\Omega) \times L^2(\Omega)]^2$

$$\begin{pmatrix} u \\ u' \\ v \\ v' \end{pmatrix}' = \begin{pmatrix} L_1 & K \\ K & L_2 \end{pmatrix} \begin{pmatrix} u \\ u' \\ v \\ v' \end{pmatrix} =: \mathcal{L} \begin{pmatrix} u \\ u' \\ v \\ v' \end{pmatrix},$$

- L_1, L_2 generators of $\mathcal{C}_0$-semigroups on $H_0^1(\Omega) \times L^2(\Omega)$
- K compact operator in $H_0^1(\Omega) \times L^2(\Omega)$

lack of exponential stability (ctnd)

$$\begin{pmatrix} u \\ u' \\ v \\ v' \end{pmatrix}' = \begin{pmatrix} L_1 & K \\ K & L_2 \end{pmatrix} \begin{pmatrix} u \\ u' \\ v \\ v' \end{pmatrix} =: \mathcal{L} \begin{pmatrix} u \\ u' \\ v \\ v' \end{pmatrix},$$

- $\omega_0(\mathcal{L})$ = type of the semigroup generated by $\mathcal{L}$
- $\omega_{\mathrm{ess}}(\mathcal{L})$ = essential growth bound (blind to compact perturbations)

$$\omega_0(\mathcal{L}) \geq \omega_{\mathrm{ess}}(\mathcal{L}) = \omega_{\mathrm{ess}} \begin{pmatrix} L_1 & K \\ 0 & L_2 \end{pmatrix} \geq \omega_{\mathrm{ess}}(L_2) = 0\,.$$

$\Rightarrow$ system cannot be exponentially stable

lack of exponential stability (ctnd)

$$\begin{pmatrix} u \\ u' \\ v \\ v' \end{pmatrix}' = \begin{pmatrix} L_1 & K \\ K & L_2 \end{pmatrix} \begin{pmatrix} u \\ u' \\ v \\ v' \end{pmatrix} =: \mathcal{L} \begin{pmatrix} u \\ u' \\ v \\ v' \end{pmatrix},$$

- $\omega_0(\mathcal{L})$ = type of the semigroup generated by $\mathcal{L}$
- $\omega_{\rm ess}(\mathcal{L})$ = essential growth bound
  (blind to compact perturbations)

$$\omega_0(\mathcal{L}) \geq \omega_{\rm ess}(\mathcal{L}) = \omega_{\rm ess}\begin{pmatrix} L_1 & K \\ 0 & L_2 \end{pmatrix} \geq \omega_{\rm ess}(L_2) = 0\,.$$

$\Rightarrow$ system cannot be exponentially stable

lack of exponential stability (ctnd)

$$\begin{pmatrix} u \\ u' \\ v \\ v' \end{pmatrix}' = \begin{pmatrix} L_1 & K \\ K & L_2 \end{pmatrix} \begin{pmatrix} u \\ u' \\ v \\ v' \end{pmatrix} =: \mathcal{L} \begin{pmatrix} u \\ u' \\ v \\ v' \end{pmatrix},$$

- $\omega_0(\mathcal{L})$ = type of the semigroup generated by $\mathcal{L}$
- $\omega_{\mathrm{ess}}(\mathcal{L})$ = essential growth bound (blind to compact perturbations)

$$\omega_0(\mathcal{L}) \geq \omega_{\mathrm{ess}}(\mathcal{L}) = \omega_{\mathrm{ess}} \begin{pmatrix} L_1 & K \\ 0 & L_2 \end{pmatrix} \geq \omega_{\mathrm{ess}}(L_2) = 0\,.$$

$\Rightarrow$ system cannot be exponentially stable

lack of exponential stability (ctnd)

$$\begin{pmatrix} u \\ u' \\ v \\ v' \end{pmatrix}' = \begin{pmatrix} L_1 & K \\ K & L_2 \end{pmatrix} \begin{pmatrix} u \\ u' \\ v \\ v' \end{pmatrix} =: \mathcal{L} \begin{pmatrix} u \\ u' \\ v \\ v' \end{pmatrix},$$

- $\omega_0(\mathcal{L}) =$ type of the semigroup generated by $\mathcal{L}$
- $\omega_{\mathrm{ess}}(\mathcal{L}) =$ essential growth bound (blind to compact perturbations)

$$\omega_0(\mathcal{L}) \geq \omega_{\mathrm{ess}}(\mathcal{L}) = \omega_{\mathrm{ess}} \begin{pmatrix} L_1 & K \\ 0 & L_2 \end{pmatrix} \geq \omega_{\mathrm{ess}}(L_2) = 0\,.$$

$\Rightarrow$ system cannot be exponentially stable

lack of exponential stability (ctnd)

$$\begin{pmatrix} u \\ u' \\ v \\ v' \end{pmatrix}' = \begin{pmatrix} L_1 & K \\ K & L_2 \end{pmatrix} \begin{pmatrix} u \\ u' \\ v \\ v' \end{pmatrix} =: \mathcal{L} \begin{pmatrix} u \\ u' \\ v \\ v' \end{pmatrix},$$

- $\omega_0(\mathcal{L})$ = type of the semigroup generated by $\mathcal{L}$
- $\omega_{\mathrm{ess}}(\mathcal{L})$ = essential growth bound
  (blind to compact perturbations)

$$\omega_0(\mathcal{L}) \geq \omega_{\mathrm{ess}}(\mathcal{L}) = \omega_{\mathrm{ess}} \begin{pmatrix} L_1 & K \\ 0 & L_2 \end{pmatrix} \geq \omega_{\mathrm{ess}}(L_2) = 0\,.$$

$\Rightarrow$ system cannot be exponentially stable

references

1993 D. Russell (J. Math.Anal. Appl.)

1999 F. Alabau-Boussouira (CRAS)

2001 F. Ammar Khodia, A. Bader (SICON)

2002
- F. Alabau-Boussouira, P. C., V. Komornik (J. evol. equ.)
- F. Alabau-Boussouira (SICON)
- A. Beyrath (Bol. Soc. Parana. Mat. 2004)
- W. Youssef (PhD Thesis 2009)

2006
- A. Batkai, K.J. Engel, J. Prüss, R. Schnaubelt (Math. Nachr.)
- P. Loreti, B. Rao (SICON)

2008 C.J.K. Batty, T. Duyckaerts (J. evol. equ.)

2010 A. Borichev, Y. Tomilov (Math. Ann.)

2011 F. Alabau-Boussouira, P. C., R. Guglielmi (arXiv:1106.0263v1)

references

1993 D. Russell (J. Math.Anal. Appl.)

1999 F. Alabau-Boussouira (CRAS)

2001 F. Ammar Khodia, A. Bader (SICON)

2002
- F. Alabau-Boussouira, P. C., V. Komornik (J. evol. equ.)
- F. Alabau-Boussouira (SICON)
- A. Beyrath (Bol. Soc. Parana. Mat. 2004)
- W. Youssef (PhD Thesis 2009)

2006
- A. Batkai, K.J. Engel, J. Prüss, R. Schnaubelt (Math. Nachr.)
- P. Loreti, B. Rao (SICON)

2008 C.J.K. Batty, T. Duyckaerts (J. evol. equ.)

2010 A. Borichev, Y. Tomilov (Math. Ann.)

2011 F. Alabau-Boussouira, P. C., R. Guglielmi (arXiv:1106.0263v1)

references

1993 D. Russell (J. Math.Anal. Appl.)

1999 F. Alabau-Boussouira (CRAS)

2001 F. Ammar Khodia, A. Bader (SICON)

2002
- F. Alabau-Boussouira, P. C., V. Komornik (J. evol. equ.)
- F. Alabau-Boussouira (SICON)
- A. Beyrath (Bol. Soc. Parana. Mat. 2004)
- W. Youssef (PhD Thesis 2009)

2006
- A. Batkai, K.J. Engel, J. Prüss, R. Schnaubelt (Math. Nachr.)
- P. Loreti, B. Rao (SICON)

2008 C.J.K. Batty, T. Duyckaerts (J. evol. equ.)

2010 A. Borichev, Y. Tomilov (Math. Ann.)

2011 F. Alabau-Boussouira, P. C., R. Guglielmi (arXiv:1106.0263v1)

references

1993 D. Russell (J. Math.Anal. Appl.)

1999 F. Alabau-Boussouira (CRAS)

2001 F. Ammar Khodia, A. Bader (SICON)

2002
- F. Alabau-Boussouira, P. C., V. Komornik (J. evol. equ.)
- F. Alabau-Boussouira (SICON)
- A. Beyrath (Bol. Soc. Parana. Mat. 2004)
- W. Youssef (PhD Thesis 2009)

2006
- A. Batkai, K.J. Engel, J. Prüss, R. Schnaubelt (Math. Nachr.)
- P. Loreti, B. Rao (SICON)

2008 C.J.K. Batty, T. Duyckaerts (J. evol. equ.)

2010 A. Borichev, Y. Tomilov (Math. Ann.)

2011 F. Alabau-Boussouira, P. C., R. Guglielmi (arXiv:1106.0263v1)

references

1993 D. Russell (J. Math.Anal. Appl.)

1999 F. Alabau-Boussouira (CRAS)

2001 F. Ammar Khodia, A. Bader (SICON)

2002
- F. Alabau-Boussouira, P. C., V. Komornik (J. evol. equ.)
- F. Alabau-Boussouira (SICON)
- A. Beyrath (Bol. Soc. Parana. Mat. 2004)
- W. Youssef (PhD Thesis 2009)

2006
- A. Batkai, K.J. Engel, J. Prüss, R. Schnaubelt (Math. Nachr.)
- P. Loreti, B. Rao (SICON)

2008 C.J.K. Batty, T. Duyckaerts (J. evol. equ.)

2010 A. Borichev, Y. Tomilov (Math. Ann.)

2011 F. Alabau-Boussouira, P. C., R. Guglielmi (arXiv:1106.0263v1)

references

1993 D. Russell (J. Math.Anal. Appl.)

1999 F. Alabau-Boussouira (CRAS)

2001 F. Ammar Khodia, A. Bader (SICON)

2002
- F. Alabau-Boussouira, P. C., V. Komornik (J. evol. equ.)
- F. Alabau-Boussouira (SICON)
- A. Beyrath (Bol. Soc. Parana. Mat. 2004)
- W. Youssef (PhD Thesis 2009)

2006
- A. Batkai, K.J. Engel, J. Prüss, R. Schnaubelt (Math. Nachr.)
- P. Loreti, B. Rao (SICON)

2008 C.J.K. Batty, T. Duyckaerts (J. evol. equ.)

2010 A. Borichev, Y. Tomilov (Math. Ann.)

2011 F. Alabau-Boussouira, P. C., R. Guglielmi (arXiv:1106.0263v1)

references

1993 D. Russell (J. Math.Anal. Appl.)

1999 F. Alabau-Boussouira (CRAS)

2001 F. Ammar Khodia, A. Bader (SICON)

2002
- F. Alabau-Boussouira, P. C., V. Komornik (J. evol. equ.)
- F. Alabau-Boussouira (SICON)
- A. Beyrath (Bol. Soc. Parana. Mat. 2004)
- W. Youssef (PhD Thesis 2009)

2006
- A. Batkai, K.J. Engel, J. Prüss, R. Schnaubelt (Math. Nachr.)
- P. Loreti, B. Rao (SICON)

2008 C.J.K. Batty, T. Duyckaerts (J. evol. equ.)

2010 A. Borichev, Y. Tomilov (Math. Ann.)

2011 F. Alabau-Boussouira, P. C., R. Guglielmi (arXiv:1106.0263v1)

references

1993 D. Russell (J. Math.Anal. Appl.)

1999 F. Alabau-Boussouira (CRAS)

2001 F. Ammar Khodia, A. Bader (SICON)

2002
- F. Alabau-Boussouira, P. C., V. Komornik (J. evol. equ.)
- F. Alabau-Boussouira (SICON)
- A. Beyrath (Bol. Soc. Parana. Mat. 2004)
- W. Youssef (PhD Thesis 2009)

2006
- A. Batkai, K.J. Engel, J. Prüss, R. Schnaubelt (Math. Nachr.)
- P. Loreti, B. Rao (SICON)

2008 C.J.K. Batty, T. Duyckaerts (J. evol. equ.)

2010 A. Borichev, Y. Tomilov (Math. Ann.)

2011 F. Alabau-Boussouira, P. C., R. Guglielmi (arXiv:1106.0263v1)

a system of second order evolutions equations

in a separable Hilbert space H

$$\begin{cases} u'' + A_1 u + Bu' + \alpha v = 0 \\ v'' + A_2 v + \alpha u = 0 \end{cases}$$

(H1) $A_i : D(A_i) \subset H \to H$ $(i = 1, 2)$ are densely defined closed linear operators such that

$$A_i = A_i^* , \qquad \langle A_i u, u \rangle \geq \omega_i |u|^2 \qquad (\omega_1, \omega_2 > 0)$$

(H2) B is a bounded linear operator on H such that

$$B = B^* , \qquad \langle Bu, u \rangle \geq \beta |u|^2 \qquad (\beta > 0)$$

(H3) $0 < |\alpha| < \sqrt{\omega_1 \omega_2}$

a system of second order evolutions equations

in a separable Hilbert space H

$$\begin{cases} u'' + A_1 u + Bu' + \alpha v = 0 \\ v'' + A_2 v + \alpha u = 0 \end{cases}$$

(H1) $A_i : D(A_i) \subset H \to H$ $(i = 1, 2)$ are densely defined closed linear operators such that

$$A_i = A_i^* , \qquad \langle A_i u, u \rangle \geq \omega_i |u|^2 \qquad (\omega_1, \omega_2 > 0)$$

(H2) B is a bounded linear operator on H such that

$$B = B^* , \qquad \langle Bu, u \rangle \geq \beta |u|^2 \qquad (\beta > 0)$$

(H3) $0 < |\alpha| < \sqrt{\omega_1 \omega_2}$

a system of second order evolutions equations

in a separable Hilbert space H

$$\begin{cases} u'' + A_1 u + Bu' + \alpha v = 0 \\ v'' + A_2 v + \alpha u = 0 \end{cases}$$

(H1) $A_i : D(A_i) \subset H \to H$ $(i = 1, 2)$ are densely defined closed linear operators such that

$$A_i = A_i^* \,, \qquad \langle A_i u, u \rangle \geq \omega_i |u|^2 \qquad (\omega_1, \omega_2 > 0)$$

(H2) B is a bounded linear operator on H such that

$$B = B^* \,, \qquad \langle Bu, u \rangle \geq \beta |u|^2 \qquad (\beta > 0)$$

(H3) $0 < |\alpha| < \sqrt{\omega_1 \omega_2}$

a system of second order evolutions equations

in a separable Hilbert space H

$$\begin{cases} u'' + A_1 u + Bu' + \alpha v = 0 \\ v'' + A_2 v + \alpha u = 0 \end{cases}$$

(H1) $A_i : D(A_i) \subset H \to H$ $(i = 1, 2)$ are densely defined closed linear operators such that

$$A_i = A_i^* , \qquad \langle A_i u, u \rangle \geq \omega_i |u|^2 \qquad (\omega_1, \omega_2 > 0)$$

(H2) B is a bounded linear operator on H such that

$$B = B^* , \qquad \langle Bu, u \rangle \geq \beta |u|^2 \qquad (\beta > 0)$$

(H3) $0 < |\alpha| < \sqrt{\omega_1 \omega_2}$

a system of second order evolutions equations

in a separable Hilbert space H

$$\begin{cases} u'' + A_1 u + Bu' + \alpha v = 0 \\ v'' + A_2 v + \alpha u = 0 \end{cases}$$

(H1) $A_i : D(A_i) \subset H \to H$ $(i = 1, 2)$ are densely defined closed linear operators such that

$$A_i = A_i^* , \qquad \langle A_i u, u \rangle \geq \omega_i |u|^2 \qquad (\omega_1, \omega_2 > 0)$$

(H2) B is a bounded linear operator on H such that

$$B = B^* , \qquad \langle Bu, u \rangle \geq \beta |u|^2 \qquad (\beta > 0)$$

(H3) $0 < |\alpha| < \sqrt{\omega_1 \omega_2}$

energies

energies associated to A_1, A_2

$$E_i(u,p) = \frac{1}{2}\left(|A_i^{1/2}u|^2 + |p|^2\right)$$

total energy of the system $U = (u, p, v, q)$

$$\mathcal{E}(U) := E_1(u,p) + E_2(v,q) + \alpha\langle u, v\rangle$$

assumptions yield

- $|u|^2 \leq \frac{2}{\omega_i} E_i(u,p)$
- $\mathcal{E}(U) \geq \nu(\alpha)\Big[E_1(u,p) + E_2(v,q)\Big]$

energies

energies associated to A_1, A_2

$$E_i(u,p) = \frac{1}{2}\left(|A_i^{1/2}u|^2 + |p|^2\right)$$

total energy of the system $U = (u,p,v,q)$

$$\mathcal{E}(U) := E_1(u,p) + E_2(v,q) + \alpha\langle u, v\rangle$$

assumptions yield

- $|u|^2 \leq \dfrac{2}{\omega_i} E_i(u,p)$
- $\mathcal{E}(U) \geq \nu(\alpha)\Big[E_1(u,p) + E_2(v,q)\Big]$

energies

energies associated to A_1, A_2

$$E_i(u,p) = \frac{1}{2}\left(|A_i^{1/2}u|^2 + |p|^2\right)$$

total energy of the system $U = (u,p,v,q)$

$$\mathcal{E}(U) := E_1(u,p) + E_2(v,q) + \alpha\langle u,v\rangle$$

assumptions yield

- $|u|^2 \leq \dfrac{2}{\omega_i} E_i(u,p)$
- $\mathcal{E}(U) \geq \nu(\alpha)\Big[E_1(u,p) + E_2(v,q)\Big]$

energies

energies associated to A_1, A_2

$$E_i(u,p) = \frac{1}{2}\left(|A_i^{1/2}u|^2 + |p|^2\right)$$

total energy of the system $U = (u,p,v,q)$

$$\mathcal{E}(U) := E_1(u,p) + E_2(v,q) + \alpha\langle u,v\rangle$$

assumptions yield

- $|u|^2 \leq \dfrac{2}{\omega_i} E_i(u,p)$
- $\mathcal{E}(U) \geq \nu(\alpha)\Big[E_1(u,p) + E_2(v,q)\Big]$

reduction to a first order system

$$\begin{aligned}\mathcal{H} &= D(A_1^{1/2}) \times H \times D(A_2^{1/2}) \times H \\ (U|\widehat{U}) &= \langle A_1^{1/2}u, A_1^{1/2}\widehat{u}\rangle + \langle p, \widehat{p}\rangle \\ &\quad + \langle A_2^{1/2}v, A_2^{1/2}\widehat{v}\rangle + \langle q, \widehat{q}\rangle + \alpha\langle u, \widehat{v}\rangle + \alpha\langle v, \widehat{u}\rangle\end{aligned}$$

system takes the equivalent form

$$\begin{cases} U'(t) = \mathcal{A}U(t) \\ U(0) = U_0 := (u^0, u^1, v^0, v^1)\,. \end{cases}$$

with $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ defined by

$$\begin{cases} D(\mathcal{A}) = D(A_1) \times D(A_1^{1/2}) \times D(A_2) \times D(A_2^{1/2}) \\ \mathcal{A}U = (p, -A_1u - Bp - \alpha v, q, -A_2v - \alpha u) \end{cases}$$

reduction to a first order system

$$\begin{aligned}\mathcal{H} &= D(A_1^{1/2}) \times H \times D(A_2^{1/2}) \times H \\ (U|\widehat{U}) &= \langle A_1^{1/2}u, A_1^{1/2}\widehat{u}\rangle + \langle p, \widehat{p}\rangle \\ &\quad + \langle A_2^{1/2}v, A_2^{1/2}\widehat{v}\rangle + \langle q, \widehat{q}\rangle + \alpha\langle u, \widehat{v}\rangle + \alpha\langle v, \widehat{u}\rangle\end{aligned}$$

system takes the equivalent form

$$\begin{cases} U'(t) = \mathcal{A}U(t) \\ U(0) = U_0 := (u^0, u^1, v^0, v^1)\,. \end{cases}$$

with $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ defined by

$$\begin{cases} D(\mathcal{A}) = D(A_1) \times D(A_1^{1/2}) \times D(A_2) \times D(A_2^{1/2}) \\ \mathcal{A}U = (p, -A_1u - Bp - \alpha v, q, -A_2v - \alpha u) \end{cases}$$

reduction to a first order system

$$
\begin{aligned}
\mathcal{H} &= D(A_1^{1/2}) \times H \times D(A_2^{1/2}) \times H \\
(U|\widehat{U}) &= \langle A_1^{1/2}u, A_1^{1/2}\widehat{u}\rangle + \langle p, \widehat{p}\rangle \\
&\quad + \langle A_2^{1/2}v, A_2^{1/2}\widehat{v}\rangle + \langle q, \widehat{q}\rangle + \alpha\langle u, \widehat{v}\rangle + \alpha\langle v, \widehat{u}\rangle
\end{aligned}
$$

system takes the equivalent form

$$
\begin{cases}
U'(t) = \mathcal{A}U(t) \\
U(0) = U_0 := (u^0, u^1, v^0, v^1)\,.
\end{cases}
$$

with $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ defined by

$$
\begin{cases}
D(\mathcal{A}) = D(A_1) \times D(A_1^{1/2}) \times D(A_2) \times D(A_2^{1/2}) \\
\mathcal{A}U = (p, -A_1u - Bp - \alpha v, q, -A_2v - \alpha u)
\end{cases}
$$

a first stability result

Theorem (ACK 2002)

Assume, for some integer $j \geq 2$,

$$|A_1 u| \leq c|A_2^{j/2} u| \qquad \forall u \in D(A_2^{j/2}) \qquad (ACK)$$

- $U_0 \in D(\mathcal{A}^{nj})$ (some $n \geq 1$) $\Rightarrow \mathcal{E}(U(t)) \leq \frac{c_n}{t^n} \sum_{k=0}^{nj} \mathcal{E}(U^{(k)}(0))$
- $\forall U_0 \in \mathcal{H}, \quad \mathcal{E}(U(t)) \to 0 \quad as \quad t \to \infty$

observe

$$(ACK) \iff D(A_2^{j/2}) \subset D(A_1) \quad \& \quad |A_1 A_2^{-j/2} u| \leq c|u|$$

a first stability result

Theorem (ACK 2002)

Assume, for some integer $j \geq 2$,

$$|A_1 u| \leq c|A_2^{j/2} u| \qquad \forall u \in D(A_2^{j/2}) \qquad (ACK)$$

- $U_0 \in D(\mathcal{A}^{nj})$ (*some* $n \geq 1$) $\Rightarrow \mathcal{E}(U(t)) \leq \dfrac{c_n}{t^n} \sum_{k=0}^{nj} \mathcal{E}(U^{(k)}(0))$
- $\forall U_0 \in \mathcal{H}, \quad \mathcal{E}(U(t)) \to 0 \quad as \quad t \to \infty$

observe

$$(ACK) \iff D(A_2^{j/2}) \subset D(A_1) \quad \& \quad |A_1 A_2^{-j/2} u| \leq c|u|$$

a first stability result

Theorem (ACK 2002)

Assume, for some integer $j \geq 2$,

$$|A_1 u| \leq c|A_2^{j/2} u| \qquad \forall u \in D(A_2^{j/2}) \qquad (ACK)$$

- $U_0 \in D(\mathcal{A}^{nj})$ (*some* $n \geq 1$) $\Rightarrow \mathcal{E}(U(t)) \leq \dfrac{c_n}{t^n} \sum_{k=0}^{nj} \mathcal{E}(U^{(k)}(0))$
- $\forall U_0 \in \mathcal{H}, \quad \mathcal{E}(U(t)) \to 0 \quad as \quad t \to \infty$

observe

$$(ACK) \iff D(A_2^{j/2}) \subset D(A_1) \quad \& \quad |A_1 A_2^{-j/2} u| \leq c|u|$$

a first stability result

Theorem (ACK 2002)

Assume, for some integer $j \geq 2$,

$$|A_1 u| \leq c|A_2^{j/2} u| \qquad \forall u \in D(A_2^{j/2}) \qquad\qquad (ACK)$$

- $U_0 \in D(\mathcal{A}^{nj})$ (*some* $n \geq 1$) $\Rightarrow \mathcal{E}(U(t)) \leq \dfrac{c_n}{t^n} \sum_{k=0}^{nj} \mathcal{E}(U^{(k)}(0))$
- $\forall U_0 \in \mathcal{H}, \quad \mathcal{E}(U(t)) \to 0 \quad as \quad t \to \infty$

observe

$$(ACK) \iff D(A_2^{j/2}) \subset D(A_1) \quad \& \quad |A_1 A_2^{-j/2} u| \leq c|u|$$

a first stability result

Theorem (ACK 2002)

Assume, for some integer $j \geq 2$,

$$|A_1 u| \leq c|A_2^{j/2} u| \qquad \forall u \in D(A_2^{j/2}) \qquad\qquad (ACK)$$

- $U_0 \in D(\mathcal{A}^{nj})$ (*some* $n \geq 1$) $\Rightarrow \mathcal{E}(U(t)) \leq \dfrac{c_n}{t^n} \sum_{k=0}^{nj} \mathcal{E}(U^{(k)}(0))$
- $\forall U_0 \in \mathcal{H}, \quad \mathcal{E}(U(t)) \to 0 \quad as \quad t \to \infty$

observe

$$(ACK) \quad \Longleftrightarrow \quad D(A_2^{j/2}) \subset D(A_1) \quad \& \quad |A_1 A_2^{-j/2} u| \leq c|u|$$

main tools

proof uses

- energy dissipation

$$\frac{d}{dt}\mathcal{E}(U(t)) = -|B^{1/2}u'(t)|^2 \qquad (U_0 \in D(\mathcal{A}))$$

- multipliers of the form $A_2^{2-j}v$ and $A_2^{1-j}A_1 u$
- an abstract decay lemma

main tools

proof uses

- energy dissipation

$$\frac{d}{dt}\mathcal{E}(U(t)) = -|B^{1/2}u'(t)|^2 \qquad (U_0 \in D(\mathcal{A}))$$

- multipliers of the form $A_2^{2-j}v$ and $A_2^{1-j}A_1 u$
- an abstract decay lemma

main tools

proof uses

- energy dissipation

$$\frac{d}{dt}\mathcal{E}(U(t)) = -|B^{1/2}u'(t)|^2 \qquad (U_0 \in D(\mathcal{A}))$$

- multipliers of the form $A_2^{2-j}v$ and $A_2^{1-j}A_1 u$
- an abstract decay lemma

main tools

proof uses

- energy dissipation

$$\frac{d}{dt}\mathcal{E}(U(t)) = -|B^{1/2}u'(t)|^2 \qquad (U_0 \in D(\mathcal{A}))$$

- multipliers of the form $A_2^{2-j}v$ and $A_2^{1-j}A_1 u$
- an abstract decay lemma

abstract decay lemma

- $A : D(A) \subset H \to H$ generator of a $\mathcal{C}_0$-semigroup
- $L : H \to [0, +\infty)$ continuous function

$$\int_0^T L(e^{tA}x)dt \leq c \sum_{k=0}^{K} L(A^k x)$$

$\Rightarrow \forall n \geq 1, \forall x \in D(A^{nK}), \forall 0 \leq s \leq T$

$$\int_s^T L(e^{tA}x) \frac{(t-s)^{n-1}}{(n-1)!} dt \leq c^n (1+K)^{n-1} \sum_{k=0}^{nK} L(e^{sA}A^k x)$$

- $L(e^{tA}x) \leq L(e^{sA}x) \Rightarrow L(e^{tA}x) \leq c^n (1+K)^{n-1} \dfrac{n!}{t^n} \sum_{k=0}^{nK} L(A^k x)$

abstract decay lemma

- $A : D(A) \subset H \to H$ generator of a $\mathcal{C}_0$-semigroup
- $L : H \to [0, +\infty)$ continuous function

$$\int_0^T L(e^{tA}x)dt \le c \sum_{k=0}^{K} L(A^k x)$$

$\Rightarrow \forall n \ge 1, \forall x \in D(A^{nK}), \forall 0 \le s \le T$

$$\int_s^T L(e^{tA}x) \frac{(t-s)^{n-1}}{(n-1)!} dt \le c^n (1+K)^{n-1} \sum_{k=0}^{nK} L(e^{sA} A^k x)$$

- $L(e^{tA}x) \le L(e^{sA}x) \Rightarrow L(e^{tA}x) \le c^n (1+K)^{n-1} \frac{n!}{t^n} \sum_{k=0}^{nK} L(A^k x)$

abstract decay lemma

- $A : D(A) \subset H \to H$ generator of a $\mathcal{C}_0$-semigroup
- $L : H \to [0, +\infty)$ continuous function

$$\int_0^T L(e^{tA}x)dt \leq c \sum_{k=0}^{K} L(A^k x)$$

$\Rightarrow$ $\forall n \geq 1$, $\forall x \in D(A^{nK})$, $\forall 0 \leq s \leq T$

$$\int_s^T L(e^{tA}x) \frac{(t-s)^{n-1}}{(n-1)!} dt \leq c^n (1+K)^{n-1} \sum_{k=0}^{nK} L(e^{sA}A^k x)$$

- $L(e^{tA}x) \leq L(e^{sA}x) \Rightarrow L(e^{tA}x) \leq c^n(1+K)^{n-1} \frac{n!}{t^n} \sum_{k=0}^{nK} L(A^k x)$

abstract decay lemma

- $A : D(A) \subset H \to H$ generator of a $\mathcal{C}_0$-semigroup
- $L : H \to [0, +\infty)$ continuous function

$$\int_0^T L(e^{tA}x)dt \leq c \sum_{k=0}^{K} L(A^k x)$$

$\Rightarrow$ $\forall n \geq 1$, $\forall x \in D(A^{nK})$, $\forall 0 \leq s \leq T$

$$\int_s^T L(e^{tA}x) \frac{(t-s)^{n-1}}{(n-1)!} dt \leq c^n (1+K)^{n-1} \sum_{k=0}^{nK} L(e^{sA}A^k x)$$

- $L(e^{tA}x) \leq L(e^{sA}x) \Rightarrow L(e^{tA}x) \leq c^n (1+K)^{n-1} \frac{n!}{t^n} \sum_{k=0}^{nK} L(A^k x)$

example 1: Dirichlet boundary conditions

$\Omega \subset \mathbb{R}^n$ bounded $\Gamma = \partial\Omega$

$$\begin{cases} \partial_t^2 u - \Delta u + \partial_t u + \alpha v = 0 \\ \partial_t^2 v - \Delta v + \alpha u = 0 \end{cases} \quad \text{in} \quad \Omega \times (0, +\infty)$$

with boundary conditions

$$u(\cdot, t) = 0 = v(\cdot, t) \quad \text{on} \quad \Gamma \quad \forall t > 0$$

in this example $A_1 = A = A_2$ with

$$D(A) = H^2(\Omega) \cap H_0^1(\Omega), \qquad Au = -\Delta u$$

so that (ACK) : $|A_1 u| \le c|A_2^{j/2} u|$ holds with $j = 2$

example 1: Dirichlet boundary conditions

$\Omega \subset \mathbb{R}^n$ bounded $\Gamma = \partial\Omega$

$$\begin{cases} \partial_t^2 u - \Delta u + \partial_t u + \alpha v = 0 \\ \partial_t^2 v - \Delta v + \alpha u = 0 \end{cases} \quad \text{in} \quad \Omega \times (0, +\infty)$$

with boundary conditions

$$u(\cdot, t) = 0 = v(\cdot, t) \quad \text{on} \quad \Gamma \quad \forall t > 0$$

in this example $A_1 = A = A_2$ with

$$D(A) = H^2(\Omega) \cap H_0^1(\Omega)\,, \qquad Au = -\Delta u$$

so that $(ACK)\ :\ |A_1 u| \le c|A_2^{j/2} u|$ holds with $j = 2$

example 1: conclusion

$$\begin{cases} \partial_t^2 u - \Delta u + \partial_t u + \alpha v = 0 \\ \partial_t^2 v - \Delta v + \alpha u = 0 \end{cases} \qquad \text{in} \qquad \Omega \times (0, +\infty)$$

$$u(\cdot, t) = 0 = v(\cdot, t) \qquad \text{on} \quad \Gamma \quad \forall t > 0$$

$$\begin{cases} u(x,0) = u^0(x)\,, \qquad u'(x,0) = u^1(x) \\ v(x,0) = v^0(x)\,, \qquad v'(x,0) = v^1(x) \end{cases} \qquad x \in \Omega$$

conclusion: for $0 < |\alpha| < C_\Omega$

$$\int_\Omega \left(|\partial_t u|^2 + |\nabla u|^2 + |\partial_t v|^2 + |\nabla v|^2 \right) dx$$

$$\leq \frac{c}{t} \left(\|u^0\|_{2,\Omega}^2 + \|u^1\|_{1,\Omega}^2 + \|v^0\|_{2,\Omega}^2 + \|v^1\|_{1,\Omega}^2 \right)$$

example 1: conclusion

$$\begin{cases} \partial_t^2 u - \Delta u + \partial_t u + \alpha v = 0 \\ \partial_t^2 v - \Delta v + \alpha u = 0 \end{cases} \quad \text{in} \quad \Omega \times (0, +\infty)$$

$$u(\cdot, t) = 0 = v(\cdot, t) \quad \text{on} \quad \Gamma \quad \forall t > 0$$

$$\begin{cases} u(x,0) = u^0(x)\,, & u'(x,0) = u^1(x) \\ v(x,0) = v^0(x)\,, & v'(x,0) = v^1(x) \end{cases} \quad x \in \Omega$$

conclusion: for $0 < |\alpha| < C_\Omega$

$$\int_\Omega \left(|\partial_t u|^2 + |\nabla u|^2 + |\partial_t v|^2 + |\nabla v|^2 \right) dx$$

$$\leq \frac{c}{t} \left(\|u^0\|^2_{2,\Omega} + \|u^1\|^2_{1,\Omega} + \|v^0\|^2_{2,\Omega} + \|v^1\|^2_{1,\Omega} \right)$$

example 1: conclusion

$$\begin{cases} \partial_t^2 u - \Delta u + \partial_t u + \alpha v = 0 \\ \partial_t^2 v - \Delta v + \alpha u = 0 \end{cases} \quad \text{in} \quad \Omega \times (0, +\infty)$$

$$u(\cdot, t) = 0 = v(\cdot, t) \quad \text{on} \quad \Gamma \quad \forall t > 0$$

$$\begin{cases} u(x,0) = u^0(x), & u'(x,0) = u^1(x) \\ v(x,0) = v^0(x), & v'(x,0) = v^1(x) \end{cases} \quad x \in \Omega$$

conclusion: for $0 < |\alpha| < C_\Omega$

$$\int_\Omega \Big(|\partial_t u|^2 + |\nabla u|^2 + |\partial_t v|^2 + |\nabla v|^2 \Big) dx \leq \frac{c}{t} \Big(\|u^0\|^2_{2,\Omega} + \|u^1\|^2_{1,\Omega} + \|v^0\|^2_{2,\Omega} + \|v^1\|^2_{1,\Omega} \Big)$$

example 2: hybrid boundary conditions

Let $\alpha \in \mathbb{R}$ and consider the problem

$$\begin{cases} \partial_t^2 u - \Delta u + \partial_t u + \alpha v = 0 \\ \partial_t^2 v - \Delta v + \alpha u = 0 \end{cases} \qquad \text{in } \Omega \times (0, +\infty)$$

with boundary conditions

$$\begin{aligned} \left(\frac{\partial u}{\partial \nu} + u\right)(\cdot, t) &= 0 \text{ on } \Gamma \\ v(\cdot, t) &= 0 \text{ on } \Gamma \end{aligned} \qquad \forall t > 0$$

example 2: hybrid boundary conditions

Let $\alpha \in \mathbb{R}$ and consider the problem

$$\begin{cases} \partial_t^2 u - \Delta u + \partial_t u + \alpha v = 0 \\ \partial_t^2 v - \Delta v + \alpha u = 0 \end{cases} \qquad \text{in } \Omega \times (0, +\infty)$$

with boundary conditions

$$\begin{aligned} \left(\frac{\partial u}{\partial \nu} + u\right)(\cdot, t) &= 0 \text{ on } \Gamma \\ v(\cdot, t) &= 0 \text{ on } \Gamma \end{aligned} \qquad \forall t > 0$$

ACK does not apply

$$D(A_1) = \left\{ u \in H^2(\Omega) : \frac{\partial u}{\partial \nu} + u = 0 \text{ on } \Gamma \right\}, \; A_1 u = -\Delta u$$

$$D(A_2) = H^2(\Omega) \cap H^1_0(\Omega), \; A_2 v = -\Delta v$$

Lemma (ACG)

$D(A_2^{k/2})$ is not included in $D(A_1)$ for any $k \geq 2$

Proof.

$$(k=2) \quad v_0 \; : \begin{cases} (-\Delta)^2 v = 1 \\ v_0 = 0 = \Delta v_0 \qquad \text{on } \Gamma \end{cases} \qquad v_1 := -\Delta v_0$$

$$D(A_2) \subset D(A_1) \Rightarrow \frac{\partial v_0}{\partial \nu}_{|\Gamma} = 0 \Rightarrow \int_\Omega v_1 \, dx = 0$$

$$\Rightarrow \int_\Omega |\nabla v_1|^2 dx = \int_\Omega (-\Delta v_1) v_1 \, dx = 0 \text{ but } -\Delta v_1 = 1$$

ACK does not apply

$$D(A_1) = \left\{ u \in H^2(\Omega) : \frac{\partial u}{\partial \nu} + u = 0 \text{ on } \Gamma \right\}, \quad A_1 u = -\Delta u$$
$$D(A_2) = H^2(\Omega) \cap H_0^1(\Omega), \quad A_2 v = -\Delta v$$

Lemma (ACG)

$D(A_2^{k/2})$ is not included in $D(A_1)$ for any $k \geq 2$

Proof.

$$(k=2) \quad v_0 \ : \begin{cases} (-\Delta)^2 v = 1 \\ v_0 = 0 = \Delta v_0 \qquad \text{on } \Gamma \end{cases} \qquad v_1 := -\Delta v_0$$

$$D(A_2) \subset D(A_1) \Rightarrow \frac{\partial v_0}{\partial \nu}_{|\Gamma} = 0 \Rightarrow \int_\Omega v_1 \, dx = 0$$

$$\Rightarrow \int_\Omega |\nabla v_1|^2 dx = \int_\Omega (-\Delta v_1) v_1 \, dx = 0 \text{ but } -\Delta v_1 = 1$$

ACK does not apply

$$D(A_1) = \left\{ u \in H^2(\Omega) : \frac{\partial u}{\partial \nu} + u = 0 \text{ on } \Gamma \right\}, \; A_1 u = -\Delta u$$
$$D(A_2) = H^2(\Omega) \cap H_0^1(\Omega), \; A_2 v = -\Delta v$$

Lemma (ACG)

$D(A_2^{k/2})$ is not included in $D(A_1)$ for any $k \geq 2$

Proof.

$$(k=2) \quad v_0 \; : \begin{cases} (-\Delta)^2 v = 1 \\ v_0 = 0 = \Delta v_0 \qquad \text{on } \Gamma \end{cases} \qquad v_1 := -\Delta v_0$$

$$D(A_2) \subset D(A_1) \Rightarrow \frac{\partial v_0}{\partial \nu}_{|\Gamma} = 0 \Rightarrow \int_\Omega v_1 \, dx = 0$$

$$\Rightarrow \int_\Omega |\nabla v_1|^2 dx = \int_\Omega (-\Delta v_1) v_1 \, dx = 0 \text{ but } -\Delta v_1 = 1$$

ACK does not apply

$$D(A_1) = \left\{ u \in H^2(\Omega) : \frac{\partial u}{\partial \nu} + u = 0 \text{ on } \Gamma \right\}, \quad A_1 u = -\Delta u$$
$$D(A_2) = H^2(\Omega) \cap H_0^1(\Omega), \quad A_2 v = -\Delta v$$

Lemma (ACG)

$D(A_2^{k/2})$ is not included in $D(A_1)$ for any $k \geq 2$

Proof.

$$(k=2) \quad v_0 : \begin{cases} (-\Delta)^2 v = 1 \\ v_0 = 0 = \Delta v_0 \quad \text{on } \Gamma \end{cases} \qquad v_1 := -\Delta v_0$$

$$D(A_2) \subset D(A_1) \;\Rightarrow\; \frac{\partial v_0}{\partial \nu}_{|\Gamma} = 0 \;\Rightarrow\; \int_\Omega v_1 \, dx = 0$$

$$\Rightarrow \int_\Omega |\nabla v_1|^2 dx = \int_\Omega (-\Delta v_1) v_1 \, dx = 0 \text{ but } -\Delta v_1 = 1$$

second stability result

Theorem (ACG 2011)

Assume

$$D(A_2) \subset D(A_1^{1/2}) \quad \& \quad |A_1^{1/2} u| \leq c|A_2 u| \quad \forall u \in D(A_2) \quad (ACG)$$

Then

- $U_0 \in D(\mathcal{A}^{4n})$ (*some* $n \geq 1$) $\Rightarrow \mathcal{E}(U(t)) \leq \dfrac{c_n}{t^n} \displaystyle\sum_{k=0}^{4n} \mathcal{E}(U^{(k)}(0))$
- $\forall U_0 \in \mathcal{H}, \quad \mathcal{E}(U(t)) \to 0 \quad as \quad t \to \infty$

observe

$$(ACG) \quad \Longleftrightarrow \quad |\langle A_1 u, v\rangle| \leq c|A_2 v| \langle A_1 u, u\rangle^{1/2}$$

second stability result

Theorem (ACG 2011)

Assume

$$D(A_2) \subset D(A_1^{1/2}) \quad \& \quad |A_1^{1/2}u| \le c|A_2 u| \quad \forall u \in D(A_2) \quad (ACG)$$

Then

- $U_0 \in D(\mathcal{A}^{4n})$ (*some* $n \ge 1$) $\Rightarrow \mathcal{E}(U(t)) \le \frac{c_n}{t^n} \sum_{k=0}^{4n} \mathcal{E}(U^{(k)}(0))$
- $\forall U_0 \in \mathcal{H}, \quad \mathcal{E}(U(t)) \to 0 \quad as \quad t \to \infty$

observe

$$(ACG) \iff |\langle A_1 u, v\rangle| \le c|A_2 v|\langle A_1 u, u\rangle^{1/2}$$

second stability result

Theorem (ACG 2011)

Assume

$$D(A_2) \subset D(A_1^{1/2}) \quad \& \quad |A_1^{1/2}u| \leq c|A_2 u| \quad \forall u \in D(A_2) \quad (ACG)$$

Then

- $U_0 \in D(\mathcal{A}^{4n})$ (*some* $n \geq 1$) $\Rightarrow \mathcal{E}(U(t)) \leq \dfrac{c_n}{t^n} \sum_{k=0}^{4n} \mathcal{E}(U^{(k)}(0))$
- $\forall U_0 \in \mathcal{H}, \quad \mathcal{E}(U(t)) \to 0 \quad as \quad t \to \infty$

observe

$$(ACG) \iff |\langle A_1 u, v\rangle| \leq c|A_2 v|\langle A_1 u, u\rangle^{1/2}$$

second stability result

Theorem (ACG 2011)

Assume

$$D(A_2) \subset D(A_1^{1/2}) \quad \& \quad |A_1^{1/2} u| \leq c|A_2 u| \quad \forall u \in D(A_2) \quad (ACG)$$

Then

- $U_0 \in D(\mathcal{A}^{4n})$ (*some* $n \geq 1$) $\Rightarrow \mathcal{E}(U(t)) \leq \dfrac{c_n}{t^n} \displaystyle\sum_{k=0}^{4n} \mathcal{E}(U^{(k)}(0))$
- $\forall U_0 \in \mathcal{H}, \quad \mathcal{E}(U(t)) \to 0 \quad as \quad t \to \infty$

observe

$$(ACG) \iff |\langle A_1 u, v\rangle| \leq c|A_2 v|\langle A_1 u, u\rangle^{1/2}$$

second stability result

Theorem (ACG 2011)

Assume

$$D(A_2) \subset D(A_1^{1/2}) \quad \& \quad |A_1^{1/2}u| \le c|A_2 u| \quad \forall u \in D(A_2) \quad (ACG)$$

Then

- $U_0 \in D(\mathcal{A}^{4n})$ (some $n \ge 1$) $\Rightarrow \mathcal{E}(U(t)) \le \dfrac{c_n}{t^n} \displaystyle\sum_{k=0}^{4n} \mathcal{E}(U^{(k)}(0))$
- $\forall U_0 \in \mathcal{H}, \quad \mathcal{E}(U(t)) \to 0 \quad as \quad t \to \infty$

observe

$$(ACG) \quad \Longleftrightarrow \quad |\langle A_1 u, v\rangle| \le c|A_2 v| \langle A_1 u, u\rangle^{1/2}$$

main tools

proof uses

- energy dissipation

$$\frac{d}{dt}\mathcal{E}(U(t)) = -|B^{1/2}u'(t)|^2 \qquad (U_0 \in D(\mathcal{A}))$$

- multipliers of the form $A_1^{-1}v$ and $A_2^{-1}u$
- abstract decay lemma

main tools

proof uses

- energy dissipation

$$\frac{d}{dt}\mathcal{E}(U(t)) = -|B^{1/2}u'(t)|^2 \qquad (U_0 \in D(\mathcal{A}))$$

- multipliers of the form $A_1^{-1}v$ and $A_2^{-1}u$
- abstract decay lemma

main tools

proof uses

- energy dissipation

$$\frac{d}{dt}\mathcal{E}(U(t)) = -|B^{1/2}u'(t)|^2 \qquad (U_0 \in D(\mathcal{A}))$$

- multipliers of the form $A_1^{-1}v$ and $A_2^{-1}u$
- abstract decay lemma

main tools

proof uses

- energy dissipation

$$\frac{d}{dt}\mathcal{E}(U(t)) = -|B^{1/2}u'(t)|^2 \qquad (U_0 \in D(\mathcal{A}))$$

- multipliers of the form $A_1^{-1}v$ and $A_2^{-1}u$
- abstract decay lemma

use interpolation

- polynomial decay estimates improved for

$$U_0 \in (\mathcal{H}, D(\mathcal{A}^{4n}))_{\theta,2} \quad \text{for some} \quad n \geq 1\,,\ 0 < \theta < 1$$

- since $\mathcal{A}$ generates a $\mathcal{C}_0$-semigroup of contractions,

$$D(\mathcal{A}^m) = (\mathcal{H}, D(\mathcal{A}^k))_{\theta,2}$$

if $\theta k = m$ for some $0 < \theta < 1$ and $k,\ m \in \mathbb{N}$

use interpolation

- polynomial decay estimates improved for

$$U_0 \in (\mathcal{H}, D(\mathcal{A}^{4n}))_{\theta,2} \quad \text{for some} \quad n \geq 1\,,\ 0 < \theta < 1$$

- since $\mathcal{A}$ generates a $\mathcal{C}_0$-semigroup of contractions,

$$D(\mathcal{A}^m) = (\mathcal{H}, D(\mathcal{A}^k))_{\theta,2}$$

if $\theta k = m$ for some $0 < \theta < 1$ and $k, m \in \mathbb{N}$

use interpolation

- polynomial decay estimates improved for

$$U_0 \in (\mathcal{H}, D(\mathcal{A}^{4n}))_{\theta,2} \quad \text{for some} \quad n \geq 1\,,\ 0 < \theta < 1$$

- since $\mathcal{A}$ generates a $\mathcal{C}_0$-semigroup of contractions,

$$D(\mathcal{A}^m) = (\mathcal{H}, D(\mathcal{A}^k))_{\theta,2}$$

if $\theta k = m$ for some $0 < \theta < 1$ and $k,\ m \in \mathbb{N}$

ACG with data in interpolation spaces

assume

$$D(A_2) \subset D(A_1^{1/2}) \quad \& \quad |A_1^{1/2}u| \leq c|A_2 u| \qquad (ACG)$$

let $\quad n \geq 1\,, \quad 0 < \theta < 1 \quad$ then

- $U_0 \in (\mathcal{H}, D(\mathcal{A}^{4n}))_{\theta,2} \Longrightarrow \|U(t)\|^2_{\mathcal{H}} \leq \frac{c_{n,\theta}}{t^{n\theta}} \|U_0\|^2_{(\mathcal{H}, D(\mathcal{A}^{4n}))_{\theta,2}}$
- $U_0 \in D(\mathcal{A}^n) \Longrightarrow \mathcal{E}(U(t)) \leq \frac{c_n}{t^{n/4}} \sum_{k=0}^{n} \mathcal{E}(U^{(k)}(0))$

ACG with data in interpolation spaces

assume

$$D(A_2) \subset D(A_1^{1/2}) \quad \& \quad |A_1^{1/2}u| \leq c|A_2 u| \qquad (ACG)$$

let $\quad n \geq 1\,, \quad 0 < \theta < 1 \quad$ then

- $U_0 \in (\mathcal{H}, D(\mathcal{A}^{4n}))_{\theta,2} \Longrightarrow \|U(t)\|_{\mathcal{H}}^2 \leq \dfrac{c_{n,\theta}}{t^{n\theta}} \|U_0\|^2_{(\mathcal{H},D(\mathcal{A}^{4n}))_{\theta,2}}$
- $U_0 \in D(\mathcal{A}^n) \Longrightarrow \mathcal{E}(U(t)) \leq \dfrac{c_n}{t^{n/4}} \sum_{k=0}^{n} \mathcal{E}(U^{(k)}(0))$

ACG with data in interpolation spaces

assume

$$D(A_2) \subset D(A_1^{1/2}) \quad \& \quad |A_1^{1/2}u| \leq c|A_2 u| \qquad (ACG)$$

let $\quad n \geq 1\,, \quad 0 < \theta < 1 \quad$ then

- $U_0 \in (\mathcal{H}, D(\mathcal{A}^{4n}))_{\theta,2} \Longrightarrow \|U(t)\|_{\mathcal{H}}^2 \leq \dfrac{c_{n,\theta}}{t^{n\theta}} \|U_0\|^2_{(\mathcal{H}, D(\mathcal{A}^{4n}))_{\theta,2}}$
- $U_0 \in D(\mathcal{A}^n) \Longrightarrow \mathcal{E}(U(t)) \leq \dfrac{c_n}{t^{n/4}} \displaystyle\sum_{k=0}^{n} \mathcal{E}(U^{(k)}(0))$

ACG with data in interpolation spaces

assume

$$D(A_2) \subset D(A_1^{1/2}) \quad \& \quad |A_1^{1/2}u| \le c|A_2u| \qquad (ACG)$$

let $\quad n \ge 1\,, \quad 0 < \theta < 1 \quad$ then

- $U_0 \in (\mathcal{H}, D(\mathcal{A}^{4n}))_{\theta,2} \Longrightarrow \|U(t)\|^2_{\mathcal{H}} \le \frac{c_{n,\theta}}{t^{n\theta}}\|U_0\|^2_{(\mathcal{H},D(\mathcal{A}^{4n}))_{\theta,2}}$
- $U_0 \in D(\mathcal{A}^n) \Longrightarrow \mathcal{E}(U(t)) \le \frac{c_n}{t^{n/4}} \sum_{k=0}^{n} \mathcal{E}(U^{(k)}(0))$

ACG with data in interpolation spaces

assume

$$D(A_2) \subset D(A_1^{1/2}) \quad \& \quad |A_1^{1/2}u| \leq c|A_2 u| \qquad (ACG)$$

let $\quad n \geq 1\,, \quad 0 < \theta < 1 \quad$ then

- $U_0 \in (\mathcal{H}, D(\mathcal{A}^{4n}))_{\theta,2} \Longrightarrow \|U(t)\|_{\mathcal{H}}^2 \leq \dfrac{c_{n,\theta}}{t^{n\theta}} \|U_0\|^2_{(\mathcal{H},D(\mathcal{A}^{4n}))_{\theta,2}}$
- $U_0 \in D(\mathcal{A}^n) \Longrightarrow \mathcal{E}(U(t)) \leq \dfrac{c_n}{t^{n/4}} \sum_{k=0}^{n} \mathcal{E}(U^{(k)}(0))$

example 2: ACG applies

the energy of the solution to the boundary-value problem

$$\begin{cases} \partial_t^2 u - \Delta u + \partial_t u + \alpha v = 0 \\ \partial_t^2 v - \Delta v + \alpha u = 0 \end{cases} \quad \text{in } \Omega \times (0, +\infty)$$

$$\begin{aligned} \left(\frac{\partial u}{\partial \nu} + u\right)(\cdot, t) &= 0 \quad \text{on } \Gamma \\ v(\cdot, t) &= 0 \quad \text{on } \Gamma \end{aligned} \qquad \forall t > 0$$

satisfies, for $0 < |\alpha| < C_\Omega$,

$$E_1(u(t), u'(t)) + E_2(v(t), v'(t)) \le \frac{c}{t^{1/4}} \left(|A_1 u^0|^2 + |A_1^{1/2} u^1|^2 + |A_2 v^0|^2 + |A_2^{1/2} v^1|^2 \right)$$

example 2: ACG applies

the energy of the solution to the boundary-value problem

$$\begin{cases} \partial_t^2 u - \Delta u + \partial_t u + \alpha v = 0 \\ \partial_t^2 v - \Delta v + \alpha u = 0 \end{cases} \quad \text{in } \Omega \times (0, +\infty)$$

$$\begin{aligned} \left(\frac{\partial u}{\partial \nu} + u \right)(\cdot, t) &= 0 \quad \text{on } \Gamma \\ v(\cdot, t) &= 0 \quad \text{on } \Gamma \end{aligned} \qquad \forall t > 0$$

satisfies, for $0 < |\alpha| < C_\Omega$,

$$\begin{aligned} &E_1(u(t), u'(t)) + E_2(v(t), v'(t)) \\ &\qquad \le \frac{c}{t^{1/4}} \left(|A_1 u^0|^2 + |A_1^{1/2} u^1|^2 + |A_2 v^0|^2 + |A_2^{1/2} v^1|^2 \right) \end{aligned}$$

proof

- recall

$$D(A_1) = \left\{ u \in H^2(\Omega) : \frac{\partial u}{\partial \nu} + u = 0 \text{ on } \Gamma \right\}, \quad A_1 u = -\Delta u$$

$$D(A_2) = H^2(\Omega) \cap H_0^1(\Omega), \quad A_2 v = -\Delta v$$

- to obtain, for all $u \in D(A_1), v \in D(A_2)$,

$$|\langle A_1 u, v \rangle| = \left| \int_\Omega \nabla u \nabla v \, dx \right| \le c \, \langle A_1 u, u \rangle^{1/2} |A_2 v|$$

since $\langle A_1 u, u \rangle = \int_\Omega |\nabla u|^2 \, dx + \int_\Gamma |u|^2 \, dS$

- yields (ACG): $D(A_2) \subset D(A_1^{1/2})$ & $|A_1^{1/2} u| \le c |A_2 u|$

proof

- recall

$$D(A_1) = \left\{ u \in H^2(\Omega) : \frac{\partial u}{\partial \nu} + u = 0 \text{ on } \Gamma \right\}, \quad A_1 u = -\Delta u$$
$$D(A_2) = H^2(\Omega) \cap H_0^1(\Omega), \quad A_2 v = -\Delta v$$

- to obtain, for all $u \in D(A_1), v \in D(A_2)$,

$$|\langle A_1 u, v\rangle| = \left| \int_\Omega \nabla u \nabla v \, dx \right| \le c \, \langle A_1 u, u\rangle^{1/2} |A_2 v|$$

since $\langle A_1 u, u\rangle = \int_\Omega |\nabla u|^2 \, dx + \int_\Gamma |u|^2 \, dS$

- yields (ACG): $D(A_2) \subset D(A_1^{1/2})$ & $|A_1^{1/2} u| \le c|A_2 u|$

proof

- recall

$$D(A_1)=\left\{u\in H^2(\Omega):\frac{\partial u}{\partial\nu}+u=0\text{ on }\Gamma\right\}\,,\ A_1u=-\Delta u$$
$$D(A_2)=H^2(\Omega)\cap H^1_0(\Omega)\,,\ A_2v=-\Delta v$$

- to obtain, for all $u\in D(A_1), v\in D(A_2)$,

$$|\langle A_1u,v\rangle|=\left|\int_\Omega\nabla u\nabla v\,dx\right|\le c\,\langle A_1u,u\rangle^{1/2}|A_2v|$$

since $\langle A_1u,u\rangle=\int_\Omega|\nabla u|^2\,dx+\int_\Gamma|u|^2\,dS$

- yields (ACG): $D(A_2)\subset D(A_1^{1/2})$ & $|A_1^{1/2}u|\le c|A_2u|$

proof

- recall

$$D(A_1) = \left\{ u \in H^2(\Omega) : \frac{\partial u}{\partial \nu} + u = 0 \text{ on } \Gamma \right\}, \; A_1 u = -\Delta u$$
$$D(A_2) = H^2(\Omega) \cap H_0^1(\Omega), \; A_2 v = -\Delta v$$

- to obtain, for all $u \in D(A_1), v \in D(A_2)$,

$$|\langle A_1 u, v\rangle| = \left| \int_\Omega \nabla u \nabla v \, dx \right| \leq c \, \langle A_1 u, u\rangle^{1/2} \, |A_2 v|$$

since $\quad \langle A_1 u, u\rangle = \int_\Omega |\nabla u|^2 \, dx + \int_\Gamma |u|^2 \, dS$

- yields (ACG): $\quad D(A_2) \subset D(A_1^{1/2}) \quad \& \quad |A_1^{1/2} u| \leq c |A_2 u|$

operators of higher order

define

$$D(A_1) = \left\{u \in H^4(\Omega) : \Delta u = 0 = \frac{\partial \Delta u}{\partial \nu} \text{ on } \Gamma\right\}, \quad A_1 u = \Delta^2 u$$

$$D(A_2) = H^2(\Omega) \cap H_0^1(\Omega), \quad A_2 v = -\Delta v$$

then

$$|\langle A_1 u, v\rangle| = \left|\int_\Omega \Delta u \Delta v \, dx\right| \le c \langle A_1 u, u\rangle^{1/2} |A_2 v|$$

deduce (ACG): $D(A_2) \subset D(A_1^{1/2})$ & $|A_1^{1/2} u| \le c|A_2 u|$

operators of higher order

define

$$D(A_1) = \left\{ u \in H^4(\Omega) : \Delta u = 0 = \frac{\partial \Delta u}{\partial \nu} \text{ on } \Gamma \right\}, \quad A_1 u = \Delta^2 u$$

$$D(A_2) = H^2(\Omega) \cap H_0^1(\Omega), \quad A_2 v = -\Delta v$$

then

$$|\langle A_1 u, v \rangle| = \left| \int_\Omega \Delta u \Delta v \, dx \right| \leq c \langle A_1 u, u \rangle^{1/2} |A_2 v|$$

deduce (ACG): $D(A_2) \subset D(A_1^{1/2})$ & $|A_1^{1/2} u| \leq c |A_2 u|$

operators of higher order

define

$$D(A_1) = \left\{ u \in H^4(\Omega) : \Delta u = 0 = \frac{\partial \Delta u}{\partial \nu} \text{ on } \Gamma \right\}, \quad A_1 u = \Delta^2 u$$

$$D(A_2) = H^2(\Omega) \cap H_0^1(\Omega), \quad A_2 v = -\Delta v$$

then

$$|\langle A_1 u, v\rangle| = \left| \int_\Omega \Delta u \Delta v \, dx \right| \leq c \, \langle A_1 u, u\rangle^{1/2} |A_2 v|$$

deduce (ACG): $D(A_2) \subset D(A_1^{1/2})$ & $|A_1^{1/2} u| \leq c |A_2 u|$

operators of higher order

define

$$\begin{aligned} D(A_1) &= \left\{ u \in H^4(\Omega) : \Delta u = 0 = \frac{\partial \Delta u}{\partial \nu} \text{ on } \Gamma \right\}, \quad A_1 u = \Delta^2 u \\ D(A_2) &= H^2(\Omega) \cap H_0^1(\Omega), \quad A_2 v = -\Delta v \end{aligned}$$

then

$$|\langle A_1 u, v\rangle| = \left| \int_\Omega \Delta u \Delta v \, dx \right| \le c \, \langle A_1 u, u\rangle^{1/2} \, |A_2 v|$$

deduce (ACG): $D(A_2) \subset D(A_1^{1/2})$ & $|A_1^{1/2} u| \le c|A_2 u|$

application

consider boundary-value problem

$$\begin{cases} \partial_t^2 u + \Delta^2 u + \partial_t u + \alpha v = 0 \\ \partial_t^2 v - \Delta v + \alpha u = 0 \end{cases} \qquad \text{in } \Omega \times (0, +\infty)$$

$$\Delta u(\cdot, t) = 0 = \frac{\partial \Delta u}{\partial \nu}(\cdot, t) \quad \text{and} \quad v(\cdot, t) = 0 \text{ on } \Gamma$$

then, for $0 < |\alpha| < C_\Omega^{3/2}$, energy decays at polynomial rate

$$E_1(u(t), u'(t)) + E_2(v(t), v'(t)) \le \frac{c}{t^{1/4}} \left(\|u^0\|_{4,\Omega}^2 + \|u^1\|_{2,\Omega}^2 + \|v^0\|_{2,\Omega}^2 + \|v^1\|_{1,\Omega}^2 \right)$$

application

consider boundary-value problem

$$\begin{cases} \partial_t^2 u + \Delta^2 u + \partial_t u + \alpha v = 0 \\ \partial_t^2 v - \Delta v + \alpha u = 0 \end{cases} \quad \text{in } \Omega \times (0, +\infty)$$

$$\Delta u(\cdot, t) = 0 = \frac{\partial \Delta u}{\partial \nu}(\cdot, t) \quad \text{and} \quad v(\cdot, t) = 0 \text{ on } \Gamma$$

then, for $0 < |\alpha| < C_\Omega^{3/2}$, energy decays at polynomial rate

$$E_1(u(t), u'(t)) + E_2(v(t), v'(t)) \le \frac{c}{t^{1/4}} \left(\|u^0\|_{4,\Omega}^2 + \|u^1\|_{2,\Omega}^2 + \|v^0\|_{2,\Omega}^2 + \|v^1\|_{1,\Omega}^2 \right)$$

application

consider boundary-value problem

$$\begin{cases} \partial_t^2 u + \Delta^2 u + \partial_t u + \alpha v = 0 \\ \partial_t^2 v - \Delta v + \alpha u = 0 \end{cases} \qquad \text{in } \Omega \times (0, +\infty)$$

$$\Delta u(\cdot, t) = 0 = \frac{\partial \Delta u}{\partial \nu}(\cdot, t) \quad \text{and} \quad v(\cdot, t) = 0 \text{ on } \Gamma$$

then, for $0 < |\alpha| < C_\Omega^{3/2}$, energy decays at polynomial rate

$$E_1(u(t), u'(t)) + E_2(v(t), v'(t)) \le \frac{c}{t^{1/4}} \left(\|u^0\|_{4,\Omega}^2 + \|u^1\|_{2,\Omega}^2 + \|v^0\|_{2,\Omega}^2 + \|v^1\|_{1,\Omega}^2 \right)$$

different coupling parameters

for general $\alpha_1, \alpha_2 \in \mathbb{R}$ consider

$$\begin{cases} u''(t) + A_1 u(t) + Bu'(t) + \alpha_1 v(t) = 0 \\ v''(t) + A_2 v(t) + \alpha_2 u(t) = 0 \,. \end{cases}$$

- above results can be generalized replacing (H3) with

$$0 < \alpha_1 \alpha_2 < \omega_1 \omega_2$$

  - $\mathcal{E}(U) := \alpha_2 E_1(u, p) + \alpha_1 E_2(v, q) + \alpha_1 \alpha_2 \langle u, v \rangle$
  - $\frac{d}{dt}\mathcal{E}(U(t)) = -\alpha_2 |B^{1/2} u'(t)|^2$
- cannot take $\alpha_1 \alpha_2 = 0$

different coupling parameters

for general $\alpha_1, \alpha_2 \in \mathbb{R}$ consider

$$\begin{cases} u''(t) + A_1 u(t) + Bu'(t) + \alpha_1 v(t) = 0 \\ v''(t) + A_2 v(t) + \alpha_2 u(t) = 0 \,. \end{cases}$$

- above results can be generalized replacing (H3) with

$$0 < \alpha_1\alpha_2 < \omega_1\omega_2$$

  - $\mathcal{E}(U) := \alpha_2 E_1(u,p) + \alpha_1 E_2(v,q) + \alpha_1\alpha_2\langle u, v\rangle$
  - $\frac{d}{dt}\mathcal{E}(U(t)) = -\alpha_2 |B^{1/2}u'(t)|^2$
- cannot take $\alpha_1\alpha_2 = 0$

different coupling parameters

for general $\alpha_1, \alpha_2 \in \mathbb{R}$ consider

$$\begin{cases} u''(t) + A_1 u(t) + Bu'(t) + \alpha_1 v(t) = 0 \\ v''(t) + A_2 v(t) + \alpha_2 u(t) = 0\,. \end{cases}$$

- above results can be generalized replacing (H3) with

$$0 < \alpha_1 \alpha_2 < \omega_1 \omega_2$$

  - $\mathcal{E}(U) := \alpha_2 E_1(u, p) + \alpha_1 E_2(v, q) + \alpha_1 \alpha_2 \langle u, v \rangle$
  - $\frac{d}{dt}\mathcal{E}(U(t)) = -\alpha_2 |B^{1/2} u'(t)|^2$
- cannot take $\alpha_1 \alpha_2 = 0$

different coupling parameters

for general $\alpha_1, \alpha_2 \in \mathbb{R}$ consider

$$\begin{cases} u''(t) + A_1 u(t) + Bu'(t) + \alpha_1 v(t) = 0 \\ v''(t) + A_2 v(t) + \alpha_2 u(t) = 0 \,. \end{cases}$$

- above results can be generalized replacing (H3) with
$$0 < \alpha_1\alpha_2 < \omega_1\omega_2$$
  - $\mathcal{E}(U) := \alpha_2 E_1(u,p) + \alpha_1 E_2(v,q) + \alpha_1\alpha_2\langle u, v\rangle$
  - $\frac{d}{dt}\mathcal{E}(U(t)) = -\alpha_2 |B^{1/2}u'(t)|^2$
- cannot take $\alpha_1\alpha_2 = 0$

different coupling parameters

for general $\alpha_1, \alpha_2 \in \mathbb{R}$ consider

$$\begin{cases} u''(t) + A_1 u(t) + Bu'(t) + \alpha_1 v(t) = 0 \\ v''(t) + A_2 v(t) + \alpha_2 u(t) = 0\,. \end{cases}$$

- above results can be generalized replacing (H3) with
$$0 < \alpha_1 \alpha_2 < \omega_1 \omega_2$$
  - $\mathcal{E}(U) := \alpha_2 E_1(u, p) + \alpha_1 E_2(v, q) + \alpha_1 \alpha_2 \langle u, v \rangle$
  - $\frac{d}{dt}\mathcal{E}(U(t)) = -\alpha_2 |B^{1/2} u'(t)|^2$
- cannot take $\alpha_1 \alpha_2 = 0$

different coupling parameters

for general $\alpha_1, \alpha_2 \in \mathbb{R}$ consider

$$\begin{cases} u''(t) + A_1 u(t) + Bu'(t) + \alpha_1 v(t) = 0 \\ v''(t) + A_2 v(t) + \alpha_2 u(t) = 0\,. \end{cases}$$

- above results can be generalized replacing (H3) with
$$0 < \alpha_1 \alpha_2 < \omega_1 \omega_2$$
  - $\mathcal{E}(U) := \alpha_2 E_1(u, p) + \alpha_1 E_2(v, q) + \alpha_1 \alpha_2 \langle u, v \rangle$
  - $\frac{d}{dt}\mathcal{E}(U(t)) = -\alpha_2 |B^{1/2} u'(t)|^2$
- cannot take $\alpha_1 \alpha_2 = 0$

why not $\alpha_1 = 0$?

- let $A_1 = A_2 =: A$
  with positive eigenvalues $\omega_k \to +\infty$ and eigenspaces $(Z_k)_{k\geq 1}$
- $B = 2\beta I$, with $0 < \beta < \sqrt{\omega_1}$, and set $\lambda_k = \sqrt{\omega_k - \beta^2}$
- the equation $u''(t) + Au(t) + 2\beta u'(t) = 0$ with initial conditions

$$u(0) = u^0 = \sum_{k\geq 1} u_k^0 \,, \quad u'(0) = u^1 = \sum_{k\geq 1} u_k^1 \,, \quad u_k^i \in Z_k$$

  admits the solution

$$u(t) = e^{-\beta t} \sum_{k\geq 1} \left[u_k^0 \cos(\lambda_k t) + \frac{u_k^1 + \beta u_k^0}{\lambda_k} \sin(\lambda_k t) \right]$$

- $u(t) \in Z_1$ for $u^0 \in Z_1$ and $u^1 \in Z_1$

why not $\alpha_1 = 0$?

- let $A_1 = A_2 =: A$
  with positive eigenvalues $\omega_k \to +\infty$ and eigenspaces $(Z_k)_{k\geq 1}$
- $B = 2\beta I$, with $0 < \beta < \sqrt{\omega_1}$, and set $\lambda_k = \sqrt{\omega_k - \beta^2}$
- the equation $u''(t) + Au(t) + 2\beta u'(t) = 0$ with initial conditions

$$u(0) = u^0 = \sum_{k\geq 1} u_k^0\,, \quad u'(0) = u^1 = \sum_{k\geq 1} u_k^1\,, \quad u_k^i \in Z_k$$

  admits the solution

$$u(t) = e^{-\beta t}\sum_{k\geq 1}\left[u_k^0\cos(\lambda_k t) + \frac{u_k^1 + \beta u_k^0}{\lambda_k}\sin(\lambda_k t)\right]$$

- $u(t) \in Z_1 \quad \text{for} \quad u^0 \in Z_1 \quad \text{and} \quad u^1 \in Z_1$

why not $\alpha_1 = 0$?

- let $A_1 = A_2 =: A$
  with positive eigenvalues $\omega_k \to +\infty$ and eigenspaces $(Z_k)_{k\geq 1}$
- $B = 2\beta I$, with $0 < \beta < \sqrt{\omega_1}$, and set $\lambda_k = \sqrt{\omega_k - \beta^2}$
- the equation $u''(t) + Au(t) + 2\beta u'(t) = 0$ with initial conditions

$$u(0) = u^0 = \sum_{k\geq 1} u_k^0 , \quad u'(0) = u^1 = \sum_{k\geq 1} u_k^1 , \quad u_k^i \in Z_k$$

  admits the solution

$$u(t) = e^{-\beta t} \sum_{k\geq 1} \left[u_k^0 \cos(\lambda_k t) + \frac{u_k^1 + \beta u_k^0}{\lambda_k} \sin(\lambda_k t) \right]$$

- $u(t) \in Z_1 \quad$ for $\quad u^0 \in Z_1 \quad$ and $\quad u^1 \in Z_1$

why not $\alpha_1 = 0$?

- let $A_1 = A_2 =: A$
  with positive eigenvalues $\omega_k \to +\infty$ and eigenspaces $(Z_k)_{k\geq 1}$
- $B = 2\beta I$, with $0 < \beta < \sqrt{\omega_1}$, and set $\lambda_k = \sqrt{\omega_k - \beta^2}$
- the equation $u''(t) + Au(t) + 2\beta u'(t) = 0$ with initial conditions

$$u(0) = u^0 = \sum_{k\geq 1} u_k^0\,, \quad u'(0) = u^1 = \sum_{k\geq 1} u_k^1\,, \quad u_k^i \in Z_k$$

  admits the solution

$$u(t) = e^{-\beta t}\sum_{k\geq 1}\left[u_k^0\cos(\lambda_k t) + \frac{u_k^1 + \beta u_k^0}{\lambda_k}\sin(\lambda_k t)\right]$$

- $u(t) \in Z_1 \quad$ for $\quad u^0 \in Z_1 \quad$ and $\quad u^1 \in Z_1$

on the other hand, the solution to

$$v''(t) + Av(t) + \alpha u(t) = 0 \tag{1}$$

is given by $v(t) = v_1(t) + v_2(t) \in Z_1 + Z_1^{\perp}$ where

$$\begin{cases} v_1''(t) + \omega_1 v_1(t) + \alpha u(t) = 0 \\ v_2''(t) + Av_2(t) = 0 \end{cases}$$

thus, the energy

$$E(v_2(t), v_2'(t)) = \frac{1}{2}\left(|v_2'(t)|^2 + \langle Av_2(t), v_2(t)\rangle\right) = \text{const}$$

hence, $v^0 \notin Z_1$, $v^1 \notin Z_1$ ensure that the system is not stabilizable

on the other hand, the solution to

$$v''(t) + Av(t) + \alpha u(t) = 0 \qquad (1)$$

is given by $v(t) = v_1(t) + v_2(t) \in Z_1 + Z_1^{\perp}$ where

$$\begin{cases} v_1''(t) + \omega_1 v_1(t) + \alpha u(t) = 0 \\ v_2''(t) + Av_2(t) = 0 \end{cases}$$

thus, the energy

$$E(v_2(t), v_2'(t)) = \frac{1}{2}\left(|v_2'(t)|^2 + \langle Av_2(t), v_2(t)\rangle\right) = \text{const}$$

hence, $v^0 \notin Z_1$, $v^1 \notin Z_1$ ensure that the system is not stabilizable

on the other hand, the solution to

$$v''(t) + Av(t) + \alpha u(t) = 0 \tag{1}$$

is given by $v(t) = v_1(t) + v_2(t) \in Z_1 + Z_1^\perp$ where

$$\begin{cases} v_1''(t) + \omega_1 v_1(t) + \alpha u(t) = 0 \\ v_2''(t) + Av_2(t) = 0 \end{cases}$$

thus, the energy

$$E(v_2(t), v_2'(t)) = \frac{1}{2}\left(|v_2'(t)|^2 + \langle Av_2(t), v_2(t)\rangle\right) = \text{const}$$

hence, $v^0 \notin Z_1$, $v^1 \notin Z_1$ ensure that the system is not stabilizable

on the other hand, the solution to

$$v''(t) + Av(t) + \alpha u(t) = 0 \tag{1}$$

is given by $v(t) = v_1(t) + v_2(t) \in Z_1 + Z_1^{\perp}$ where

$$\begin{cases} v_1''(t) + \omega_1 v_1(t) + \alpha u(t) = 0 \\ v_2''(t) + Av_2(t) = 0 \end{cases}$$

thus, the energy

$$E(v_2(t), v_2'(t)) = \frac{1}{2}\left(|v_2'(t)|^2 + \langle Av_2(t), v_2(t)\rangle\right) = \text{const}$$

hence, $v^0 \notin Z_1$, $v^1 \notin Z_1$ ensure that the system is not stabilizable

open problems

- study localized damping with hybrid boundary conditions
- consider boundary control with hybrid boundary conditions
- obtain similar decay rates for problems in exterior domains

Thank you for your attention
Danke!

open problems

- study localized damping with hybrid boundary conditions
- consider boundary control with hybrid boundary conditions
- obtain similar decay rates for problems in exterior domains

Thank you for your attention
Danke!